Proton-proton bremsstrahlung and elastic nucleon-nucleon scattering
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Two-pion-exchange contributions: the formalism

5.1 Introduction

The phenomenological description of the $NN$ interaction based on the one-boson exchange (OBE) approximation has been of great success in the past [8, 9, 124]. The OBE model of Fleischer and Tjon [10, 75, 76] is based on a quasipotential approximation of the Bethe-Salpeter equation with the kernel consisting of a sum of one-boson exchanges. Higher order corrections would have to mimic the intermediate-range central attraction of the strong force which phenomenologically is thought to be given by correlated two-pion exchanges. This corrections to the potential will enter at one loop level and higher. There is a large number of interactions which would lead to an attractive medium range central potential. It has been suggested by Weinberg [26, 27] that chiral symmetry supplemented with a power counting scheme provides the means for separating the dominant contributions in strong interaction reactions.

In this chapter the formalism for supplementing the OBE potential with one-loop two-pion corrections will be developed. The quasipotential approach to relativistic scattering will firstly be reviewed and the OBE approximation briefly presented. Then some aspects of chiral symmetry with their implications on low energy nucleon-nucleon interaction are described. The basis of chiral perturbation theory is introduced and the leading-order chiral Lagrangian for $NN$ scattering derived [125]. The calculation of two-pion chiral loops follows the procedure developed by Passarino and Veltman [126] and used for the study of the strong force by Zuilhof [127]. For the numerical evaluation of one-loop integrals the $ff$ package of Oldenborgh [128] has been employed. Some further details of the formalism for two-nucleon scattering are also presented.

5.2 The quasipotential approach to $NN$ scattering

In relativistic field theories the T-matrix for scattering of two nucleons is a solution of the inhomogeneous Bethe-Salpeter ($BS$) equation

\[ T(p, p'; P) = V(p, p') - i \int \frac{d^4k}{(2\pi)^4} V(p, k) G_2(k, P) T(k, p'; P) , \]  

(5.1)
where $G_2(p, P)$ is the two-body propagator which is given by the direct product of two one-particle free-fermion propagators with the relative momentum $p$ and total momentum $P$. In principle, the kernel $V$ has to be taken as the sum of all irreducible diagrams. To solve such an equation is an impossible task and therefore a quasipotential approximation is usually employed. The full BS equation, written now in symbolical form,

$$T = V + VGT$$

is replaced by a set of two coupled equations

$$T = W + WgT,$$
$$W = V + V(G - g)W,$$

equivalent with the original one. The new propagator $g$ is chosen by restricting the relative energy in some way, but preserving properties like two-particle unitarity and relativistic covariance. Out of several possibilities (e.g. [129, 130, 131]), we will be using the one due to Blankenbecler-Sugar-Logunov-Tavkhelidze [83] commonly known as the BSLT approximation. The BSLT equation has been applied successfully in the description of electron-deuteron scattering [72, 73, 74] and relativistic one-boson exchange calculations for the coupled $NN - N\Delta$ scattering [132, 133, 134]. It consists of replacing the scalar part of the two-nucleon propagator

$$G_0 = \frac{1}{(\frac{1}{2}p + p)^2 - M^2 + i\epsilon} \frac{1}{(\frac{1}{2}P - p)^2 - M^2 + i\epsilon}$$

by

$$G_2^{BSLT} = i\pi \frac{1}{E_p - E} \frac{1}{(E_p + E)^2} \delta(p_0).$$

The two-particle propagator becomes

$$G_2^{BSLT}(p, P) = \frac{1}{2} (E_p - E) \delta(p_0) S^{(1)}(p, P) S^{(2)}(p, P),$$

where $E = \frac{1}{2}P_0$ and $E_p = \sqrt{p^2 + M^2}$. Using the above form of the propagator the integration over the relative energy can be performed in the BS equation. One is left with the BSLT equation, which can be handled more easily from a practical point of view

$$T(p, \tilde{p}'; P) = W(p, \tilde{p}') - i \int \frac{d^4k}{(2\pi)^4} W(p, \tilde{k}) G_2^{BSLT}(\tilde{k}, P) T(\tilde{k}, \tilde{p}'; P),$$

where the four-momentum $\tilde{k}$ is restricted by the $\delta$-function in $S_2$ such that in the center-of-mass frame of the two nucleons its time component is zero, i.e. $\tilde{k}_0=0$.

The kernel $W$ of the BSLT equation, known as a quasipotential, can be determined from Eq. (5.4), represented graphically on the third line of Fig. (5.1). It is well established that the long range part of the $NN$ force is dominated by one pion exchange.
5.3. Chiral symmetry and the $NN$ interaction

Quantum chromodynamics (QCD) is the fundamental theory of strong interactions, with the Lagrangian given by (the gluon field terms have been omitted)

$$\mathcal{L} = i\bar{q} \gamma^\mu \partial_\mu q - \bar{q} mq$$

$$= \mathcal{L}_0 + \mathcal{L}^{\chi_{SB}} \quad (5.9)$$

(OPE) contributions, and the medium range by heavier one-meson exchanges. Therefore, historically the OBE approximation to the kernel $V$ of the full BS has been used. Within this approximation the kernel $V$ is taken as the sum of the tree level exchange diagrams of the lower mesons. Contributions of the $\pi$, $\rho$, $\delta$, $\omega$, $\eta$ and $\epsilon$ mesons have been considered. As an example, in Appendix A we list the contributions of these mesons to the potential $V$ as considered in the OBE model of Fleischer and Tjon [75]. In this case, for the quasipotential $W$ we have $W = V$.

It has been shown conclusively that in the medium-range part of the $NN$ interaction two-pion exchange contributions play an important role [122]. Theoretically, the relevance of the two-pion exchange potential has already been studied a long time ago within various frameworks [13, 15]. To consider them within the presented framework the potential $V$ has to be extended to include also the one-loop irreducible diagrams.

5.3 Chiral symmetry and the $NN$ interaction

Figure 5.1: Graphical representation of the BS equation (first line), of the BSLT equation (second line) and of the integral equation satisfied by the quasipotential. The quasipotential propagator $g$ is denoted by a crossed line.
where \( q \) and \( m \) are given, in the three flavor case, respectively by

\[
q = \begin{pmatrix} q_u \\ q_d \\ q_s \end{pmatrix}, \quad m = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}.
\]

The QCD Lagrangian Eq. (5.10) exhibits a global \( U_V(1) \) symmetry and when the values of quark masses are set equal an additional \( SU_V(3) \) symmetry is observed

\[
U(1)_V: \quad q \rightarrow q' = \exp[-i\omega^V]q, \\
SU(3)_V: \quad q \rightarrow q' = \exp[-i\theta^V_T^a]q.
\] (5.10)

The former corresponds to conservation of the baryon number, while the latter is a generalization of the \( SU(2) \) isospin symmetry to three-flavored systems. In the realistic case, isospin breaking terms are proportional with the difference of the quark masses. For example, in the case of two-flavored systems one can write

\[
-\bar{q}mq = -m_u\bar{u}u - m_d\bar{d}d \\
= -1/2(m_u + m_d)(\bar{u}u + \bar{d}d) - 1/2(m_u - m_d)(\bar{u}u - \bar{d}d),
\] (5.11)

only the second term being responsible for the breaking of isospin symmetry. One could expect that the value of the parameter

\[
\eta_{SU(2)} = \frac{|m_u - m_d|}{m_u + m_d} = 0.0 \div 0.8
\] (5.12)

represents a criterion for how good the realization of the isospin symmetry in nature really is. A similar parameter for the three flavor case can be defined

\[
\eta_{SU(3)} = \frac{m_s - \frac{m_u + m_d}{2}}{m_s + \frac{m_u + m_d}{2}} = 0.8 \div 1.0.
\] (5.13)

The estimates for the two \( \eta \) parameters were obtained by using the following values of the current quark masses [135]: \( m_u = 1\div 5 \text{ MeV}, m_d = 3\div 9 \text{ MeV} \) and \( m_s = 75\div 170 \text{ MeV} \). The value of the \( \eta_{SU(2)} \) parameter is consistent with small values, while \( \eta_{SU(3)} \), due to the large value of the strange quark mass as compared with the up and down quark masses, necessarily takes values close to unity. This implies that the isospin symmetry should be badly broken for particles with strange quark content, which is contrary to what is observed in nature. One is forced to conclude that these terms in the Lagrangian are strongly suppressed by a mass scale \( \Lambda_{\chi SB} \) [136], whose precise meaning will be specified later, for which the following holds

\[
m_u, m_d, m_s \ll \Lambda_{\chi SB}.
\] (5.14)

A consequence of this constraint is that the quark mass terms in the QCD Lagrangian can be viewed as a small correction to the kinetic term \( L_0 \). Neglecting the quark mass
term for the moment, the remaining Lagrangian possesses a further axial $U(1)_A \times SU(3)_A$ symmetry

\begin{align*}
U(1)_A &: \quad q \longrightarrow q' = \exp[-i\omega A \gamma_5]q \\
SU(3)_A &: \quad q \longrightarrow q' = \exp[-i\theta^A_a \gamma_5 T^a]q.
\end{align*}

