Chapter 4

\( \mathcal{N} = 4 \ d = 4 \) Supergravity and its Scalar Potential

In the previous chapter we have investigated the relation between higher-dimensional supergravities and lower-dimensional supergravities. One can ask the question whether all four-dimensional supergravities can be deduced from a ten-dimensional supergravity. The answer is by the time of writing of this thesis still not known. To obtain an answer both the dimensional reductions of ten-dimensional supergravity and the features of four-dimensional supergravity need to be investigated.

For any theory the most important objects are the solutions of the equations of motion. Within the solutions the most prominent place is taken by the vacuum solution(s). Most vacuum solutions are characterized by the vanishing of all fields except the metric and the scalars, of which the latter take constant values. The scalars contribute to the curvature through the scalar potential. The value of the scalars at the vacuum determine how much supersymmetry is broken. Therefore the vacua of a theory can be studied by the scalars and their potential.

In four dimensional \( \mathcal{N} = 4 \) supergravity it is possible to introduce so-called \( SU(1,1) \)-angles. These angles influence the scalar potential and hence the vacua of the theory. A ten-dimensional origin of the \( SU(1,1) \)-angles is not known.

In this chapter we study \( \mathcal{N} = 4 \ d = 4 \) supergravity and we focus on the potential of the theory. A premature discussion on \( \mathcal{N} = 4 \ d = 4 \) supergravity was given in section 2.4.2, which we extend in this chapter. In section 4.1 we introduce the four-dimensional \( \mathcal{N} = 4 \) supergravity multiplet, give its (gauged and ungauged) action and introduce the \( SU(1,1) \)-angles. In section 4.2 we introduce the matter multiplets and in section 4.2.2 we study the symplectic embedding. The symplectic embedding provides a technique for determining the coupling between the scalars and the vectors in a supergravity. In section 4.3 we briefly discuss the matter coupled gauged \( \mathcal{N} = 4 \)
\[d = 4\] supergravity and in section 4.4 we introduce the scalar potential of \(\mathcal{N} = 4\) \(d = 4\) supergravity. In sections 4.5 and 4.6 we study the potential for semisimple and CSO-gaugings. We always assume that the number of space-time dimensions is four, unless otherwise stated.

### 4.1 The Pure \(\mathcal{N} = 4\) Supergravity

In this section we review the basics of pure \(\mathcal{N} = 4\) \(d = 4\) supergravity. We do not consider the construction of the theory, which has been done in the end of the seventies, see e.g. \([58,59,140]\).

The supergravity multiplet of \(\mathcal{N} = 4\) supergravity consists of a metric, four gravitini, six vectors, four spin-1/2 fermions and two scalars. All fields carry a representation of \(\mathfrak{su}(4) \cong \mathfrak{so}(6)\); the metric and the scalars are singlets under \(\mathfrak{su}(4)\), the fermions are in the vector representation \(4\) of \(\mathfrak{su}(4)\) and the vector fields are in the real \(6\) representation of \(\mathfrak{su}(4)\), which is the same as the vector representation of \(\mathfrak{so}(6)\).

#### The Scalar Manifold \(SU(1,1)/U(1)\)

The two scalars \(\phi^\alpha, \alpha = 1, 2\), parameterize an \(SU(1,1)/U(1)\) coset. We define \(\phi_\alpha\) by \(\phi_1 = (\phi^1)^*\) and \(\phi_2 = -(\phi^2)^*\). We introduce an \(SU(1,1)\)-matrix \(\mathcal{V}\) by:

\[
\mathcal{V} = \begin{pmatrix} \phi^1 & \phi^2 \\ \phi^2 & \phi^1 \end{pmatrix}; \quad |\phi^1|^2 - |\phi^2|^2 = \phi^\alpha \phi_\alpha = 1, \quad (4.1.1)
\]

and which satisfies \(\mathcal{V}^\dagger \eta_{1,1} \mathcal{V} = \eta_{1,1}\) where

\[
\eta_{1,1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (4.1.2)
\]

We decompose the Lie algebra \(\mathfrak{su}(1,1)\) into its compact part and its noncompact part; \(\mathfrak{su}(1,1) = \mathfrak{u}(1) \oplus \mathfrak{u}(1)^\perp\). In the vector representation the subalgebra \(\mathfrak{u}(1)\) is spanned by \(t_1 = i\sigma_3\), whereas \(\mathfrak{u}(1)^\perp\) is spanned by \(t_2 = \sigma_1\) and \(t_3 = \sigma_2\), where the \(\sigma_i\) are the Pauli-matrices (see Appendix A). We call \(\mathbb{P}\) the projection onto \(\mathfrak{u}(1)^\perp\).

We find

\[
\mathbb{P} \mathcal{V}^{-1} \partial_\mu \mathcal{V} = \begin{pmatrix} 0 & \phi^2 \partial_\mu \phi_1 + \phi_1 \partial_\mu \phi_2 \\ -\phi^2 \partial_\mu \phi_1 + \phi_1 \partial_\mu \phi_2 & 0 \end{pmatrix}, \quad (4.1.3)
\]

and thus for the \(SU(1,1)\)-scalars we find the kinetic Lagrangian:

\[
\frac{1}{2} \text{Tr} (\mathcal{V}^{-1} \partial_\mu \mathcal{V} \mathcal{P} \mathcal{V}^{-1} \partial^\mu \mathcal{V}) = \partial_\mu \phi_\alpha \partial^\mu \phi^\alpha + \phi^\alpha \partial_\mu \phi_\alpha \phi^\beta \partial^\mu \phi_\beta. \quad (4.1.4)
\]

The kinetic term 4.1.4 is the kinetic term of the \(SU(1,1)\)-scalars of \(\mathcal{N} = 4\) supergravity found in the literature \([36,59]\).
Using a $U(1)$-transformation we can make $\phi^1$ real. Hence we take the following parametrization of $SU(1,1)/U(1)$

$$\phi^1 = \frac{1}{\sqrt{1-r^2}}, \quad \phi^2 = \frac{re^{i\chi}}{\sqrt{1-r^2}}. \quad (4.1.5)$$

The parameter $r$ is restricted to the interval $[0,1)$ and $\chi$ is an angular parameter running from 0 to $2\pi$. Substituting the parametrization 4.1.5 into the kinetic term 4.1.4 gives rise to the scalar kinetic Lagrangian:

$$L[r,\chi] = -\frac{1}{(1-r^2)^2} (\partial_{\mu}r \partial^{\mu}r + r^2 \partial_{\mu}\chi \partial^{\mu}\chi). \quad (4.1.6)$$

We now briefly show the connection with other formulations of the scalar manifold $SU(1,1)/U(1)$. We first put $z = re^{-i\chi}$ and define

$$\tau = i\frac{1-z}{1+z}, \quad (4.1.7)$$

and obtain the scalar Lagrangian

$$L[\tau] = -\frac{1}{4} \partial_{\mu}\tau \partial^{\mu}\bar{\tau}. \quad (4.1.8)$$

The action 4.1.8 is invariant under Möbius-transformations: $\tau \mapsto (a\tau + b)(c\tau + d)^{-1}$, where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. Putting $\tau = \sigma + i e^{-\phi}$ we obtain the Lagrangian of the coset $SL(2;\mathbb{R})/SO(2)$

$$L[\sigma,\phi] = -\frac{1}{4} \left( \partial_{\mu}\phi \partial^{\mu}\phi + e^{2\phi} \partial_{\mu}\sigma \partial^{\mu}\sigma \right). \quad (4.1.9)$$

In the following we use that formalism that gives the most elegant result.

### The Bosonic Lagrangian

Ungauged $\mathcal{N} = 4$ supergravity admits a formulation in which $SU(4) \cong SO(6)$ is a global symmetry. Using the $SU(1,1)$-variables $\sigma, \phi$ the bosonic Lagrangian reads in the $SO(6)$-formulation

$$e^{-1}L = R(\omega) - \frac{1}{2} \left( \partial_{\mu}\phi \partial^{\mu}\phi + e^{2\phi} \partial_{\mu}\sigma \partial^{\mu}\sigma \right)$$

$$- \frac{1}{8} e^{-\phi} \sum_{a=1}^{6} F_{\mu\nu}^{a} F^{\mu\nu,a} - \frac{1}{8} \sigma \epsilon^{\mu\nu\rho\sigma} \sum_{a=1}^{6} F_{\mu\nu}^{a} F_{\rho\sigma}^{a}. \quad (4.1.10)$$

where $F_{\mu\nu}^{a} = \partial_{\mu}A_{\nu}^{a} - \partial_{\nu}A_{\mu}^{a}$ are the abelian field strengths. The global $SO(6)$-symmetry rotates the field strengths $F_{\mu\nu}^{a}$ and the fermions. To go to the $SU(4)$-formulation one
uses the 't Hooft symbols (see appendix A.3.2) to rewrite the field strengths as:

$$F^{ij}_{\mu\nu} = \frac{1}{2}(G_a)^{ij}F^a_{\mu\nu}.$$ 

The six vectors $A^a_{\mu}$ can be promoted to nonabelian gauge fields by making a six-dimensional subgroup $H$ of $SO(6)$ a local symmetry group. The gauged theory has no longer a global $SO(6)$-symmetry, but only the local $H$-symmetry. We associate with every gauge field $A^a_{\mu}$ a generator $T_a$ such that the structure constants are given by:

$$[T_a, T_b] = f^{abc}T_c.$$ 

The tensors $f^{abc} \equiv f^{abd}\delta_{dc}$ have to be completely antisymmetric.

To remain supersymmetric the Lagrangian 4.1.10 acquires a potential. The potential $V(\sigma, \phi)$ is given by:

$$V(\sigma, \phi) = -e^{\phi} \sum_{a,b,c} (f^{abc})^2.$$ 

(4.1.11)

The potential 4.1.11 is unbounded from below. In theories that are described in flat space-times a potential that is unbounded from below is a problem, since there is no stable vacuum with the lowest energy. For theories that involve gravity the stability of an anti-de Sitter vacuum is guaranteed if the Breitenlohner–Freedman bound is satisfied [141,142].

The $SU(1,1)$-angles

There exists a formulation of ungauged pure $\mathcal{N} = 4$ supergravity in which the global $SO(6)$-symmetry is broken; all vector fields are coupled to the $SU(1,1)$-scalars in another way. To see what happens it is convenient to rewrite the scalars $\phi$ and $\sigma$ into the complex scalar $z$ as defined above; $\sigma + i e^{-\phi} = i(1 - z)(1 + z)^{-1}$. The different couplings are then obtained by putting $z \mapsto e^{-2i\alpha_a}z$ in each coupling to the gauge field $A^a_{\mu}$. The Lagrangian is given by [36,143]

$$e^{-1} \mathcal{L} = R(\omega) - \frac{2}{(1 - |z|^2)^2} \partial_{\mu} z \partial^{\mu} z - \frac{1}{4} \sum_{a=1}^6 \frac{1 - |z|^2}{|1 + z e^{-2i\alpha_a}|^2} F^a_{\mu\nu} F^a_{\mu\nu} - 1 - \frac{\text{Im}(z e^{-2i\alpha_a})}{|1 + z e^{-2i\alpha_a}|^2} F^a_{\mu\nu} F^a_{\rho\sigma}.$$ 

(4.1.12)

The factor of 2 for the $SU(1,1)$-angles $\alpha_a$ is for later convenience. One can show that the action 4.1.12 is related to the action 4.1.10 by a duality rotation, which we discuss in section 4.2.2.

It is clear that the gauging is affected by the introduction of the $SU(1,1)$-angles. To have local $H$-symmetry, the $SU(1,1)$-angles need to be chosen such that they respect the $H$-symmetry. This is discussed in more detail in the sections 4.5.1 and 4.6.2. For now we remark that if the structure constant $f_{ab}^c$ is nonzero, then $\alpha_a =$
\[ \alpha_b = \alpha_c. \] The potential that arises from the gauging becomes [36,143]:

\[ V(z) = -\frac{1 + ze^{-2i\alpha_a}}{1 - |z|^2} \sum_{a,b,c} (f_{ab}^c)^2 - \frac{1}{36} \sum_{a,...,f} \epsilon^{abcdef} f_{ab}^c f_{de}^f \sin(\alpha_a - \alpha_d). \] (4.1.13)

The potential 4.1.13 can have an extremum [36], which was also noticed by [144]. The fact that due to the introduction of the \( SU(1,1) \)-angles the potential of pure \( N = 4 \) supergravity can have an extremum, makes the \( SU(1,1) \)-angles appealing. However, to the knowledge of the author it has never been shown how to obtain the \( SU(1,1) \)-angles from a dimensional reduction of a ten-dimensional supergravity. Therefore there seems to be no string theoretical origin of the \( SU(1,1) \)-angles.

### 4.2 The Vector Multiplet

For \( N = 4 \) supersymmetry there is only one matter multiplet that can be coupled to the supergravity multiplet; the vector multiplet, which is the subject of this section. We first consider in general the \( N = 4 \) supersymmetric Yang–Mills theory, then we consider the symplectic embedding, which provides us with useful information about the coupling to the supergravity multiplet. In particular, we construct the coupling of the scalars to the vector fields when all \( SU(1,1) \)-angles are equal.

#### 4.2.1 \( N = 4 \) \( d = 4 \) Super-Yang–Mills

The \( N = 4 \) Super-Yang–Mills theory can be obtained from a dimensional reduction of the ten-dimensional \( N = 1 \) Super-Yang–Mills theory. In Super-Yang–Mills theory gravity is not included and hence the theory is formulated in a Minkowski space-time.

We first discuss important aspects of the ten-dimensional \( N = 1 \) Super-Yang–Mills theory and subsequently discuss the dimensional reduction to four dimensions. The conventions, notations and some useful formulas for spinors are explained in section appendix C.

### The Ten-Dimensional Theory

In ten dimensions the nonabelian \( N = 1 \) Super-Yang–Mills multiplet consists of a gauge potential \( A_\mu \) and a Majorana–Weyl spinor \( \psi \) satisfying \( \Gamma_{11} \psi = P_+ \psi \), where \( P_+ \) is a chiral projection operator. Both fields take values in a compact Lie algebra \( \mathfrak{g} \). The Lagrangian is given by:

\[ \mathcal{L}_{SYM} = \int d^{10}x \left( \frac{i}{4} \text{Tr}_{ad}(F_{\mu\nu}F^{\mu\nu}) + \frac{i}{2} \text{Tr}_{ad} \bar{\psi} \Gamma^\mu D_\mu \psi \right), \] (4.2.1)
where $\bar{\psi} = \psi^T C$, with $C$ the antisymmetric charge conjugation matrix, and where
\[ D_\mu \psi = \partial_\mu \psi + g_{YM} [A_\mu, \psi], \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g_{YM} [A_\mu, A_\nu]. \quad (4.2.2) \]
The field strength $F_{\mu\nu}$ satisfies the Bianchi identity $D_{[\mu} F_{\nu\rho]} = 0$. An infinitesimal Yang–Mills transformation with parameter $\Lambda$ taking values in $g$ acts on the fields as:
\[ \delta_\Lambda \psi = [\Lambda, \psi], \quad \delta_\Lambda A_\mu = - \frac{1}{g_{YM}} F_{\mu\nu} \Lambda. \quad (4.2.3) \]

The awkward looking sign in front of the Lagrangian 4.2.1 is due to the fact that for compact Lie algebras the trace in the adjoint representation is negative definite. The Lagrangian 4.2.1 is real due to the identity $C = BC^* B$, where $B$ is the symmetric complex conjugation matrix. The Lagrangian 4.2.1 is invariant under the supersymmetry transformations:
\[ \delta_\epsilon A_\mu = i \Gamma_\mu \epsilon, \quad \delta_\epsilon \psi = - \frac{1}{2} \Gamma_{\mu\nu} F^{\mu\nu} \epsilon. \quad (4.2.4) \]
The $i$ in the supersymmetry identity for two Majorana–Weyl fermions $\epsilon_1, \epsilon_2$ of positive chirality
\[ \epsilon_2 \bar{\epsilon}_1 - \epsilon_1 \bar{\epsilon}_2 = - \frac{1}{8} \epsilon_1 \Gamma^\mu \epsilon_2 \Gamma_\mu \epsilon_1 - \frac{1}{960} \epsilon_1 \Gamma^{\mu_1 \cdots \mu_5} \epsilon_2 \Gamma_{\mu_1 \cdots \mu_5} \epsilon_1, \quad (4.2.5) \]
we find that the supersymmetry algebra is given by:
\[ [\delta_1, \delta_2] \psi = - 2 i \epsilon_1 \Gamma^\mu \epsilon_2 \partial_\mu \psi + [- 2 i g_{YM} \epsilon_1 \Gamma^\sigma A_\sigma \epsilon_2, \psi] + \frac{7}{8} \epsilon_1 \Gamma^\mu \epsilon_2 \Gamma^\nu \epsilon_2 \Gamma^\rho \epsilon_2 g_{YM} \Gamma_\mu \epsilon_1 \Gamma_\nu \epsilon_1 \Gamma_\rho \epsilon_1 \Gamma_\sigma \epsilon_2 \Gamma_\sigma \epsilon_1 \Gamma_\tau \epsilon_1 \Gamma_\tau \epsilon_1 \Gamma_\rho \epsilon_1, \quad \]  
\[ [\delta_1, \delta_2] A_\mu = - 2 i \epsilon_1 \Gamma^\nu \epsilon_2 \bar{\sigma}_\nu \epsilon_2 \partial_\mu A_\mu + D_\mu (2 i \epsilon_1 \Gamma^\nu \epsilon_2 \bar{\sigma}_\nu \epsilon_2). \quad (4.2.6) \]
The supersymmetry algebra closes up to gauge transformations with gauge parameter $\Lambda = - 2 i g_{YM} \epsilon_1 \Gamma^\mu \epsilon_2 A_\mu$ and up to the equation of motion of the fermion, $\Gamma^\rho D_\rho \psi = 0$.