(5.15)

The $U(1)_A$ does not lead to additional conserved quantities due to the fact that it is broken by quantum corrections, this phenomenon being known as the $U(1)$ anomaly [137, 138]. The algebra of the $SU(3)_A$ is not closed due to the appearance of the $\gamma_5$ matrix. By projecting the left- and right-hand components of the spinors,

\begin{equation}
q_{R,L} = \frac{1}{2}(1 \pm \gamma_5)q, \tag{5.16}
\end{equation}

the $SU(3)_V \times SU(3)_A$ algebra can be brought into a block-diagonal form, denoted $SU(3)_L \times SU(3)_R$. The $SU(3)_L$ and $SU(3)_V$ multiplets are connected by parity transformations

\begin{equation}
P q_{L,R}(t,\vec{x}) P^{-1} = \gamma_0 q_{R,L}(t,-\vec{x}). \tag{5.17}
\end{equation}

If the axial (chiral) symmetry were realized in nature one would expect partners, of opposite parity and all the other quantum numbers the same (most importantly mass), to each already known particle found experimentally. Such chiral partners have not been found. This is not due to a heavily broken chiral symmetry, but merely to the different way it is realized. Isospin, as well as other global symmetries, is realized in a Wigner-Weyl way [139, 140], i.e. the multiplets belong to linear representations of the symmetry group. Chiral symmetry is realized in the Goldstone-Nambu way [140]. In this case even though the Lagrangian is invariant under the respective symmetry transformation, the state of lowest energy (vacuum) is not, leading to a phenomenon known as spontaneous symmetry breaking [141, 142, 143]. In this scenario the symmetry is realized non-linearly with important consequences: the vacuum is degenerate with the generators belonging to the broken part acting in this subspace, each of them corresponding in the particle spectrum to a massless spin-0 boson.

### 5.4 Effective theory of strong interactions

To solve QCD with conventional techniques at low energies has proved impossible, due to its non-perturbative character in this region. At energies of interest for nuclear physics, quarks are strongly bound into hadrons. A transition from the fundamental theory to an effective one, with the degrees of freedom represented by the mesons and baryons is of interest. This can be achieved through the hadronization of QCD [144], a technically very difficult process. One is left with a theory with both light and heavy particles. In the case of low-energy nuclear physics the latter are of no interest and one has to get rid of them somehow. The resolution of this problem lies in the “decoupling theorem” [145, 146, 147] which states that if the low theory is renormalizable, all heavy particle effects appear either as renormalization of the coupling constants in the theory or else they are
suppressed by powers of heavy particles masses. Integrating the heavy particles out, the resulting effective action is a highly non-local object in both the light fields and the heavy particles propagators. The latter ones peak at short distances and by expanding them around these points one obtains a series expansion in inverse powers of heavy particles masses for the action which is now also made up of local operators. A similar procedure can be applied in a region close to the on-shell point of a heavy particle [148]. In the case of non-renormalizable theories one has to make sure that in the local version of the effective Lagrangian a complete set of operators is used in order for the renormalization procedure to be carried out systematically order by order in the effective expansion.

The mass $M$ of the lightest integrated out particle serves as a scale for the “new” physics, not included in the effective Lagrangian. The amplitudes computed within such a framework have the form of a Taylor series with the expansion parameter $Q/M$, where $Q$ denotes the typical momentum exchanged in the reaction. In an effective theory describing the low energy QCD physics the parameter $M$ should be replaced with the QCD chiral symmetry breaking scale $\Lambda_{\chi SB}$. Its value can be estimated as follows: due to explicit chiral symmetry breaking the Goldstone bosons of low energy QCD are massive ($m_\pi = 140$ MeV, $m_K = 497$ MeV, $m_\eta = 548$ MeV) and the lowest non-Goldstone boson particle in the spectrum is the $\rho$ vector meson. In the chiral limit the masses of the Goldstone bosons vanish while the $\rho$ remains massive. This behavior is a sign of nonperturbative effects in the effective theory. From this one concludes that $\Lambda_{\chi SB} \approx m_\rho \approx 770$ MeV.

In principle the number of terms in the effective Lagrangian that satisfy certain symmetry constraints is infinite. In order for an effective theory to be useful a procedure should be defined in order to specify which of these terms are to be kept for a predefined accuracy to be reached. Such means are provided by a power counting scheme [26, 27, 30] together with the assumption of naturalness for the coupling constants [29, 30]. The latter states that on basis of naive dimensional analysis each coupling constant can be put in the form $g_i = \tilde{g}_i / \Lambda_{\chi SB}^{\delta_i}$ with the modified coupling constants adimensional and close to unity. Each Feynman diagram contributes an amplitude proportional to a certain power $\nu$ of the small parameter $Q/\Lambda_{\chi SB}$, the proportionality factor given by coupling constants. In the following we will concentrate on an effective theory of nucleons and pions. In order to derive the power counting scheme one has to decide what are the scaling properties of internal nucleon and pion lines and of the vertices of the effective theory. The pion propagator $D_F = 1/(k^2 - m_\pi^2)$ contributes two negative powers of the characteristic momentum $Q$. For the nucleon propagator, a close to on-shell form is used, which facilitates its reduction to the non-relativistic form [27]

$$S_F = \frac{P + Q + m_N}{(P + Q)^2 - m_N^2 + i\epsilon} \approx \frac{P + m_N}{2P \cdot Q + i\epsilon} = \frac{\Lambda}{Q^0 + i\epsilon}$$

with $P = (m_N, 0, 0, 0)$ and $\Lambda = (P + m_N)/2m_N$ is the projection operator onto positive-energy zero-momentum Dirac wave function. It is seen that the nucleon propagator contributes $-1$ powers of $Q$ to the amplitude. In the interaction Lagrangian, time derivatives of the nucleon wave function can appear leading to factors proportional to the nucleon energy, which are large compared to $Q$. Such terms can easily be eliminated.
by using the equations of motion for nucleons [27] or equivalently by using a decoupling procedure of the eigenstates of the four-velocity operator [149] often used in heavy-quark calculations. After the elimination of the nucleon time derivatives each vertex gives rise to terms proportional to some power of $Q$. Putting all pieces together one can write the following expression for the scaling parameter $\nu$ [27]

$$\nu = 4L - I_n - 2I_p + \sum_i V_i d_i.$$  \hspace{1cm} (5.19)

The various parameters present in the above relation have the following meaning:

- $L$ - number of loops,
- $I_n$ - number of internal nucleon lines,
- $I_p$ - number of internal pion lines,
- $V_i$ - number of vertices of type $i$,
- $d_i$ - number of derivatives appearing at a vertex of type $i$,
- $E_n$ - number of external nucleon lines,
- $n_i$ - number of nucleon fields in an interaction of type $i$.

The last two parameters will appear in later. Using the topological relations

$$L = I_n + I_p - \sum_i V_i d_i,$$

$$2I_n + E_n = \sum_i V_i n_i,$$  \hspace{1cm} (5.20)

the expression in Eq. (5.19) can be put in a more useful form

$$\nu = 2 - \frac{1}{2}E_n + 2L + \sum_i V_i \Delta_i,$$

$$\Delta_i = d_i + \frac{1}{2}n_i - 2.$$  \hspace{1cm} (5.21)

For the case of two-nucleon scattering, which is of interest in this chapter, the above relation reduces to $\nu = 2L + \sum_i V_i \Delta_i$. For the terms that are chiral invariant or break chiral symmetry proportional to the quark mass is can be shown [30] that $\Delta_i \geq 0$, meaning that $\Delta_i$ provides a useful ordering scheme for the chiral Lagrangian

$$\mathcal{L} = \sum_n \mathcal{L}^{(n)}, \quad n \equiv \Delta_i.$$  \hspace{1cm} (5.22)
5.5 Construction of the effective Lagrangian

In this work we would like to consider the leading and next-to-leading order contributions due to two-pion-exchange (TPE) graphs to the nucleon-nucleon potential. As can be seen from Eq. (5.21) the lowest value possible for the scaling parameter is \( \nu = 0 \) and corresponds to the tree-level one-pion exchange diagram. We will not consider contact four-fermion contributions to the Lagrangian and therefore \( n_i = 2 \). The case \( \nu = 1 \) cannot be realized for nucleon-nucleon scattering. Two-pion contributions can first appear at \( \nu = 2 \) order through terms with \( \Delta_i = 0 \) and thus containing one derivative of the pion fields and one loop integral. One-loop two-pion terms also appear at order \( \nu = 3 \) through terms with \( d_i = 2 \) \( (\Delta_i = 1) \). We will sketch in the following the constructions of these terms following [125], to which we refer for more details.