The Reduction

We now perform a toroidal reduction to obtain the four dimensional Super-Yang–Mills $N = 4$ multiplet. All ten-dimensional fields and indices are hatted. The ten-dimensional space-time indices $\hat{\mu}, \hat{\nu}, \ldots$ are split into $0 \leq \mu, \nu, \ldots \leq 3$ and $1 \leq a, b, \ldots \leq 6$. The coordinates on the four-dimensional space-time are denoted $x^a$ and the coordinates on the six-dimensional torus are denoted by $y^a$. All lower-dimensional fields are independent of the coordinates $y^a$.

The reduction Ansatz for the gauge potential is given by $\hat{A} = A_\mu(x) dx^\mu + Z_a(x) dy^a$. From this we find the following components of the ten-dimensional field strength $\hat{F}_{\mu\nu}$:
\[ \hat{F}_{\mu\nu} = F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g_{YM} [A_\mu, A_\nu], \]
\[ \hat{F}_{\mu a} = D_\mu Z_a = \partial_\mu Z_a + g_{YM} [A_\mu, Z_a], \]
\[ \hat{F}_{ab} = g_{YM} [Z_a, Z_b]. \quad (4.2.7) \]
4.2 The Vector Multiplet

The reduction of the fermions is done by choosing a particular representation of the Clifford algebra of \( \mathfrak{so}(1,9) \). We take:

\[
\hat{\Gamma}_\mu = \gamma_\mu \otimes \mathbb{1}_{8 \times 8}, \quad \hat{\Gamma}_a = \gamma_5 \otimes \Gamma_a, \tag{4.2.8}
\]

where \( \gamma_\mu \) represent the Clifford algebra of \( \mathfrak{so}(1,3) \) and \( \Gamma_a \) represent the Clifford algebra of \( \mathfrak{so}(6) \) by means of the ’t Hooft symbols \( G_a \) as explained in appendix C.3.

The Clifford algebra of \( \mathfrak{so}(1,3) \) can be taken to be real \( \gamma^*_\mu = \gamma_\mu \) and an explicit real representation is given in C.2. Hence \( B^{(4)}_\mu \sim \mathbb{1}_{4 \times 4} \) and we choose a basis for the fermions such that \( B^{(4)}_\mu = -i\mathbb{1}_{4 \times 4} \). The charge conjugation matrix of \( \mathfrak{so}(1,3) \), denoted \( C \), we therefore take to be \( C = i\gamma_0 \). The chirality matrix \( \gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 \) is purely imaginary.

For the charge conjugation matrix \( C \) and the complex conjugation matrix \( B \) we take the decomposition:

\[
C = i\gamma_0 \otimes \tilde{C}, \quad B = -i\mathbb{1}_{4 \times 4} \otimes \tilde{C}, \tag{4.2.9}
\]

where \( \tilde{C} \) is the symmetric charge conjugation matrix of \( \mathfrak{so}(6) \). The explicit representation of \( \tilde{C} \) is given by equation C.3.2. The chirality matrix is given by \( \hat{\Gamma}_{11} = \gamma_5 \otimes \Gamma_7 \). The explicit representation of \( \Gamma_7 \) is given by equation C.3.3.

The spinors \( \hat{\psi} \) are decomposed as:

\[
\hat{\psi}(x,y) = \sum_{i=1}^{8} \psi^{(i)}(x) \otimes \theta^i. \tag{4.2.10}
\]

The spinors \( \theta^i \) are the real basis spinors given by: \( (\theta^i)_j = \delta_{ij} \) and hence a ten-dimensional spinor \( \hat{\psi} \) is effectively written as:

\[
\hat{\psi} = \begin{pmatrix} 
\psi^{(1)} \\
\vdots \\
\psi^{(8)} 
\end{pmatrix}. \tag{4.2.11}
\]

Imposing the Majorana constraint \( \hat{\psi}^* = iB\psi \) gives:

\[
(\psi^{(1)})^* = \psi^{(5)}, \quad (\psi^{(2)})^* = \psi^{(6)}, \quad (\psi^{(3)})^* = \psi^{(7)}, \quad (\psi^{(4)})^* = \psi^{(8)}. \tag{4.2.12}
\]

Therefore we write the first four \( \psi^{(i)} \) with a upper index as \( \psi^i \), \( i = 1,2,3,4 \), and the last four with a lower index \( \psi_i \), \( i = 1,2,3,4 \); we then have \( \psi^* = \psi \). We thus decompose the ten-dimensional spinor \( \hat{\psi} \) as a doublet of a quartet of fermions, where one doublet is the complex conjugate of the other:

\[
\hat{\psi} = \begin{pmatrix} 
\psi^1 \\
\psi_1 
\end{pmatrix}. \tag{4.2.13}
\]
Imposing the chirality constraint $\hat{\Gamma}_{11} \hat{\psi} = \hat{\psi}$ gives:

$$\gamma_5 \psi^i = \psi^i \quad \gamma_5 \psi_i = -\psi_i,$$

(4.2.14)

which is consistent with $\gamma_5^* = -\gamma_5$.

We define for the four-dimensional spinor $\psi^i$ the conjugates $\bar{\psi}_i = \psi^T \gamma_0$ and for $\psi_i$ similarly $\bar{\psi}^i = \psi_i^T \gamma_0$. With this definition we have $(\bar{\psi}_i)^* = \bar{\psi}_i$ and

$$\hat{\psi} = i(\bar{\psi}_i, \bar{\psi}^i).$$

(4.2.15)

Under the action of $SU(4)$ the spinors $\bar{\psi}_i$ and $\psi_i$ both transform in the $4$ and $\bar{\psi}^i$ and $\psi^i$ transform in the $\bar{4}$.

Putting the expressions together one finds:

$$L_{SYM} = \text{Tr} \left[ \frac{1}{4} F_{\mu \nu} F^{\mu \nu} + \frac{1}{2} \sum_{a=1}^{6} D_\mu Z_a D^\mu Z_a - \frac{1}{2} \bar{\psi}_i \gamma^\mu D_\mu \psi^i - \frac{1}{2} \bar{\psi}^i \gamma^\mu D_\mu \psi_i ight. \\
+ \frac{1}{2} g_{YM}^2 \sum_{a,b=1}^{6} [Z_a, Z_b] [Z_a, Z_b] + g_{YM} \sum_{a=1}^{6} Z_a \text{Im} \left( \bar{\psi}_i \phi_{ijk} \psi^j \right) \left( \bar{\psi}^i \phi_{ijk} \psi^j \right) \right].$$

(4.2.16)

When we use the ’t Hooft symbols $G^{ij}_a$ to define

$$\phi^{ij} = \frac{1}{2} \sum_a G^{ij}_a Z_a$$

(4.2.17)

and its complex conjugate $\phi_{ij} = \phi^{ij*}$ and use the properties of the ’t Hooft symbols (see appendix A.3.2) we obtain the result:

$$L_{SYM} = \text{Tr} \left[ \frac{1}{4} F_{\mu \nu} F^{\mu \nu} + \frac{1}{2} D_\mu \phi_{ijk} D^\mu \phi^{ijk} - \frac{1}{2} \bar{\psi}_i \gamma^\mu D_\mu \psi^i - \frac{1}{2} \bar{\psi}^i \gamma^\mu D_\mu \psi_i \\
+ \frac{g_{YM}^2}{4} [\phi_{ijk}, \phi_{kl}] [\phi^{ij}, \phi^{kl}] + i g_{YM} \left( \bar{\psi}_i \phi^{ij} \psi^j - \bar{\psi}^i \phi_{ij} \psi^j \right) \right].$$

(4.2.18)

The supersymmetry transformation under which the action 4.2.18 is invariant are given by:

$$\delta A_\mu = -\bar{\epsilon}_i \gamma_\mu \psi_i - \epsilon_i \gamma_\mu \psi^i,$$
$$\delta \phi^{ij} = -i \left( \bar{\epsilon}_i \psi^j - \epsilon^j \psi^i + \epsilon^{ijkl} \epsilon_k \psi_l \right),$$
$$\delta \psi^i = -\frac{1}{2} F^{\mu \nu} \gamma_\mu \epsilon^i + 2i (D_\mu \phi^{ij}) \gamma^\mu \epsilon_j + 2[\phi^{ij}, \phi_{kl}] \epsilon^j.$$  

(4.2.19)

The supersymmetry transformation rules 4.2.19 can be found by reducing the ten-dimensional supersymmetry transformation rules 4.2.4.
4.2 The Vector Multiplet

The action 4.2.18 contains a scalar potential that is proportional to $g_Y^2$. When we take the abelian limit - that is, setting the derived algebra $\mathfrak{g}'$ to zero - the scalar potential vanishes. In the abelian limit only the free theory survives and the Lagrangian becomes a sum of kinetic terms.