The construction of chiral invariant terms proceeds most easily once the chiral covariant derivatives for both the pion and nucleon are introduced [30]

\[
\vec{D}_\mu = \frac{1}{1 + \pi^2/F_\pi^2} \partial_\mu \vec{\pi}/F_\pi = D^{-1} \partial_\mu \vec{\pi}/F_\pi, \tag{5.23}
\]

\[
\mathcal{D}_\mu \psi = (\partial_\mu + \frac{i}{F_\pi} c_0 \vec{\tau} \cdot \vec{D}_\mu) \psi = (\partial_\mu + \vec{\tau} \cdot \vec{E}_\mu) \psi. \tag{5.24}
\]

Chiral symmetry requires that \( c_0 = 1, F_\pi = 185 \text{ MeV} \) is the pion decay constant, \( g_A = 1.2573 \) is the Gamow-Teller coupling and \( \tau \) are the isospin Pauli matrices. Next-to-leading order two-pion diagrams can also originate from Lagrangians involving two derivatives which are of the form

\[
\mathcal{D}_\nu \vec{D}_\mu, \quad \mathcal{D}_\nu \mathcal{D}_\mu \psi. \tag{5.25}
\]

Out of these building blocks a large number of terms consistent with chiral symmetry and Lorentz invariance can be constructed. Their number is greatly reduced by imposing that parity and charge conjugation invariance also hold. The following terms survive for \( \Delta_i = 0 \) [125]

\[
\vec{D}_\mu \cdot \vec{D}_\nu, \quad \bar{\psi} \gamma_\mu \mathcal{D}_\mu \psi, \quad \psi \gamma_\mu \gamma_5 \vec{\tau} \cdot \vec{D}_\mu \psi, \quad m_\pi^2 D^{-1} \vec{\pi}^2. \tag{5.26, 5.27}
\]

The first two terms give rise to the kinetic energy pieces for the pions and nucleon respectively and to the Weinberg-Tomozawa two-pion interaction term. The third represents the well known pseudovector coupling of pions to nucleons while the last one is a chiral symmetry breaking term giving rise to the pion mass. Contributions for the \( \Delta_i = 1 \) case are a bit more numerous [125]

\[
\bar{\psi} \vec{D}_\mu \cdot \vec{D}_\nu \psi, \quad \bar{\psi} \vec{\tau} \cdot \vec{D}_\mu \times \vec{D}_\nu \sigma^{\mu\nu} \psi, \quad \bar{\psi} \gamma_5 \vec{\tau} \cdot \vec{D}_\mu \mathcal{D}_\mu \psi, \quad \bar{\psi} \mathcal{D}_\mu \vec{D}_\mu \psi, \quad \bar{\psi} \mathcal{D}_\mu \vec{D}_\mu \psi, \quad \bar{\psi} \vec{D}_\mu \gamma_5 \vec{D}_\mu \psi + i \bar{\psi} \vec{D}_\mu \vec{\tau} \cdot \vec{D}_\mu \gamma_5 \psi. \tag{5.28, 5.29, 5.30, 5.31}
\]
Construction of the effective Lagrangian

Figure 5.2: Tree level and loop diagrams contributing to the OPE and TPE potentials. The \( NN2\pi \) WT vertex is represented by a full circle.

\[
i\bar{\psi}\vec{\tau}\cdot\vec{D}\rightarrow\bar{\psi}\vec{D}\psi,\quad m_\pi^2D^{-1}\pi F_\pi\bar{\psi}\psi,\quad im_\pi^2\frac{1}{F_\pi}D^{-1}\bar{\psi}\gamma_5\vec{\tau}\cdot\vec{\pi}\psi.
\]

The arrow on \( D \) shows in which direction the derivative operator present in the nucleon chiral derivative should act. The last two terms break chiral symmetry explicitly. Not all of these interaction terms will be kept in our calculation. Only those that will give the leading contributions when a non-relativistic reduction is performed will be kept and terms contributing to the three or higher pion interactions will be omitted, i.e. the factor \( D^{-1} \) will be developed in powers of the pion field and only the leading term kept. Whether a term will survive or not after the non-relativistic reduction is performed depends to which of the two following categories of Dirac space operators it belongs to:

\[
\Gamma_1 = \{I, \gamma^0, \gamma^5, \gamma^5\gamma^5, \sigma^{ij}\} \approx \mathcal{O}(1),
\]

\[
\Gamma_2 = \{\gamma^k, \gamma^5, \gamma^5\gamma^5, \sigma^{ij}\} \approx \mathcal{O}\left(\frac{Q}{m_N}\right) \approx \mathcal{O}\left(\frac{Q^2}{\Lambda_{\chi}^2}\right).
\]

The second group of matrices \( \Gamma_2 \) mix the small and big components of Dirac spinors and terms made out of them will be suppressed with respect to those from the set \( \Gamma_1 \). Using this argument the interaction terms in Eq. (5.29a),(5.31) and (5.33b) can safely be ignored. The term in Eq. (5.32) can be shown [125] to give rise to interactions with at least three pions and will also be neglected. The term in Eq. (5.30) gives rise to three-pion interactions as well as to a term which resembles the one given by Eq. (5.29a) [125]. We will neglect it in our calculation. The expression in Eq. (5.29b) is the term associated in chiral perturbation with the coupling constant \( c_2 \). Due to the appearance of the nucleonic energy it will only contribute to higher orders in the effective expansion [122, 125]. Finally, we have been left with three interactions, the ones in Eq. (5.28) and (5.33a). We want to stress that we have only used results from chiral perturbation theory to decide which of
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Figure 5.3: One-loop diagrams with $\nu=3$ contributing to the TPE potential. The WT interaction is represented by a full circle while the sub-leading order $NN2\pi$ vertices with coupling constants $c_1, c_3$ or $c_4$ by a full square.

The possible two-pion-interactions would give the dominant contribution in the low energy nucleon-nucleon interaction but in the actual calculations the full relativistic version of the selected ones will be used. Referring back to Eq. (5.22) the corresponding Lagrangians for $n=0$ and $n=1$ take respectively the form

\begin{align}
\mathcal{L}^{(0)} &= \bar{\psi}(i\gamma_\mu \partial^\mu - m_N - \frac{c_0}{F_\pi^2} \gamma_\mu \vec{\tau} \cdot \vec{\pi} \times \partial^\mu \vec{\pi} + \frac{gA}{F_\pi} \gamma_\mu \gamma_5 \vec{\tau} \cdot \partial^\mu \vec{\pi} ) \psi, \\
\mathcal{L}^{(1)} &= \bar{\psi}(\frac{8 c_1}{F_\pi^2} m_N^2 \vec{\pi}^2 - \frac{4 c_3}{F_\pi^2} \partial_\mu \vec{\pi} \cdot \partial^\mu \vec{\pi} + \frac{2 c_4}{F_\pi^2} \sigma_{\mu\nu} \vec{\tau} \cdot \partial^\mu \vec{\pi} \times \partial^\nu \vec{\pi} ) \psi.
\end{align}

The coupling constants of the $n=1$ Lagrangian have been assigned in agreement with conventions used $\pi N$ [150, 151] as well as $NN$ [32, 33, 122] scattering. In an equivalent approach [151, 152] but with a different choice for the fundamental building blocks a similar Lagrangian has been obtained, with three extra couplings, labeled $c_5, c_6, c_7$, due isospin symmetry breaking and external electromagnetic field respectively [153, 154]. Instead of the $c_1, c_3, c_4$ coupling constants other authors [30] use $B_1=4c_3, B_2=-4c_1-1/m_N$ and $B_3=8c_1$.

The $\mathcal{L}^{(0)}$ Lagrangian gives rise to contributions that scale with the powers $\nu=0$ and $\nu=2$ of the expansion parameter $Q/\Lambda_{\chi SB}$. The only diagram with $\nu=0$ is the tree-level one-pion exchange depicted in Fig. (5.2a). The other ones, (Fig. 5.2b-i), are next-to-leading order (NLO) contributions with $\nu=2$. Some of them don’t have to be fully included in the kernel $V$: the direct box diagram(Fig. 5.2b) is reducible and part of it is generated by iterating the BSLT equation once; the ones in Fig. (5.2d) and Fig. (5.2i) represent vertex corrections to the OPE potential [155]. Terms with $\nu=3$ arise from diagrams involving terms from both $\mathcal{L}^{(0)}$ and $\mathcal{L}^{(1)}$, corresponding to the next to next-to-leading order (NNLO) contributions to the $NN$ potential Fig. (5.3). They are triangular graphs with one of the vertex a $NN2\pi$ interaction with the coupling $c_3$ or $c_4$ and football graphs with one of the vertex the WT interaction and the other one...
5.6. Explicit expressions for the two-pion diagrams

To solve the BSLT equation, the quasipotential \( W \) will have to be determined once \( V \) is known. For that, besides the irreducible one-loop diagrams contributing to the TPE potential, the expressions of the full direct-box and the quasi-potential direct-box diagrams will also have to be considered (see Eq. (5.4) or Fig. (5.1)). For sake of completeness the expressions of the former are presented in this section. The labeling convention used in this section for the momenta associated to a generic one-loop diagram is exemplified in Fig. (5.4) for the direct-box diagram.

The direct and crossed box diagrams are given respectively by

\[
I^{(DB)} = \alpha^{(DB)} \int d^4k \left[ \frac{\bar{u}(q_1) (\vec{k} + \vec{p}_1 - \vec{q}_1) \gamma_5 (\vec{k} + \vec{q}_1 + m_N) \vec{k} \gamma_5 u(p_1)}{(k + p_1)^2 - m_N^2} \right] \times \frac{[\bar{u}(q_2) (\vec{k} + \vec{q}_2 - \vec{p}_2) \gamma_5 (\vec{k}_2 - \vec{k} + m_N) \vec{k} \gamma_5 u(p_2)]}{(k - p_2)^2 - m_N^2} \cdot D_{\pi_1 \pi_2},
\]

and

\[
I^{(CB)} = \alpha^{(CB)} \int d^4k \left[ \frac{\bar{u}(q_1) (\vec{k} + \vec{p}_1 - \vec{q}_1) \gamma_5 (\vec{k} + \vec{q}_1 + m_N) \vec{k} \gamma_5 u(p_1)}{(k + p_1)^2 - m_N^2} \right] \times \frac{[\bar{u}(q_2) \gamma_5 (\vec{k} + \vec{q}_2 + m_N) (\vec{k} + \vec{p}_1 - \vec{q}_1) \gamma_5 u(p_2)]}{(k + q_2)^2 - m_N^2} \cdot D_{\pi_1 \pi_2},
\]

The coefficients \( \alpha^{(DB)} \) and \( \alpha^{(CB)} \) contain coupling constants, normalization and isospin factors. For the two diagrams above they are given by

\[
\alpha^{(DB)} = \frac{1}{4\pi^4} \left( \frac{g_{PV}^2}{4\pi} \right)^2 \left[ 9 - 4 \cdot I (I + 1) \right],
\]

\[
\alpha^{(CB)} = \frac{1}{4\pi^4} \left( \frac{g_{PV}^2}{4\pi} \right)^2 \left[ 4 \cdot I (I + 1) - 3 \right],
\]

with \( g_{PV} \) the pseudovector \( NN\pi \) coupling constant, which can be determined from the Goldberger-Treiman relation \( g_{PV} = g_A/F_\pi \) in terms of the pion decay constant and the Gamow-Teller coupling; \( I \) is the total isospin of the two-nucleon system. The pion propagators are included in the factor \( D_{\pi_1 \pi_2} \)

\[
D_{\pi_1 \pi_2} = \frac{1}{k^2 - m_\pi^2} \cdot \frac{1}{(k + p_1 - q_1)^2 - m_\pi^2}.
\]
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Figure 5.4: Labeling of the momenta for the direct-box diagram. For any diagram consider here, the integration momentum $k$ is always assigned to the meson line attached to the incoming nucleon 1.

Next, the expression of the other diagrams stemming from the minimal chiral model ($\mathcal{L}^{(0)}$) will be presented. The expression of the $c_0$ triangle diagram in Fig. (5.2e) is given by

$$I^{(TRc_0)} = \frac{\alpha^{(TRc_0)}}{\mathcal{F}_\pi^2} \int d^4k \left[ \bar{u}(q_1) \left( \frac{k + p_1 - q_1}{k + p_1 + m_N} \right) \gamma_5 u(p_1) \right]^{(1)} \times$$

$$\left[ \bar{u}(q_2) \left( 2k + p_1 - q_1 \right) u(p_2) \right]^{(2)} \cdot \mathcal{D}_{\pi,\pi_2},$$

where $\alpha^{(TRc_0)} = -\frac{1}{16\pi^5} \frac{c_0}{\mathcal{F}_\pi^2} \frac{g_{PV}^2}{4\pi} \left[ 4I(I+1) - 6 \right].$

It suffices to only list the expression of one of the triangle diagrams. The other one can easily be obtain by replacing $p_1$ and $q_1$ with $p_2$ and $q_2$ respectively and changing the sign of the integration variable $k$. The expression for the $c_0$ football diagram is

$$I^{(FTc_0)} = \frac{\alpha^{(FTc_0)}}{\mathcal{F}_\pi^2} \int d^4k \left[ \bar{u}(q_1) \left( 2k + p_1 - q_1 \right) u(p_1) \right]^{(1)} \times$$

$$\left[ \bar{u}(q_2) \left( 2k - p_2 + q_2 \right) u(p_2) \right]^{(2)} \cdot \mathcal{D}_{\pi,\pi_2},$$

For these two diagrams the normalization coefficients $\alpha$ are given respectively by

$$\alpha^{(TRc_0)} = -\frac{1}{16\pi^5} \frac{c_0}{\mathcal{F}_\pi^2} \frac{g_{PV}^2}{4\pi} \left[ 4I(I+1) - 6 \right],$$

$$\alpha^{(FTc_0)} = \frac{1}{2} \left( \frac{c_0}{\mathcal{F}_\pi^2} \right)^2 \left[ 4I(I+1) - 6 \right].$$

Some of the next-to-leading order contributions to the TPE potential are given by the triangle diagrams in Fig. (5.3a). Their expressions are given by

$$I^{(TRc_i)} = \frac{\alpha^{(TRc_i)}}{\mathcal{F}_\pi^2} \int d^4k \left[ \bar{u}(q_1) \left( \frac{k + p_1 - q_1}{k + p_1 + m_N} \right) \gamma_5 u(p_1) \right]^{(1)} \times$$

$$\left[ \bar{u}(q_2) O_i u(p_2) \right]^{(2)} \cdot \mathcal{D}_{\pi,\pi_2},$$

with $i=1,3,4$. The operators $O_i$ are respectively given by

$$O_1 = 1,$$

$$O_3 = k \cdot (k - p_2 + q_2),$$

$$O_4 = (q_2 - q_1) k - k (q_2 - q_1),$$

where $k = k + p_1 - q_1$.
and with $\alpha$ coefficients given by

$$
\alpha^{(TRc_1)} = \frac{3}{\pi^5} \frac{g_{PV}^2 c_1 m^2_{\pi}}{F_{\pi}^2},
$$

$$
\alpha^{(TRc_3)} = -\frac{3}{2\pi^5} \frac{g_{PV}^2 c_3}{F_{\pi}^2},
$$

$$
\alpha^{(TRc_4)} = -\frac{1}{8\pi^5} \frac{g_{PV}^2 c_4}{F_{\pi}^2} [4I(I + 1) - 6].
$$

As before, the expression of the triangle diagrams of Fig. (5.3b) can be obtained from the ones already presented by a permutation of the variables referring to nucleon 1 and 2 with each other.

We are left with the evaluation of the $c_0 - c_i$ football diagrams ($i=1,3,4$). Out of these three only the $c_0 - c_4$ football diagram will give a finite answer, the other two being identically zero due to their particular analytic structure. The relevant expression is

$$
I^{(FTc_0,c_4)} = \alpha^{(FTc_0,c_4)} \int d^4k \left[ \bar{u}(q_1) (2\not{k} + \not{p}_1 - \not{q}_1) u(p_1) \right]^{(1)} \times
$$

$$
\left[ \bar{u}(q_2) (\not{p}_2 - \not{q}_2)(\not{k} - \not{k} (\not{p}_2 - \not{q}_2)) u(p_2) \right]^{(2)} \cdot D_{\pi_1\pi_2},
$$

with

$$
\alpha^{(FTc_0,c_4)} = -\frac{1}{(2\pi)^6} \frac{c_0 c_4}{F_{\pi}^4} [4I(I + 1) - 6].
$$

In this section all the meson-nucleon vertices were considered to be pointlike. In the actual calculations, a dipole form-factor which depends only on the momentum of the exchanged meson has been considered at each vertex. To arrive at the corresponding expressions for the loop integrals one has to make the following substitution for the pion propagator

$$
\frac{1}{k^2 - \mu^2} \rightarrow \frac{1}{k^2 - \mu^2} \cdot \left( \frac{\Lambda^2}{k^2 - \Lambda^2} \right)^2.
$$

5.7 The quasipotential direct box

The quasipotential direct box can be obtained by substituting in the expression of the full direct box, Eq. (5.37), the scalar part of the intermediate two-fermion propagator with its BSLT version in Eq. (5.6). The $k_0$ integral can then be trivially performed. In principle at this stage a simple three-dimensional numerical integration will provide the final answer (taking care properly of the principle value singularities) but a decomposition of the remaining integral in smaller pieces will allow the implementation of a numerical dimensional regularization besides the cut-off one. In this section a different labeling convention, more appropriate for this particular calculation, for the integration four-momentum $k$ has been made Fig. (5.5). The first step is the decomposition of the operators belonging to each of the two fermion lines into monomials of the length of
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**Figure 5.5:** The labeling convention used for the computation of the quasipotential direct box. $P=(2E,0)$ is the total momentum in the CM frame and $p, q, k$ are relative momenta. Due to the quasipotential approximation $k_0=0$ while for the other two momenta similar relations only hold on-shell.