To obtain matter coupled $\mathcal{N} = 4$ supergravity the free theory is coupled to the ungauged $\mathcal{N} = 4$ supergravity. To gauge the coupled theory one uses the global symmetry group $G$ of which a subgroup $H$ is promoted to a local symmetry. The gauge fields of the supergravity multiplet and the gauge fields from the Super-Yang–Mills multiplet are then promoted to the Yang–Mills fields of the group $H$ and all derivatives are made covariant with respect to $H$.

The gauging procedure in general breaks supersymmetry and to restore supersymmetry one has to modify the supersymmetry transformation rules and the Lagrangian. The modification of the Lagrangian involves adding a potential. The appearance of a scalar potential looks natural in the light of the action 4.2.18, since this action has a local gauge symmetry and is supersymmetric (though globally supersymmetric). The potential in gauged matter coupled supergravity is also proportional to the square of the coupling constant of the gauge group $H$. However, the two potentials are different in nature; the potential of 4.2.18 is contained in the potential of gauged matter coupled $\mathcal{N} = 4$ supergravity.

4.2.2 The Symplectic Embedding

In ungauged supergravity all gauge fields are abelian gauge fields. The gauge fields couple to the scalars in the following way:

\[ \mathcal{L} = -f(\Phi) \star F \wedge F + g(\Phi)F \wedge F, \quad (4.2.20) \]

where $f$ and $g$ are functions depending on the scalars, collectively denoted $\Phi$. The functions $f$ and $g$ can be found when the scalar manifold is known by a trick called the ‘symplectic embedding’. We now present how the symplectic embedding works and apply it to ungauged $\mathcal{N} = 4$ supergravity, where the scalar manifold is $SU(1,1)/U(1) \otimes SO(6,n)/SO(6) \times SO(n)$. This section is based on [36, 72]. To make the presentation less spoilt by a large number of indices, we first introduce a compact and more abstract notation. We then show how the symplectic groups arise as duality groups, rotating equivalent but different Lagrangians into each other. Then we use the duality group to obtain the coupling of the vector fields to the scalars.

We introduce $N$ abelian gauge field strengths $F^I_{\mu\nu}$, $I = 1, \ldots, N$. The scalars $\Phi$ parameterize a coset $G/K$. The most general Lagrangian for the field strengths $F^I_{\mu\nu}$ coupled to the scalars $\Phi$ is\(^1\)

\[ \mathcal{L}_{vec}[\Phi, F] = -\frac{1}{4} \gamma_{IJ} F^I_{\mu\nu} F^J_{\mu\nu} + \frac{1}{8} \theta_{IJ} \epsilon^{\mu\nu\lambda\rho} F^I_{\mu\nu} F^J_{\lambda\rho}, \quad (4.2.21) \]

\(^1\)The role of gravity is immaterial and hence we will take a flat space-time Lagrangian. The extension to curved space-times is straightforward.
where $\gamma_{IJ}$ and $\theta_{IJ}$ are $\Phi$-dependent symmetric matrices.

The field strengths $F^I$ are two-forms and the space of two-forms we denote $\Omega_2$. Let us define the linear transformation $j$ on $\Omega_2$ by

\[
(jF)_{\mu \nu} = \frac{1}{2} \epsilon_{\mu \nu \lambda \rho} F^{\lambda \rho},
\]
for any $F \in \Omega_2$. We have $j^2 = -1$.

We assemble the field strengths $F^I$ into the vector $F = (F^1, \ldots, F^N)$, which is an element of $\mathcal{W} = \mathbb{R}^N \otimes \Omega_2$. We equip $\mathcal{W}$ with the inner product $(,)$ defined by

\[
(X, Y) = \sum_{I=1}^N X^I_{\mu \nu} Y^I_{\mu \nu}, \quad X, Y \in \mathcal{W}.
\]

The linear transformation $j$ is extended to the linear transformation $J = \mathbb{I} \otimes j$ on $\mathcal{W}$. Since $J$ squares to minus the identity on $\mathcal{W}$ it has eigenvalues $+i$ and $-i$ and therefore we pass to the complex extension of $\mathcal{W}$, denoted $\mathcal{W}^\mathbb{C}$. The inner product $(,)$ is extended to a bilinear form on $\mathcal{W}^\mathbb{C}$ by $\mathbb{C}$-linearity; if $U, V, X, Y \in \mathcal{W}$, then on $\mathcal{W}^\mathbb{C}$ we have

\[
(U + iV, X + iY) = (U, X) + i (V, X) + i (U, Y) - (V, Y).
\]

The vector space $\mathcal{W}^\mathbb{C}$ splits into the direct sum of the eigenspaces of $J$. We define for any $F \in \mathcal{W}^\mathbb{C}$ the projections $F^\pm$ onto the $\pm i$-eigenspaces of $J$ by

\[
F = F^+ + F^-, \quad F^\pm = \frac{1}{2} (F \mp iJF), \quad J F^\pm = \pm i F^\pm.
\]

With respect to the inner product $(,)$ the $\pm i$-eigenspaces of $J$ are orthogonal to each other.

We introduce the symmetric kinetic matrix $K$ and its complex conjugate $\bar{K}$ by

\[
K = i\gamma + \theta, \quad \bar{K} = -i\gamma + \theta,
\]
where the components of $\gamma$ and $\theta$ are $\gamma_{IJ}$ and $\theta_{IJ}$ respectively. We put $\mathcal{K} = K \otimes \mathbb{I}$. If $J\mathcal{X} = \pm i\mathcal{X}$ then we have $J\mathcal{K}\mathcal{X} = \pm i\mathcal{K}\mathcal{X}$.

With these definitions the Lagrangian 4.2.21 can be written as

\[
\mathcal{L}_{\text{vec}}[\Phi, F] = \frac{i}{4} (F^+, \mathcal{K}F^+) - \frac{i}{4} (F^-, \bar{K}F^-)
\]

We define $\mathcal{G}$ by

\[
J \mathcal{G} = 2 \frac{\partial \mathcal{L}_{\text{vec}}}{\partial F} = (-\gamma \otimes \mathbb{I} + \theta \otimes j) F,
\]

which gives $\mathcal{G} = (\gamma \otimes j + \theta \otimes \mathbb{I}) F$. One finds the following $J$-eigenspace decomposition of $\mathcal{G}$:

\[
\mathcal{G}^+ = \mathcal{K}F^+, \quad \mathcal{G}^- = \bar{K}F^-.
\]
The equation of motion and Bianchi identity now read
\[ \partial^\mu (j F)_{\mu \nu} = 0, \quad \partial^\mu (j G)_{\mu \nu} = 0. \] (4.2.30)

The Bianchi equation and the equation of motion are invariant under \( GL(2N, \mathbb{R}) \)-transformations:
\[ (F^+_{\mu \nu} + G^+_{\mu \nu})' = (A B C D) (F^+_{\mu \nu} + G^+_{\mu \nu}), \] (4.2.31)
where \( A, B, C, D \) are \( N \times N \)-matrices. However for general \( GL(2N, \mathbb{R}) \)-rotations the transformation of \( G^\pm \) is inconsistent with its definition 4.2.28. We now show that when we restrict to \( Sp(N, \mathbb{R}) \)-rotations and transform the scalars, using \( G \)-transformations, the inconsistency can be cured. We thus show that an isometry of the scalar manifold \( G/K \) accompanied by an \( Sp(N, \mathbb{R}) \)-rotation on the electric and magnetic field strengths, gives rise to an equivalent set of equations of motions and Bianchi identities. The combination of the isometry of \( G/K \) with the \( Sp(N, \mathbb{R}) \)-rotation is called a duality transformation. The duality transformations do not describe symmetries of the Lagrangian, but symmetries of the equations of motion and Bianchi identities.