the integration three momentum $\vec{k}$. We will limit ourselves to the on-shell case here. A generalization to the half-off-shell case needed for the iteration of the BSLT equation is straightforward. For the fermion line 1 (see Fig. (5.5)) one has

$$O_1 = \sum_{i=0,3} O_{ij} k^i,$$  

$$O_{ij} = \begin{cases} (E^3 + 3m_N E)\gamma_0 - (m_N^3 + 3m_N E^2), & j = 1 \\ -(E^2 + 3m_N^2)\vec{\gamma} \cdot \vec{k}, & j = 2 \\ -E\gamma_0 + 3m_N, & j = 3 \\ \vec{\gamma} \cdot \vec{k}. & j = 4 \end{cases}$$  

The fermion line 2 is identical to fermion line 1 with the exception of a minus sign in front of each appearance of $\vec{k}$

$$O_2 = \sum_{i=0,3} O_{i2} k^i,$$  

$$O_{i2} = (-1)^i O_{ij}.$$  

The expression of the quasipotential direct box can then be written

$$I^{QDB} = \alpha^{QDB} \sum_{i,j=0}^3 \int d\Omega_{\vec{k}} \left[ \bar{u}(\frac{1}{2}P + q) O_{ij} u(\frac{1}{2}P + p) \right] \times$$  

$$\left[ \bar{u}(\frac{1}{2}P - q) O_{ij} u(\frac{1}{2}P - p) \right] \cdot I(d, i + j),$$  

$$I(d, l) = -\frac{1}{2} \int_0^\infty dk \frac{k^{d+l-1}}{(E + E_\vec{k})(E - E_\vec{k})(\vec{k} - \vec{q})^2 + m_N^2 (\vec{k} - \vec{p})^2 + m_N^2}.$$  

The symbol $d$ represents the number of spatial dimensions and $\alpha^{QDB} = \alpha^{DB}$. Divergences can only occur from the 1 dimensional integral $I(d, l)$. It can be regulated with
5.8. Method of evaluation of one-loop integrals

a form-factor, *i.e.* of dipole form, or via dimensional regularization. In the latter case the divergent integrals have to be decomposed further in order to isolate the ultraviolet divergence from the principal value pole and then with specific techniques the integrals can be evaluated and the infinities removed analytically. A simpler method is to subtract from the divergent integrands their asymptotic form (with sufficient terms in the asymptotic expansion) to render finite integrals that can be evaluated numerically. The asymptotic integrands are of a Laurent series type and the respective integrals can easily be evaluated and dimensionally regularized. This second method has been employed in the actual calculations.

5.8 Method of evaluation of one-loop integrals

The evaluation of the two-pion exchange diagrams in this thesis is based on the general method for evaluating one-loop integrals devised by Passarino and Veltman [126, 157]. It has already been applied to the case of relativistic NN scattering by Zuilhof and Tjon [127] in the process of evaluation of the direct-box and crossed-box diagrams with both pseudoscalar and pseudovector coupling. We follow closely the procedure described in [127]; for completeness its relevant steps are reproduced in this section. Similar techniques have been applied by Celenza et al. [158] in a study of nucleon-nucleon interaction and chiral symmetry.

Following the work of Passarino and Veltman [126] we define the scalar and tensor two-point, three-point and four-point functions

\[
B_0; B_\mu; B_{\mu\nu} = \int d^4k \frac{1; k_\mu; k_{\mu\nu}}{[k^2 - m_1^2] \left[(k + p)^2 - m_2^2\right]}, \tag{5.54}
\]

\[
C_0; C_\mu; C_{\mu\nu}; C_{\mu\nu\rho} = \int d^4k \frac{1; k_\mu; k_{\mu\nu}; k_{\mu\nu\rho}}{[k^2 - m_1^2] \left[(k + p)^2 - m_2^2\right] \left[(k + p + q)^2 - m_3^2\right] \left[(k + p + q + r)^2 - m_4^2\right]},
\]

\[
D_0; D_\mu; D_{\mu\nu}; D_{\mu\nu\rho} = \int d^4k \frac{1; k_\mu; k_{\mu\nu}; k_{\mu\nu\rho}}{[k^2 - m_1^2] \left[(k + p)^2 - m_2^2\right] \left[(k + p + q)^2 - m_3^2\right]} \left[1 \prod_{i=4}^n \left[(k + p + q + \cdots)^2 - m_i^2\right]\right] \nonumber
\]

The scalar point functions \(B_0, C_0\) and \(D_0\) have to be evaluated explicitly, either numerically or, where possible, using their analytic expressions. The tensor point functions can be written in terms of scalar moments, which in turn can be evaluated in terms of lower moments and/or scalar point functions. Some details about this reduction, nomenclature and a simple but illustrative example for the whole machinery are given in Appendix B. For the complete set of formulas the reader is referred to [126]. The convention for labeling of the external momenta and internal masses is given in Fig. (5.6) and agrees with the convention used in [126] and [128]. In the present study scalar moments up to \(D_2\) and \(C_3\) were needed. They were computed using the \(ff\) package developed by van Oldenborgh et al. [128].

Any of the one-loop diagram with at most four external legs can be brought in the
Figure 5.6: Convention used for the labeling of four momenta for the four-point, three-point and respectively two-point scalar functions.

following form

$$L = \sum_{r=0}^{N_B} O_B^{\mu_1 \ldots \mu_r} [r] B_{\mu_1 \ldots \mu_r} + \sum_{r=0}^{N_C} O_C^{\mu_1 \ldots \mu_r} [r] C_{\mu_1 \ldots \mu_r} + \sum_{r=0}^{N_D} O_D^{\mu_1 \ldots \mu_r} [r] D_{\mu_1 \ldots \mu_r} \quad \text{(5.55)}$$

by separating the $k_{\mu_1 \ldots \mu_r}$ from the spinor part of the numerator and then together with the scalar parts of the internal propagators identifying them with the scalar and tensor loop integrals in Eq. (5.55). Here $N_B$, $N_C$ and $N_D$ are the highest rank appearing for the given diagram for the tensor two-point, three-point and four-point functions respectively. By making use of the expression of tensor loop integrals in terms of scalar moments (see Appendix B), the expression of the one-loop diagram can be written

$$L = \sum_{r=0}^{N_B} \sum_{i=1}^{n_B^r} O_B[r,i] B_{r,i} + \sum_{r=0}^{N_C} \sum_{i=1}^{n_C^r} O_C[r,i] C_{r,i} + \sum_{r=0}^{N_D} \sum_{i=1}^{n_D^r} O_D[r,i] D_{r,i} \quad \text{(5.56)}$$

The coefficients $n_B^r, n_C^r$ and $n_D^r$ represent for a given $r$ the number of moments $Br$, $Cr$ and $Dr$. For example $n_C^1 = 2$ and $n_C^2 = 4$ as can be seen from Eq. (B-5.2). The operators $O_B$, $O_C$ and $O_D$ contain the whole spin structure of the diagram in questions, and for the case of two-fermion scattering can be of put the form

$$O_M[r,i] = O_M^{(1)}[r,i] \times O_M^{(2)}[r,i], \quad M = B, C, D \quad \text{(5.57)}$$

with the superscripts denoting the fermion line number. Their matrix elements between the two-particle helicity states defined in Appendix C are then evaluated in the center-of-mass frame of the two-nucleons. The two-particle states are the direct product of one-particle helicity spinors which satisfy the Dirac equation

$$\not{p} u^\pm(\not{p}) = [m_N + \gamma^0 (p_0 \mp E_p)] u^\pm(\not{p}) \quad \text{(5.58)}$$

where the upper index $\pm$ label the positive or the negative energy solutions.
In the two-fermion spin space the following sixteen operators form a basis \[127\]

\[
\begin{align*}
O_1 &= 1^{(1)} 1^{(2)} \\
O_5 &= \gamma_5^{(1)} \gamma_{\mu}^{(2)} \\
O_9 &= \gamma_5^{(1)} \gamma_5^{(2)} \\
O_{13} &= \gamma_5^{(1)} O_5 \gamma_5^{(2)} \\
O_2 &= \gamma_0^{(1)} 1^{(2)} \\
O_6 &= \gamma_0^{(1)} O_5 \\
O_{10} &= \gamma_0^{(1)} O_9 \\
O_{14} &= \sigma_{\mu\nu}^{(1)} \sigma_{\mu\nu}^{(2)} \\
O_3 &= 1^{(1)} \gamma_0^{(2)} \\
O_7 &= O_5 \gamma_0^{(2)} \\
O_{11} &= O_9 \gamma_0^{(2)} \\
O_{15} &= \gamma_0^{(1)} O_14 \\
O_4 &= \gamma_0^{(1)} \gamma_0^{(2)} \\
O_8 &= \gamma_0^{(1)} O_5 \gamma_0^{(2)} \\
O_{12} &= \gamma_0^{(1)} O_9 \gamma_0^{(2)} \\
O_{16} &= O_{14} \gamma_0^{(2)}
\end{align*}
\]

By applying the off-shell Dirac’s equation Eq. (5.58) the spin structure can be easily reduced to a linear combination of the above operators. One can write

\[
O_M[r,i] = \sum_{j=1}^{16} b^M_{r,j} O_j, \quad M = B, C, D \tag{5.59}
\]

leading to the final expression for the considered diagram

\[
L = \sum_{j=1}^{16} c_j O_j \tag{5.60}
\]

Evaluating the \(c_j\) coefficients involves cumbersome algebraic operations. Also due the great number of terms, an automated evaluation of these coefficients was desirable. For these purpose the computer algebra program FORM [159] has been used to write a code that determines the analytical expression for each of the \(c_j\)’s for a particular diagram in terms of the scalar moments and of the kinematical variables of the process. Next, the values of the scalar moments are computed using the \(ff\) package, which in turn allows the numerical evaluation of the coefficients \(c_j\). Then the matrix elements of the sixteen \(O_j\) operators in the two-particle helicity basis are computed and using Eq. (5.60) the numerical value of the one-loop diagram is obtained. For the chosen interaction Lagrangian only the \(c_1 \cdots c_8\) coefficients take nonzero values.

### 5.9 An example: the \(c_0\) triangle diagram

In this section an explicit example for the decomposition of a Feynman diagram into scalar moments will be presented. The \(c_0\) triangle diagram Eq. (5.41) has been chosen since its decomposition requires all the elements presented in the previous section. The decomposition of the football diagrams is simple, with a short final result, while the evaluation of the box, crossed-box and subleading triangle diagrams is quite involved resulting to up to a thousand terms in the final result for each. A similar example, for the pseudoscalar crossed-box, has been presented by Zuilhof [127].

The first step is to bring the expression of the loop integral in the form shown in Eq. (5.55). In this particular case due to the fact that each nucleon lines contains an even number of \(\gamma_5\) matrices we can eliminate them by using the anticommutation

\[
\begin{align*}
\end{align*}
\]
relations with $\gamma_\mu$ matrices. The numerator contains four powers of the integration momentum $k$ which means that the final result would contain C4 scalar moments. Since the \textit{ff} package only provides numerical values of moments up to C3 the powers of $k$ are reduced by the use of the following identity

$$k^2 = [(k + p_1)^2 - m_N^2] - 2k \cdot p_1 - p_1^2 + m_N^2.$$

This has the effect of cancellation of the nucleon propagator from some of the terms leading to the appearance of the scalar two-point function moments (B) in the final result. Only nucleon propagators should be canceled in this way. Pion propagators are leading to the appearance of the scalar two-point function moments (B) in the final result. The expressions for the \textit{O} integrals in Eq. (5.54) presented in Appendix B are then used.

\begin{align*}
O^\mu[2]_B &= 2 \left[ \gamma^\mu \right]^{(1)} \times \left[ \gamma^\nu \right]^{(2)}, \\
O^\mu[1]_B &= - \left[ \gamma^\mu \right]^{(1)} \times \left[ \not{p}_2 - \not{q}_2 \right]^{(2)} - 2 \left[ \not{q}_1 + m_N \right]^{(1)} \times \left[ \gamma^\mu \right]^{(2)}, \\
O_B[0] &= \left[ \not{q}_1 + m_N \right]^{(1)} \times \left[ \not{p}_2 - \not{q}_2 \right]^{(2)}. \\
\end{align*}

The expressions for the $O^{\mu_1 \cdots \mu_r}[r]$ are somewhat more involved

\begin{align*}
O^\mu[2]_B &= 2 \left[ (m_N \gamma^\mu - \gamma^\mu \not{q}_1 + 2q_1^\mu)(\not{p}_1 + m_N) \right]^{(1)} \times \left[ \gamma^\nu \right]^{(2)}, \\
O^\mu[1]_B &= \left[ (m_N \gamma^\mu - \gamma^\mu \not{q}_1 + 2q_1^\mu)(\not{p}_1 + m_N) \right]^{(1)} \times \left[ \not{p}_2 - \not{q}_2 \right]^{(2)} + 2 \left( p_1^2 - m_N^2 \right) \left[ \not{q}_1 + m_N \right]^{(1)} \times \left[ \gamma^\mu \right]^{(2)}, \\
O_B[0] &= \left( p_1^2 - m_N^2 \right) \left[ \not{q}_1 + m_N \right]^{(1)} \times \left[ \not{p}_2 - \not{q}_2 \right]^{(2)}. \\
\end{align*}

The expressions for the integrals in Eq. (5.54) presented in Appendix B are then used. After the use of the off-shell Dirac equation Eq. (5.58) and some rearrangements one arrives at the final result for the $c_1 \ldots c_8$ coefficients

\begin{align*}
c_1 &= 4m_N^2 \left( m_N^2 - p_1^2 \right)(C_{11} - C_{12}) - 2m_N^2(E_1 E_1' + 2m_N^2 + 2p_1^2)C_{21}, \\
&- 2m_N^2(E_1 E_1' + 2m_N^2 + 2q_1^2)C_{22} + 4m_N^2(E_1 E_1' + 2m_N^2 + p_1^2 + q_1^2)C_{23}, \\
c_2 &= 2m_N E_1'(m_N^2 - p_1^2)(C_{11} - C_{12}) - 2m_N(2m_N^2 E_1 + m_N^2 E_1' + p_1^2 E_1')C_{21} - 2m_N(m_N^2 E_1 + 2m_N^2 E_1' + q_1^2 E_1)C_{22} + 2m_N(3m_N^2 E_1 + 3m_N^2 E_1' + q_1^2 E_1)C_{23}, \\
c_3 &= 2m_N(E_2 - E_2')B_0 + 4m_N(E_2 - E_2')B_{11} + 2m_N(m_N^2 E_2 - p_1^2 E_2)C_{11} + 2m_N(2\sqrt{s} - E_2)(p_1^2 - m_N^2)C_{11} - m_N(E_1 E_1' + 2p_1^2)(E_2 - E_2')C_{11} + m_N(E_1 E_1' + 2m_N^2 + 2q_1^2)(E_2 - E_2')C_{12} - 4m_N(\sqrt{s} - E_2')(p_1^2 - m_N^2)C_{12} + 2m_N(E_1 E_1' + 2m_N^2 + 2p_1^2)(\sqrt{s} - E_2)(C_{21} - C_{23}) + 2m_N(E_1 E_1' + 2m_N^2 + 2q_1^2)(\sqrt{s} - E_2')(C_{22} - C_{23}) + 8m_N E_1 C_{24}, \\
c_4 &= E_1'(E_2 - E_2')B_0 + (3E_1' - E_1)(E_2 - E_2')B_{11} + 2(E_1' - E_1)(E_2 - E_2')B_{21}.
\end{align*}
\[ -E_i'(E_2 - E_{2'})p_1^2 - m_N^2 C_0 + E_i'(m_N^2 E_{2'} - p_1^2 E_2)C_{11} \]
\[ -2m_N^2 E_1 + p_1^2 E_i'(E_2 - E_{2'})C_{11} + (2\sqrt{s}E_1 - E_2 E_i'(p_1^2 - m_N^2)C_{11} \]
\[ + (2m_N^2 E_i' + m_N^2 E_1 + q_1^2 E_1)(E_2 - E_{2'})C_{12} - 2E_i'(\sqrt{s} - E_{2'})C_{12} \]
\[ + (4m_N^2 E_1 + 2m_N^2 E_i' + 2p_1^2 E_i'[(\sqrt{s} - E_{2})(C_{21} - C_{23}) \]
\[ + (4m_N^2 E_i' + 2m_N^2 E_1 + 2q_1^2 E_1)(\sqrt{s} - E_{2'})C_{21} - C_{23}) + 4E_i E_i'C_{24}, \]
\[ c_5 = 2B_{22} + (8m_N^2 - 2E_1 E_i')C_{24}, \]
\[ c_6 = -4m_N(E_1 - E_i')C_{24}, \]
\[ c_7 = 0, \]
\[ c_8 = 0. \]

In the above relations \( s = (p_1 + p_2)^2 = (q_1 + q_2)^2 \) is the total energy; the labeling of momenta corresponds to the case depicted in Fig. (5.4) and one has \( E_1 = E_{p_1}, E_i' = E_{q_1} \), etc. This is the full off-shell result for the amplitude. On-shell a lot of terms cancel, since all the energy factors are equal, leading to a much simpler final answer.

## 5.10 Partial-wave decomposition of the amplitude

After the new one-loop contributions to the potential \( V \) are determined, adding the irreducible part of the direct box to it the quasipotential \( W \) is obtained. The BSLT equation can then be solved in a partial waves basis, in which it reduces to a set of one- and two-dimensional integral equations once the contributions of the negative energy states are neglected. For completeness, in this section the partial-wave projection formalism as presented by Kubis [160] is outlined.