A duality transformation acts on the vector fields as a \( GL(2N, \mathbb{R}) \)-transformation mapping \( F, G \) to \( F', G' \) respectively such that the following relation holds:
\[ J (G')^\prime = 2 \partial \mathcal{L}_{\text{vec}}^{\prime} / \partial F', \] (4.2.32)
where \( \mathcal{L}_{\text{vec}}' \) is the new vector Lagrangian: \( \mathcal{L}_{\text{vec}}' = \mathcal{L}_{\text{vec}}[\Phi', \mathcal{F}'] \). Imposing the constraint 4.2.32 gives
\[ K' = (C + DK) (A + BK)^{-1}, \] (4.2.33)
where the \( A, B, C, D \) are the \( N \times N \)-matrix as in equation 4.2.31. Transformations of the kind 4.2.33 are called fractional linear transformation and they form a group.

What singles out the symplectic group is the requirement that the new kinetic matrix \( K' \) is again symmetric and that the duality rotations form a group, called the duality group. Working out the requirement of symmetry of \( K' \) gives
\[ 0 = A^T C + K^T (B^T C - D^T A) + K^T B^T D K - \text{Transpose}. \] (4.2.34)
Since \( K \) is an arbitrary complex symmetric matrix, we find \( A^T C = C^T A, D^T B = B^T D \) and \( A^T D - C^T B = \sigma I, \sigma \in \mathbb{R} \). The constraints can be summarized by
\[ \Lambda^T \begin{pmatrix} 0 & 1 \\ -I & 0 \end{pmatrix} \Lambda = \sigma \begin{pmatrix} 0 & 1 \\ -I & 0 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \] (4.2.35)

The number \( \sigma \) must equal 1 in order that the duality transformations describe a group. The duality group is then fixed and is \( Sp(N, \mathbb{R}) \). We have thus proved that
any theory involving scalars and $N$ abelian field strengths the maximal symmetry

group of the equations of motion and Bianchi identities is $Sp(N, \mathbb{R})$. The $Sp(N, \mathbb{R})$-
rotation is fixed by the isometry of the scalar manifold. Hence we can see the duality
rotations as an embedding map from the set of isometries of the scalar manifold to
the group $Sp(N, \mathbb{R})$.

We now present a way to find the kinetic matrix $K$. The idea is to embed the
isometry group $G$ into $Sp(N, \mathbb{R})$ such that the compact subgroup $K$ gets embedded
into the compact subgroup of $Sp(N, \mathbb{R})$.

The group $Sp(N, \mathbb{R})$ is isomorphic to the group $USp(N,N) \equiv Sp(N, \mathbb{C}) \cap U(N,N)$.
A complex $2N \times 2N$-matrix $M$ describes an element of $USp(N,N)$ if

$$M^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \quad M^\dagger \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (4.2.36)$$

The general form of $M \in USp(N,N)$ is

$$M = \begin{pmatrix} T & V^* \\ V & T^* \end{pmatrix}, \quad \text{where } T^\dagger T - V^\dagger V = 1 \text{ and } T^\dagger V^* = V^\dagger T^*, \quad (4.2.37)$$

If $\Lambda$ is a $Sp(N, \mathbb{R})$-matrix, then the matrix $M$, given by

$$M = C\Lambda C^{-1} \quad \text{where, } \quad C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \mathbb{I} \\ 1 & -i \mathbb{I} \end{pmatrix}, \quad (4.2.38)$$

is a $USp(N,N)$-matrix. Using the $USp(N,N)$-language has the advantage that the
compact subgroup is ‘on the diagonal’; the compact subgroup of $USp(N,N)$ is $U(N)$
and is obtained by putting $V = 0$ in equation 4.2.37.

Let $V(\Phi)$ be a representative of $G/K$, then we embed $V$ into a $2N \times 2N$-matrix
$V(\Phi) \in USp(N,N)$, such that if $\Phi \neq \Phi'$, there is no $U(N)$-matrix $W$ such that

$$V(\Phi') = V(\Phi) \begin{pmatrix} W & 0 \\ 0 & W^* \end{pmatrix}. \quad (4.2.39)$$

The $USp(N,N)$-element $V(\Phi)$ describes the symplectic embedding of $G/K$ into the
coset $USp(N,N)/U(N)$, such that $K$ is embedded in $U(N)$. We write the matrix
$V(\Phi)$ in block form as

$$V(\Phi) = \begin{pmatrix} U_0(\Phi) & U_1^*(\Phi) \\ U_1(\Phi) & U_0^*(\Phi) \end{pmatrix}. \quad (4.2.40)$$

If we perform a duality transformation, mapping $\Phi$ to $\Phi'$, the matrix $V(\Phi)$ gets
rotated by a $USp(N,N)$-matrix $S$ such that

$$SV(\Phi) = V(\Phi') \begin{pmatrix} W(\Phi) & 0 \\ 0 & W^*(\Phi) \end{pmatrix}, \quad S = \begin{pmatrix} T & V^* \\ V & T^* \end{pmatrix}. \quad (4.2.41)$$
4.2 The Vector Multiplet

where $W$ is a $U(N)$-matrix. Re-expressing this in $Sp(N, \mathbb{R})$-language, using 4.2.38 and the expression for $\mathbb{V}$ 4.2.40 one finds

$$
U^\dagger_0 (\Phi') + U^\dagger_1 (\Phi') = W^\dagger (\Phi) \left( U^\dagger_0 (\Phi) (A^T + iB^T) + U^\dagger_1 (\Phi) (A^T - iB^T) \right),
$$

(4.2.42)

$$
U^\dagger_0 (\Phi') - U^\dagger_1 (\Phi') = W^\dagger (\Phi) \left( U^\dagger_0 (\Phi) (D^T - iC^T) - U^\dagger_1 (\Phi) (D^T + iC^T) \right),
$$

(4.2.43)

where $A, B, C, D$ are given by inverting 4.2.38:

$$
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix} = C^{-1} S C.
$$

(4.2.44)

From 4.2.42 we see that putting

$$
K = i \left( U^\dagger_0 + U^\dagger_1 \right)^{-1} \left( U^\dagger_0 - U^\dagger_1 \right)
$$

(4.2.45)

solves the transformation rule 4.2.33. Using that $\mathbb{V}$ is a $USp(N,N)$-matrix one verifies that $K$ is symmetric. Hence we have found a way to find the kinetic matrix. The result 4.2.44 is the result derived in 1981 by Gaillard and Zumino [145].

We now apply the procedure outlined above and obtain the vector-scalar couplings in ungauged $\mathcal{N} = 4$ supergravity coupled to $n$ vectors with all $SU(1,1)$-angles equal.

There are $6 + n$ vectors and hence the symplectic group of interest is $Sp(6+n, \mathbb{R}) \cong USp(6 + n, 6 + n)$. The scalar manifold is given by:

$$
G/K = \frac{SU(1,1)}{U(1)} \otimes \frac{SO(6,n)}{SO(6) \times SO(n)}.
$$

(4.2.46)

Since the global symmetry group consists of the direct product of the two groups $SU(1,1)$ and $SO(6,n)$, the symplectic embedding must be such that the embedded $SU(1,1)$ commutes with the embedded $SO(6,n)$. The embedding map into $USp(6 + n, 6 + n)$ is denoted $I$.

The precise embedding is contained in the decompositions of the irreducible representations of $Sp(N, \mathbb{R})$ into irreducible representations of the global symmetry group. The decomposition is often dictated by supersymmetry. We take the decomposition as given and proceed from there to give the scalar vector couplings. The decomposition of the vector representation $12 + 2n$ of $Sp(6+n, \mathbb{R})$ decomposes into the direct sum of two vector representations of $SO(6,n)$: $12 + 2n \rightarrow (6 + n) \oplus (6 + n)$. The decomposition of the vector representation $12 + 2n$ of $Sp(6+n, \mathbb{R})$ decomposes into the irreducible representations of $SU(1,1)$ as $12 + 2n \rightarrow \bigoplus_{i=1}^{6+n} 2$, where $2$ denotes the vector representation of $SU(1,1)$.