For the elastic scattering of two spin-1/2 particle 16 helicity amplitudes can be formed. Not all are independent due to symmetries. For example, parity conservation

\[ \langle \lambda_{1'}, \lambda_{2'} | \phi(p, \theta, \varphi) | \lambda_1, \lambda_2 \rangle = \eta\langle -\lambda_{1'}, -\lambda_{2'} | \phi(p, \theta, \pi - \varphi) | -\lambda_1, -\lambda_2 \rangle, \]

reduces the number of independent amplitudes to eight. Time reversal invariance reduces this number to six and in the case when particles are identical only five independent amplitudes remain. In the following only the constraints set by parity invariance will be used; the set of eight independent amplitudes is chosen to be

\[ \phi_1 = \langle ++ | \phi | ++ \rangle, \quad \phi_5 = \langle ++ | \phi | ++ \rangle, \]
\[ \phi_2 = \langle ++ | \phi | -- \rangle, \quad \phi_6 = \langle ++ | \phi | -- \rangle, \]
\[ \phi_3 = \langle +- | \phi | +- \rangle, \quad \phi_7 = \langle +- | \phi | ++ \rangle, \]
\[ \phi_4 = \langle +- | \phi | -- \rangle, \quad \phi_8 = \langle +- | \phi | ++ \rangle. \]

The full amplitudes can be expanded in partial waves of definite total angular momentum \( J \)

\[ \phi_i = \sum_J (2J + 1) \phi_i^J d_{\lambda\lambda'}^J(\theta), \]
where \( \lambda = \lambda_1 - \lambda_2 \), \( \lambda' = \lambda_{1'} - \lambda_{2'} \) and \( d^J(\theta) \) are the Jacobi polynomials. To extract the partial wave amplitudes Eq. (5.73) has to be inverted. One can avoid the complexity of the Jacobi polynomials by introducing a new set of amplitudes which are linear combinations of the original ones and at the same time are free from kinematical singularities originating in the momentum-transfer variable [92, 160]

\[
\begin{array}{ll}
f_1 &= \phi_1 - \phi_2 , \\
f_2 &= \phi_1 + \phi_2 , \\
f_3 &= \frac{1}{2} [\phi_3 / \cos^2(\theta/2) - \phi_4 / \sin^2(\theta/2)] , \\
f_4 &= \frac{1}{2} [\phi_3 / \cos^2(\theta/2) + \phi_4 / \sin^2(\theta/2)] , \\
f_5 &= (\phi_5 - \phi_6) / \sin \theta , \\
f_6 &= (\phi_5 + \phi_6) / \sin \theta , \\
f_7 &= -(\phi_7 - \phi_8) / \sin \theta , \\
f_8 &= -(\phi_7 + \phi_8) / \sin \theta .
\end{array}
\] (5.74)

These amplitudes possess simple partial wave expansions in terms of the Legendre polynomials:

\[
\begin{array}{ll}
\phi_1^J - \phi_2^J &= \frac{1}{2} \int_{-1}^{1} dz P_J(z) f_1 , \\
\phi_1^J + \phi_2^J &= \frac{1}{2} \int_{-1}^{1} dz P_J(z) f_2 , \\
\phi_3^J - \phi_4^J &= \frac{1}{2} \int_{-1}^{1} dz \{ [c_J^2 P_{J+1}(z) + c_{J+1}^2 P_{J-1}(z)] f_3 + P_J(z) f_4 \} , \\
\phi_3^J + \phi_4^J &= \frac{1}{2} \int_{-1}^{1} dz \{ [c_J^2 P_{J+1}(z) + c_{J+1}^2 P_{J-1}(z)] f_4 + P_J(z) f_3 \} , \\
\phi_5^J + \phi_6^J &= \frac{1}{2} c_J c_{J+1} \int_{-1}^{1} dz \{ P_{J+1}(z) - P_{J-1}(z) \} f_5 , \\
\phi_5^J - \phi_6^J &= \frac{1}{2} c_J c_{J+1} \int_{-1}^{1} dz \{ P_{J+1}(z) - P_{J-1}(z) \} f_6 , \\
\phi_7^J + \phi_8^J &= \frac{1}{2} c_J c_{J+1} \int_{-1}^{1} dz \{ P_{J+1}(z) - P_{J-1}(z) \} f_7 , \\
\phi_7^J - \phi_8^J &= \frac{1}{2} c_J c_{J+1} \int_{-1}^{1} dz \{ P_{J+1}(z) - P_{J-1}(z) \} f_8 ,
\end{array}
\] (5.75)

with \( z = \cos \theta \) and \( c_J = [J/(2J + 1)]^{1/2} \), \( c_{J+1} = [(J + 1)/(2J + 1)]^{1/2} \). For the case \( J = 0 \) the only nonzero amplitude are \( \phi_1^J \pm \phi_2^J \). The amplitudes in the previous relations represent transitions between states of definite total angular momentum and parity. They are defined as

\[
| J, r, \lambda_1, \lambda_2 \rangle = (1/\sqrt{2}) \{ | J, \lambda_1, \lambda_2 \rangle + r | J, \lambda_1, \lambda_2 \rangle \}
\] (5.76)
with $r = \pm 1$. For the definition of the states of definite total angular momentum (and helicity, since helicity is invariant under rotations) $| J, \lambda_1, \lambda_2 \rangle$ we refer to Section (8.4.2) of Ref. [161]. For the case when spatial parity is conserved the following transitions between states of definite parity are possible

$$ T_1^J = \langle + + | T_1^J | + + \rangle = \phi_1^J - \phi_2^J \quad \text{(singlet, } L = J) \quad (5.77) $$

$$ T_3^J = \langle + - | T_3^J | + - \rangle = \phi_3^J - \phi_4^J \quad \text{(triplet, } L = J) $$

$$ T_5^J = \langle + + | T_5^J | + + \rangle = \phi_5^J - \phi_6^J \quad \text{(singlet - triplet, } L = J) $$

$$ T_8^J = \langle + - | T_8^J | + + \rangle = \phi_8^J - \phi_9^J \quad \text{(triplet - singlet, } L = J) $$

and

$$ T_2^J = \langle + + | T_2^J | + + \rangle = \phi_1^J + \phi_2^J \quad \text{(triplet, } L = J \pm 1) \quad (5.78) $$

$$ T_4^J = \langle + - | T_4^J | + - \rangle = \phi_3^J + \phi_4^J \quad \text{(triplet, } L = J \pm 1) $$

$$ T_6^J = \langle + + | T_6^J | + + \rangle = \phi_5^J + \phi_6^J \quad \text{(triplet, } L = J \pm 1) $$

The transition from the $| J, M; \lambda_1, \lambda_2 \rangle$ basis to the $| J, M; L, S \rangle$ one can be performed with the help of the relation [162]

$$ \langle J, M; L, S | J, M; \lambda_1, \lambda_2 \rangle = \left( \frac{2L + 1}{2J + 1} \right)^{1/2} C(LS; 0, \lambda_1 - \lambda_2)C(S_1, S_2; \lambda_1, -\lambda_2). $$

The following expressions for the partial wave amplitudes are obtained [160]

- Singlet, $L = J$:
  $$ T_s^J = T_1^J, $$

- Singlet - triplet, $L = J$:
  $$ T_5^J = -T_6^J, \quad T_8^J = -T_8^J $$

- Triplet, $L = J$:
  $$ T_t^J = T_3^J, $$

- Triplet, $L = J \pm 1$:
  $$ T_{t,J-1}^J = c_2^J T_2^J + c_{J+1}^J T_4^J + c_J c_{J+1}(T_5^J + T_7^J), $$
  $$ T_{t,J+1}^J = c_J c_{J+1} T_2^J + c_{J+1}^J T_4^J - c_J c_{J+1}(T_5^J + T_7^J), $$
  $$ T_{t,J-1,J+1}^J = c_J c_{J+1}(-T_5^J + T_7^J) + c_J^2 T_5^J - c_J^2 T_7^J, $$
  $$ T_{t,J+1,J-1}^J = c_J c_{J+1}(-T_2^J + T_4^J) - c_J^2 T_2^J + c_J^2 T_4^J. $$
Appendix A: Tree-level potentials in the OBE model

In the OBE model of Fleischer and Tjon contribution of the following mesons have been included: \(\pi, \rho, \delta, \eta, \omega\) and \(\epsilon\). The tree-level potentials of the isovector mesons \(\pi, \rho\) and \(\delta\) are given respectively by

\[
V_{\pi}(k,p) = -ig^2_{\pi}\frac{\gamma_5(k-p)}{4M^2} \cdot \vec{\tau}_1 \cdot \vec{\tau}_2,
\]

\[
\Delta_{\pi}(k-p) \cdot (\gamma_5(k-p))^{(2)} \cdot \vec{\tau}_1 \cdot \vec{\tau}_2,
\]

\[
\Delta_{\rho}(k-p) = -i g_T^\rho \gamma_\alpha^T \frac{(k-p)^\alpha}{2M} \cdot \Delta_{\rho}^\alpha(k-p) \cdot (\gamma_\beta^{(2)} \cdot \frac{i g_T^\rho}{2M} \sigma^\beta_\mu(k-p)^\mu) \cdot \vec{\tau}_1 \cdot \vec{\tau}_2,
\]

\[
V_{\delta}(k,p) = -ig^2_{\delta} \Delta_{\delta}(k-p) \cdot \vec{\tau}_1 \cdot \vec{\tau}_2,
\]

and for the isoscalar meson \(\eta, \omega\) and \(\epsilon\) by

\[
V_{\eta}(k,p) = -ig^2_{\eta}\frac{\gamma_5(k-p)}{4M^2} \cdot \vec{\tau}_1 \cdot \vec{\tau}_2,
\]

\[
V_{\omega}(k,p) = -ig^2_{\omega} \Delta_{\omega}(k-p) \cdot \vec{\tau}_1 \cdot \vec{\tau}_2,
\]

\[
V_{\epsilon}(k,p) = -ig^2_{\epsilon} \Delta_{\epsilon}(k-p) \cdot \vec{\tau}_1 \cdot \vec{\tau}_2,
\]

where the bracketed upper indices denote the nucleon on which operators act, \(\Delta(p)\) is the propagator of scalar \((\delta, \epsilon)\) and isoscalar \((\pi, \eta)\) mesons, while \(\Delta^{\mu\nu}(p)\) is the propagator of vector mesons \((\rho, \omega)\); \(k\) and \(p\) are the four momenta of respectively the final and the initial nucleons. To ensure the correct behavior at high momenta, a cutoff of the dipole form,

\[
F(p^2) = \left(\frac{\Lambda^2}{\Lambda^2 - p^2}\right)^2,
\]

is introduced at each nucleon-meson vertex, with \(\Lambda\) being the cutoff mass. In the present OBE model the same cutoff mass is taken for each meson.