Let $\mathcal{V}_1$ be an $(6 + n) \times (6 + n)$ $SO(6,n)$-matrix satisfying:

$$
\mathcal{V}_1^T \eta_{6,n} \mathcal{V}_1 = \eta, \quad \eta_{6,n} = \begin{pmatrix}
-\mathbb{I}_{6 \times 6} & 0 \\
0 & \mathbb{I}_{n \times n}
\end{pmatrix}.
$$

(4.2.47)
Then the symplectic embedding of $V_1$ into $Sp(6 + n, \mathbb{R})$ can be taken to be:

$$V_1 \hookrightarrow \begin{pmatrix} V_1 & 0 \\ 0 & V_1^{-T} \end{pmatrix} \in Sp(6 + n, \mathbb{R}).$$

(4.2.47)

Hence the embedding $\mathcal{J}(V_1)$ of $V_1$ into $USp(6 + n, 6 + n)$ is

$$\mathcal{J} : V_1 \mapsto \frac{1}{2} \begin{pmatrix} V_1 + \eta_{6,n} V_1 \eta_{6,n} & V_1 - \eta_{6,n} V_1 \eta_{6,n} \\ V_1 - \eta_{6,n} V_1 \eta_{6,n} & V_1 + \eta_{6,n} V_1 \eta_{6,n} \end{pmatrix}.$$

(4.2.48)

A general $SU(1,1)$-matrix is given by

$$V_2(\phi^\alpha) = \begin{pmatrix} \phi_1^1 & \phi_2^2 \\ \phi_1^2 & \phi_1^1 \end{pmatrix}; \quad |\phi_1^1|^2 - |\phi_2^2|^2 = \phi^\alpha \phi_{\alpha} = 1.$$  

(4.2.49)

The embedding of this matrix into $USp(6 + n, 6 + n)$ is given by

$$\mathcal{J}(V_2(\phi^\alpha)) = \begin{pmatrix} \text{Re} \phi_1^1 \mathbb{1} + i \text{Im} \phi_2^1 \eta_{6,n} & \text{Re} \phi_2^2 \mathbb{1} - i \text{Im} \phi_2^1 \eta_{6,n} \\ \text{Re} \phi_2^2 \mathbb{1} + i \text{Im} \phi_2^1 \eta_{6,n} & \text{Re} \phi_1^1 \mathbb{1} - i \text{Im} \phi_1^1 \eta_{6,n} \end{pmatrix},$$

(4.2.50)

and the embeddings commute.

The embedding is completed by putting

$$\mathbb{V}(\phi^\alpha, Z) = \mathcal{J}(V_1) \circ \mathcal{J}(V_2) = \begin{pmatrix} U_0(\phi^\alpha, Z) & U_1^* (\phi^\alpha, Z) \\ U_1(\phi^\alpha, Z) & U_0^* (\phi^\alpha, Z) \end{pmatrix}$$

(4.2.51)

and inserted in the master equation 4.2.44 gives:

$$K(\phi^\alpha, Z) = \frac{i}{|\phi_1^1 + \phi_2^2|^2} (V_1 V_1^T)^{-1} + \frac{2 \text{Im}(\phi^1 \phi^2^*)}{|\phi_1^1 + \phi_2^2|^2} \eta_{6,n}.$$  

(4.2.52)

We now choose a particular representative for the $SO(6,n)$-scalars. We put

$$\eta_{6,n} V_1^{-1} = \begin{pmatrix} X & Y \\ U & V \end{pmatrix}, \quad V_1^{-T} \eta_{6,n} V_1^{-1} = \eta_{6,n},$$

(4.2.53)

from which we find

$$\eta_{6,n} V_1^{-T} V_1^{-1} \eta_{6,n} = \eta_{6,n} + 2 \begin{pmatrix} X^T \\ Y^T \end{pmatrix} (X \quad Y).$$

(4.2.54)

We call $(X, Y)_a^R = Z_a^R$ with $1 \leq a, b, \ldots \leq 6$ and $1 \leq R, S, \ldots \leq 6 + n$, and we define $Z^{RS} = \sum_a Z_a^R Z_a^S$. For the kinetic matrix we find

$$K(\phi^\alpha, Z_a^R)_{RS} = \frac{i}{|\phi_1^1 + \phi_2^2|^2} (\eta_{RS} + 2Z_{RS}) + \frac{2 \text{Im}(\phi^1 \phi^2^*)}{|\phi_1^1 + \phi_2^2|^2} \eta_{RS},$$

(4.2.55)
4.3 Matter Coupled Gauged $\mathcal{N} = 4$ Supergravity

where $\eta_{RS}$ are the components of $\eta_{6,n}$, and $Z_{RS} = Z^{TU} \eta_{RT} \eta_{US}$. If $L$ is an $SO(6,n)$-matrix, then so is $L^T$ and hence we have $Z_a^R \eta_{RS} Z_b^S = -\delta_{ab}$, and thus $\eta_{RS} + 2Z_{RS}$ is an $SO(6,n)$-matrix.

The coupling between the scalars and the vectors is given by:

$$L_{vec} = -\frac{\eta_{RS} + 2Z_{RS}}{4|\phi^1 + \phi^2|^2} F_{\mu\nu} F^{\mu\nu} + \frac{\text{Im}(\phi^1 \phi^{2*})}{4|\phi^1 + \phi^2|^2} \eta_{RS} F_{\mu\nu} F^{\mu\sigma} \epsilon^{\rho\sigma\rho\sigma}, \quad (4.2.56)$$

which is the same scalar-vector coupling as found with the conformal programme by De Roo for all $SU(1,1)$-angles equal [60].

We now show that the result 4.2.56 is compatible with the Lagrangian of pure supergravity. When $n = 0$ the manifold $SO(6,n)/SO(6) \times SO(n)$ becomes trivial; the scalars $Z_a^R$ can be put to $Z_a^R = \delta_a^R$ using the local $SO(6)$. Hence we have $\eta_{RS} + 2Z_{RS} = \delta_{RS}$. When we parameterize the $SU(1,1)$-scalars as

$$\phi^1 = \frac{1}{\sqrt{1 - |z|^2}}, \quad \phi^2 = \frac{z}{\sqrt{1 - |z|^2}}, \quad z \in \mathbb{C}, \ |z| < 1, \quad (4.2.57)$$

we obtain the vector-scalar couplings of 4.1.12 for all $SU(1,1)$-angles equal.

We thus have obtained by using duality symmetries the vector scalar couplings. When the $SU(1,1)$-angles are nonzero the coupling 4.2.56 becomes a little different, but is a straightforward extension of 4.2.56.

4.3 Matter Coupled Gauged $\mathcal{N} = 4$ Supergravity

We now discuss briefly the $\mathcal{N} = 4$ matter coupled gauged supergravity, where the number of matter multiplets is $n$. We restrict ourselves mostly to the bosonic sector. For the full result and a more complete discussion see e.g. [36]. We do not present the Lagrangian since it is not very instructive. We also do not give the most general gaugings; our gaugings are a subsector of the most general gaugings as discussed in [146] (in terms of reference [146] our gaugings are characterized by the vanishing of the parameter $\xi_{aM}$).

The field content of the theory is: one metric, four gravitini of spin 3/2, 6+n gauge fields, 4+4n fermions of spin 1/2 and 6n+2 scalars. The scalars are the $SU(1,1)$-scalars and the $SO(6,n)$-scalars. The $SU(1,1)$-scalars can be combined in a complex scalar $z$ as shown in section 4.1. The $SO(6,n)$-scalars are the $Z_a^R$, as in section 4.2.2. The kinetic term of the $SU(1,1)$-scalars is as in section 4.1 while the kinetic term of the $SO(6,n)$-scalars is expressed in terms of the symmetric $SO(6,n)$-matrix $M_{RS} = \eta_{RS} + 2Z_{RS}$ as

$$L_{kin}[Z_a^R] = \frac{1}{16} \text{Tr} \left( \partial_\mu M^{-1} \partial^\mu M \right). \quad (4.3.1)$$
The ungauged theory has a global symmetry group $G = SO(6,n) \times SU(1,1)$ of which the maximal compact subgroup $K = SO(6) \times SO(n) \times U(1)$ is a local symmetry group. The gauge fields are all abelian and the scalars parameterize the coset $G/K$. The $U(1)$-symmetry is used to write the $SU(1,1)$-scalars as $z$ and the $SO(n)$-subgroup of $K$ acts trivially on the scalars $Z_a^R$, while the global $SO(6,n)$ acts as $Z_a^R \mapsto O^R S Z_a^S$. 

The coupling of the scalars to the vector fields in the ungauged theory was discussed section 4.2.2 for equal $SU(1,1)$-angles. As in section 4.1 the extension to different $SU(1,1)$-angles can be done by replacing $z$ with $ze^{2i\alpha R}$ for each vector field $A^R\mu$, or equivalently putting $\phi^1 \mapsto \phi^1 e^{i\alpha R}$ and $\phi^2 \mapsto \phi^2 e^{-i\alpha R}$ for each vector field $A^R\mu$.

A subgroup $H$ of $G$ can be promoted to a local symmetry; this destroys the global symmetry $G$. The number of vector fields is $6 + n$ and hence at most a $(6 + n)$-dimensional group can be gauged. With every gauge field $A^R\mu$ we associate a generator $T^R$ and we define the gauge group $H$ by its structure constants $f_{RS T}^R$: 

$$\left[ T^R, T^S \right] = f_{RS T}^R T^T.$$ 

The structure constants are not arbitrary but have to satisfy 

$$f_{RS T}^R \eta_{TU} + f_{RU T}^R \eta_{TS} = 0.$$ 

Hence the adjoint representation of the Lie algebra $\mathfrak{h}$ of the group $H$ has to be embedded in the vector representation of $so(6,n)$. This severely restricts the possibilities of gaugings and is discussed more in detail when we investigate the scalar potential. The scalar potential factorizes in an $SU(1,1)$-part and an $SO(6,n)$-part. The potential arises when the ungauged supergravity is gauged while keeping the theory supersymmetric.

### 4.4 The Scalar Potential

The solutions of a theory are the most important features of a theory and within the solutions the vacua play the most dominant role. The vacua are the backgrounds around which the quantum field theory is developed.

We are mainly interested in vacua where the geometry is maximally symmetric and all fields except the metric vanish. In this situation the number of Killing vectors is maximal. If we denote the Killing vectors by $K_I$, $I = 1, 2, \ldots$, the metric satisfies $(\mathcal{L}_{K_I} g)_{\mu \nu} = \nabla_\mu K^I_\nu + \nabla_\nu K^I_\mu = 0$. For a given metric the maximal number of Killing vectors is thus 10. It turns out that there are only three distinct maximally symmetric geometries. Either space-time is Minkowski or de Sitter or anti-de Sitter. In a Minkowski space-time there is no curvature and the 10 Killing vectors generate an $SO(3,1) \ltimes \mathbb{R}^4$-group. In a de Sitter space-time the curvature is positive and the Killing vectors form an $so(4,1)$ Lie algebra, while in an anti-de Sitter the curvature is negative and the Lie algebra formed by the Killing vectors is $so(3,2)$.

The configuration of the fields has to be compatible with the symmetry and therefore the Lie derivative along a Killing vector on a field has to vanish. For gauge
fields one of course demands that the Lie derivative along a Killing vector on the field strength vanishes. However, working out the Lie derivatives along the Killing vectors is quite tedious for the de Sitter and anti-de Sitter vacua.

All three isometry groups contain the Lorentz group. Therefore it is a necessary condition that all fields (or field strengths for gauge fields) are invariant under local Lorentz symmetry. For a scalar this implies that it has to be constant since $\partial_\alpha \Phi = E_\mu \partial_\mu \Phi$, where $\Phi$ is any scalar, is not invariant under local Lorentz transformations unless $\partial_\alpha \Phi = 0$. The requirement that the scalars are constant is precisely what one obtains when demanding that the Lie derivative along the Killing vectors vanishes on the scalars. The fermions are in irreducible representations of the Lorentz group and hence in a Lorentz invariant configuration all fermions vanish. Along similar lines, all two-form field strengths have to vanish. Hence for $N = 4 \ d = 4$ supergravity a solution is a vacuum solution if the scalars are constants, the metric describes a maximally symmetric space and all other fields (field strengths for gauge fields) vanish.

The scalar potential in the vacuum configuration $V_0$ is the only contribution of the nonmetric fields to the energy-momentum tensor. The vacuum potential $V_0$ therefore determines the geometry. If $V_0 > 0$ the geometry is de Sitter, if $V_0 < 0$ the geometry is the anti-de Sitter and if $V_0 = 0$ the geometry is Minkowskian.

Recent observations have shown that the universe is at present in an accelerating phase [147–150]. Therefore our universe has (on large scale) a de Sitter geometry or is evolving into a de Sitter geometry [151]. On the other hand, there are no-go theorems that state that no dimensional reduction of a ten-dimensional supergravity theory admits a four-dimensional de Sitter vacuum [152, 153].

Within the context of string theory some attempts have been made to evade the no-go theorems, for example in [154–156], but in these cases it is a delicate issue to stabilize all the scalars. When not all scalars are stabilized, the vacuum is not stable and the theory will roll into another vacuum, which in most cases is not a de Sitter vacuum (see e.g. [157]).

In four dimensions it is possible to obtain stable de Sitter vacua from $N = 2$ supergravities [158, 159]. But also in five dimensional $N = 2$ supergravities stable de Sitter vacua have been constructed [160].

In order to obtain a de Sitter vacuum it seems necessary to gauge a noncompact group. It has been shown that the noncompact gaugings can in some cases be associated with a dimensional reduction of a higher dimensional supergravity over a noncompact manifold [161]. But not for all gauged supergravities a higher dimensional origin is known; for example, for the anomaly-free six-dimensional Salam–Sezgin model [162–164] and for the matter coupled $N = 4$ supergravity with nontrivial $SU(1, 1)$-angles no higher dimensional origin is yet known. Maybe it is not possible to obtain all lower-dimensional supergravities from a ten-dimensional supergravity.

There exist vacua where spacetime geometry is not maximally symmetric and where vector fields have nonvanishing field strengths. Examples are provided by the
electrovac and magnetovac solutions [165–167]. However, we focus on maximally symmetric spacetimes.

In the following sections we try to find stable de Sitter vacua in $\mathcal{N} = 4$ supergravity. We first give the potential and give various definitions that come in handy at later points in the discussion. In section 4.5 we discuss semisimple gaugings and in section 4.6 we study the potential obtained from gauging a $\text{CSO}$-group, the concept of which is discussed in the same section. The possible extensions of the analysis are discussed in section 4.7.

The Potential of $\mathcal{N} = 4$ Supergravity

We first give the potential and then discuss the ingredients. The potential is given by

$$V(Z_a^R, \phi^\alpha) = \frac{1}{4} \left[ Z^{RU} Z^{SV} (\eta^{TW} + \frac{2}{3} Z^{TW}) \right] \text{Re} \left( \Phi^*_R \Phi(U) \right) g(R) g(U) f_{RST} f_{UVW}$$

$$+ \frac{1}{36} Z^{RSTUVW} \text{Im} \left( \Phi^*_R \Phi(U) \right) g(R) g(U) f_{RST} f_{UVW}, \quad (4.4.1)$$

where $Z^{RS} = Z_a^R Z_a^S$ and $Z^{RSTUVW} = \epsilon_{abcdel} Z^a R Z^b S Z^c T Z^d U Z^e V Z^f W$ are $SO(6)$-invariant combinations of the $SO(6, n)$-scalars. Therefore the scalar potential is $SO(6)$-invariant. The compact subgroup $SO(6) \times SO(n)$ acts on the $SO(6, n)$-scalars $Z_a^R$ only through the first $SO(6)$ subgroups and hence the potential is invariant under the local symmetry group $SO(6) \times SO(6)$.

The structure constants are not arbitrary; $f_{RST} = \eta_{RU} f_{ST}^U$ is completely antisymmetric, which is equivalent to the constraint 4.3.2. The coupling constants are contained in the numbers $g(R)$. For every subgroup of the gauge group we allow different coupling constants. With every generator $T_R$ we associate a coupling constant $g(R)$, such that if $T_R$ and $T_S$ belong to the same subgroup, then $g(R) = g(S)$.

The $SU(1, 1)$-scalars are contained in $\Phi^*_R \Phi(U)$ through

$$\Phi^*_R \Phi(U) = e^{i\alpha_R} \phi^1 + e^{-i\alpha_R} \phi^2. \quad (4.4.2)$$

If all $\alpha_R$ angles are the same we have $\text{Im} \Phi^*_R \Phi(U) = 0$ and the last term in the potential 4.4.1 vanishes. In this situation the symmetry group of the potential is enlarged to $O(6)$.

4.5 The Potential with Semisimple Gaugings

In this section we study the potential with semisimple gaugings. Most of the discussion can be found in references [A, B]. Before we study the potential we analyze which gauge groups are allowed.
4.5 The Potential with Semisimple Gaugings

4.5.1 Semisimple Gaugings

Let us denote the gauge group with $H$. If we demand that $H$ is semisimple, $H$ is the direct product of simple groups: $H = H_1 \otimes \ldots \otimes H_n$. We denote the Lie algebras of the simple factors $H_i$ by $\mathfrak{h}_i$; $\mathfrak{h}$ is the direct sum of the $\mathfrak{h}_i$. A basis of $\mathfrak{h}$ is given by $T_1, T_2, \ldots$ such that we have $[T_R, T_S] = f_{RS}^T T_T$. The coefficients $f_{RS}^T$ are real and are called the structure constants of $\mathfrak{h}$.

With each gauge field $A_R^\mu$ we associate a generator $T_R$ in the gauge algebra and an $SU(1,1)$-angle $\alpha_R$. The gauge group rotates the gauge fields associated to the same factor into each other. All the generators that can be obtained by rotating the generator $T_R$ need to have the same $SU(1,1)$-angle $\alpha_R$ for the gauge group to be a symmetry. Hence along the gauge orbit of $T_R$, denoted by $\Gamma[T_R]$ and defined by

$$\Gamma[T_R] = \{ \text{ad}(A)|A \in \mathfrak{h}\} \quad (4.5.1)$$

the $SU(1,1)$-angle has to be constant. If $\Gamma[T_R] \cap \Gamma[T_S] \neq 0$ we need $\alpha_R = \alpha_S$. For semisimple groups the gauge orbits are the simple factors and hence with each simple factor we associate a single $SU(1,1)$-element.