Appendix B: Scalar moments

In this appendix we will present the conventions used in expressing the tensor integrals in Eq. (5.54) in terms of scalar moments. The procedure is very simple: one has to meet in the final answer all Minkowski space tensorial quantities of the same rank as the starting tensor integral. We start with the two point functions, for which one has

\[
B_{\mu}(p,m_1,m_2) = p_{\mu} B_{11},
\]

\[
B_{\mu\nu}(p,m_1,m_2) = p_{\mu} p_{\nu} B_{21} + g_{\mu\nu} B_{22}.
\]

For the two-point functions analytical expressions can be obtained [126, 157]. For the calculations in this thesis C moments up to the third rank were used

\[
C_{\mu} = p_{\mu} C_{11} + q_{\mu} C_{12},
\]

\[
C_{\mu\nu} = p_{\mu} p_{\nu} C_{21} + g_{\mu\nu} C_{22}.
\]
Appendix B: Scalar moments

\[ C_{\mu \nu} = p_\mu p_\nu C_{21} + q_\mu q_\nu C_{22} + \{p,q\}_{\mu \nu} C_{23} + g_{\mu \nu} C_{24}, \]

\[ C_{\mu \nu \rho} = p_\mu p_\nu p_\rho C_{31} + q_\mu q_\nu q_\rho C_{32} + \{p q q\}_{\mu \nu \rho} C_{33} + \{p p q\}_{\mu \nu \rho} C_{34} + \{q g\}_{\mu \nu \rho} C_{35} + \{q q g\}_{\mu \nu \rho} C_{36}, \]

Finally, for the \( D \) tensor integrals one writes the following decomposition (again only up to the third rank)

\[ D_{\mu} = p_\mu D_{11} + q_\mu D_{12} + r_\mu D_{13}, \]

\[ D_{\mu \nu} = p_\mu p_\nu D_{21} + q_\mu q_\nu D_{22} + r_\mu r_\nu D_{23} + \{p q\}_{\mu \nu} D_{24} + \{p r\}_{\mu \nu} D_{25} + \{q r\}_{\mu \nu} D_{26} + g_{\mu \nu} D_{27}, \]

\[ D_{\mu \nu \rho} = p_\mu p_\nu p_\rho D_{31} + q_\mu q_\nu q_\rho D_{32} + r_\mu r_\nu r_\rho D_{33} + \{p q q\}_{\mu \nu \rho} D_{34} + \{p p r\}_{\mu \nu \rho} D_{35} + \{p q r\}_{\mu \nu \rho} D_{36} + \{q q r\}_{\mu \nu \rho} D_{37} + \{q q q\}_{\mu \nu \rho} D_{38} + \{q q g\}_{\mu \nu \rho} D_{39} + \{q q q\}_{\mu \nu \rho} D_{310} + \{p g\}_{\mu \nu \rho} D_{311} + \{q g\}_{\mu \nu \rho} D_{312} + \{r g\}_{\mu \nu \rho} D_{313}, \]

The kinematical conventions for the two-point, three-point and four-point functions are the ones displayed in Fig. (5.6). The three-point and four-point scalar moments cannot be expressed in a closed form. For the \( C_0 \) and \( D_0 \) moments, formulas in terms of Spence functions can be derived for certain kinematical situations [157]. The value of the higher scalar moments can be found once the values of the zero order moments are known. A simple method to achieve this is presented in [126]. We will only present the very simple case of \( C_1 \) moments. One begins by contracting the first line of Eq. (B-5.2) with the available tensors of rank 1 (\( p^\mu \) and \( q^\mu \) in this case). The same is done for the corresponding expression for \( C_\mu \) in Eq. (5.54) and by reducing terms like \( k^2, k \cdot p \) and \( k \cdot q \) against the propagators in the denominator, \( p^\mu C_\mu \) and \( q^\mu C_\mu \) can be expressed in terms of the \( C_0 \) and \( B \) moments. A linear systems of equations results

\[ C_{11} p^2 + C_{12} p \cdot q = \frac{1}{2} \left[ f_1 C_0 + B_0(1, 3) - B_0(2, 3) \right], \]

\[ C_{11} p \cdot q + C_{12} q^2 = \frac{1}{2} \left[ f_2 C_0 + B_0(1, 2) - B_0(1, 3) \right], \]

\[ f_1 = m_1^2 - m_2^2 - p^2, \]

\[ f_2 = m_2^2 - m_3^2 - (p + q)^2 + p^2. \]

The moment \( B_0(1, 3) \) was obtained from the \( C \) moments by a cancellation of the propagator with mass \( m_2 \). It is therefore a two-point function with internal particles of mass \( m_1 \) and \( m_3 \) and external momentum \( p+q \). Similar reasonings apply to the other \( B_0(i,j) \) moments. This linear system allows the extraction of the \( C_{11} \) and \( C_{12} \) moments. Then
Chapter 5: Two-pion-exchange contributions: the formalism

one can determine the $C_2$ moments in a similar fashion and so on. This method cannot
be applied when the determinant of the system is zero or very close to zero. To avoid
such problems the evaluation of the scalar moments in the package $fj$ is implemented
using different, more refined algorithms [128, 163].

Appendix C: Spinors

In this work spinors in the convention due to Kubis [160] have been used. The difference
from the more popular convention of Bjorken and Drell lies in a different normalization
and in the description of the negative energy states. The positive and negative spinors
are respectively given by

$$
\begin{align*}
    u_p^{(+)}(\lambda) &= N_p \left[ \frac{1}{\sqrt{E_p+m_N}} \right] \chi_\lambda(\theta, \phi), \\
    u_p^{(-)}(\lambda) &= N_p \left[ \frac{-\vec{\sigma} \cdot \vec{p}}{2 \sqrt{E_p+m_N}} \right] \chi_\lambda(\theta, \phi),
\end{align*}
$$

with $N_p = [2E_p(E_p + m_N)^{-1}]$ ensuring the proper normalization

$$
    u_\lambda^{(\rho)}(\vec{p}) u_\lambda^{(\rho')}(-\vec{p}) = \delta_{\rho\rho'} \delta_{\lambda\lambda'}.
$$

The two-component Pauli spinors have the following expressions

$$
\begin{align*}
    \chi_{1/2}(\theta, \phi) &= \left( \begin{array}{c} \cos \theta/2 \\ e^{i\phi} \sin \theta/2 \end{array} \right), \\
    \chi_{-1/2}(\theta, \phi) &= \left( \begin{array}{c} -e^{-i\phi} \sin \theta/2 \\ \cos \theta/2 \end{array} \right).
\end{align*}
$$

For the two-particle states the convention of Jacob and Wick [162] is used. The spatial momentum $\vec{p}$ of particle 1 (see Fig. (5.4)) defines the polar direction. The final
momentum of particle 1, $\vec{q}$, is chosen to lie in the $xz$ plane at an angle $\theta$ from the polar
direction. For the particle 2 the sign of the helicity in the Pauli spinor is reversed for
both the initial and the final state. Using the Kubis spinors the two-particle state can
be written [160]

$$
\begin{align*}
    \tilde{\psi}(q_1, q_2) &\equiv \tilde{\psi}(p_1, \lambda_1', \rho_1'; p_2, \lambda_2', \rho_2') \\
    &= \tilde{u}(q, \theta)_{\lambda_1'} \tilde{u}(-q, \theta)_{-\lambda_2'}, \\
    \tilde{\psi}(p_1, p_2) &\equiv \psi(p_1, \lambda_1, \rho_1; p_2, \lambda_2, \rho_2) \\
    &= u(p)_{\lambda_1} u(-q)_{-\lambda_2}.
\end{align*}
$$

Primed quantities refer to final states. For the kinematic variables in the center of mass
we have used the notation: $p_1 = (E_p, \vec{p})$, $p_2 = (E_p, -\vec{p})$ and the corresponding ones for
the final state with $\vec{p}$ replace by $\vec{q}$.