Let us define a nondegenerate symmetric bilinear form $\Omega : \mathfrak{h} \times \mathfrak{h} \to \mathbb{R}$ on $\mathfrak{h}$ by

$$\Omega(T_R, T_S) = \eta_{RS} \quad (4.5.2)$$

The relation 4.3.2 then reads $\Omega([T_R, T_S], T_U) = \Omega(T_R, [T_S, T_U])$ and $\Omega$ is thus an invariant bilinear symmetric nondegenerate form on $\mathfrak{h}$. From now on we abbreviate 'symmetric nondegenerate bilinear form' as 'metric'. A group $H$ can be a gauge group if $\Omega$ is an invariant metric on the Lie algebra of $H$.

For simple complex Lie algebras there is up to a multiplicative constant just one invariant metric. In appendix B we prove this statement. Since the Cartan–Killing metric is invariant, all invariant metrics on a simple complex Lie algebra are proportional to the Cartan–Killing metric. For real Lie algebras of which the complex extension is simple, the same result holds. This can be seen by going to the complex extension; if the real Lie algebra would have more than one invariant metric, so would the complex Lie algebra, which is a contradiction.

However, there exist real Lie algebras of which the complex extension is not simple and in this case the complex extension is the direct sum of two identical simple complex Lie algebras [94]. The real Lie algebra is then complex; it is a complex Lie algebra, written as a real direct sum of the real and imaginary part$^2$. An example is given by $\mathfrak{so}(1,3)$, for which we have $\mathfrak{so}(1,3)^C = \mathfrak{so}(3)^C \oplus \mathfrak{so}(3)^C$ and $\mathfrak{so}(1,3) \cong \mathfrak{so}(3) \oplus_{\mathbb{R}} \mathfrak{is}(3)$. One can show that for real Lie algebras of this kind there exist a two-parameter family of invariant metrics. In the discussion that follows we ignore this and take as invariant metric on the Lie algebra the metric that is proportional to

$^2$Note that the decomposition into the real and imaginary part is not unique.
\[ N = 4 \ d = 4 \] Supergravity and its Scalar Potential

Table 4.5.1: The real simple Lie algebras of dimension not exceeding twelve. For every real simple Lie algebra the dimension of the compact subalgebra \( \dim \mathfrak{k}_i \) and of the noncompact part \( \dim \mathfrak{p}_i \) are denoted.

<table>
<thead>
<tr>
<th>Real Forms</th>
<th>( \dim \mathfrak{t}_i )</th>
<th>( \dim \mathfrak{p}_i )</th>
<th>Real Forms</th>
<th>( \dim \mathfrak{t}_i )</th>
<th>( \dim \mathfrak{p}_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathfrak{su}(2) )</td>
<td>3</td>
<td>0</td>
<td>( \mathfrak{so}(1,3) )</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>( \mathfrak{sl}(2, \mathbb{R}) )</td>
<td>1</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \mathfrak{su}(3) )</td>
<td>8</td>
<td>0</td>
<td>( \mathfrak{so}(5) )</td>
<td>10</td>
<td>0</td>
</tr>
<tr>
<td>( \mathfrak{su}(2,1) )</td>
<td>4</td>
<td>4</td>
<td>( \mathfrak{so}(1,4) )</td>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td>( \mathfrak{sl}(3, \mathbb{R}) )</td>
<td>3</td>
<td>5</td>
<td>( \mathfrak{so}(2,3) )</td>
<td>4</td>
<td>6</td>
</tr>
</tbody>
</table>

For semisimple Lie algebras consisting of \( n \) simple factors that are not complex, the space of invariant metrics is \( n \)-dimensional. The Cartan–Killing metric of each simple factor \( \mathfrak{h}_i \) has to be proportional to the metric \( \Omega \).

To make contact with string theory we will take 6 vector multiplets coupled to \( \mathcal{N} = 4 \) supergravity. This corresponds to ten-dimensional \( \mathcal{N} = 1 \) supergravity where the Yang–Mills fields are truncated away. With the choice of six additional vector multiplets the total number of gauge vectors is twelve and hence the gauge group \( H \) can at most be 12-dimensional.

Every simple Lie algebra \( \mathfrak{h}_i \) can be decomposed as \( \mathfrak{h}_i = \mathfrak{t}_i \oplus \mathfrak{p}_i \), where \( \mathfrak{t}_i \) is the compact part and where \( \mathfrak{p}_i \) is the noncompact part. In Table 4.5.1 we have put up to isomorphisms the real simple Lie algebras of dimension less than twelve. The relevant isomorphisms are:

\[
\begin{align*}
\mathfrak{su}(2) &\cong \mathfrak{so}(3) \cong \mathfrak{sp}(1), \\
\mathfrak{su}(1,1) &\cong \mathfrak{sl}(2, \mathbb{R}) \cong \mathfrak{sp}(1, \mathbb{R}) \cong \mathfrak{so}(1,2), \\
\mathfrak{so}(2,2) &\cong \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}), \\
\mathfrak{so}(1,3) &\cong \mathfrak{sp}(1,3).
\end{align*}
\]

The real Lie algebra \( \mathfrak{sl}(2, \mathbb{C})_{\mathbb{R}} \) is obtained by writing the complex Lie algebra \( \mathfrak{sl}(2, \mathbb{C}) \) as the direct sum of its real part and its complex part \( \mathfrak{sl}(2, \mathbb{C})_{\mathbb{R}} \cong \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{C}) \) (also see Appendix B).

The Cartan–Killing metric is negative definite on the compact part and positive definite on the noncompact part. Therefore the dimensions of each \( \mathfrak{t}_i \) and \( \mathfrak{p}_i \) cannot exceed six. From Table 4.5.1 we see that the simple algebras that can be used for gauging are: \( \mathfrak{su}(2), \mathfrak{so}(2,3), \mathfrak{so}(1,4), \mathfrak{sl}(8, \mathbb{R}), \mathfrak{su}(1,1), \mathfrak{su}(1,2), \mathfrak{so}(1,3) \). In this list only \( \mathfrak{so}(1,3) \) is a complex Lie algebra. For reasons of simplicity, we ignore the fact of its complex structure.
4.5 The Potential with Semisimple Gaugings

that \( \mathfrak{so}(1, 3) \) admits a two-parameter family of invariant metrics and fix the invariant metric of \( \mathfrak{so}(1, 3) \) to be the Cartan–Killing metric.

The gauge Lie algebra can be split into a part on which \( \Omega \), (defined in equation 4.5.2) is positive definite, denoted \( \mathfrak{h}_+ \), and a part on which \( \Omega \) is negative definite, denoted \( \mathfrak{h}_- \). For every simple factor there are two distinct ways how to embed them into the gauge group. For the noncompact simple algebras \( \mathfrak{h}_i \) either \( k_i \) lies in \( \mathfrak{h}_+ \) and \( p_i \) in \( \mathfrak{h}_- \) or \( p_i \) lies in \( \mathfrak{h}_+ \) and \( k_i \) in \( \mathfrak{h}_- \). The compact simple Lie algebras \( \mathfrak{h}_i \) have to be embedded either completely in \( \mathfrak{h}_+ \) or completely in \( \mathfrak{h}_- \). A semisimple gauging is determined by the choice of simple factors and the choice of embedding in the gauge algebra.

Taking into account the different ways to embed the simple groups into the gauge group, the possible Lie algebras that can be the gauge algebra \( \mathfrak{h} \) are:

\[
\begin{align*}
\mathfrak{so}(2, 3), & \quad \mathfrak{so}(1, 4), & \quad 4 \times \mathfrak{su}(2), & \quad 4 \times \mathfrak{sl}(2, \mathbb{R}), \\
\mathfrak{su}(2) \oplus 3 \times \mathfrak{sl}(2, \mathbb{R}), & \quad 2 \times \mathfrak{su}(2) \oplus 2 \times \mathfrak{sl}(2, \mathbb{R}), \\
2 \times \mathfrak{su}(2) \oplus \mathfrak{so}(1, 3), & \quad \mathfrak{su}(2, 1) \oplus \mathfrak{sl}(2, \mathbb{R}), \\
\mathfrak{sl}(3, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}), & \quad \mathfrak{so}(1, 3) \oplus 2 \times \mathfrak{sl}(2, \mathbb{R}),
\end{align*}
\]

(4.5.4)

and subalgebras of these algebras. The notation \( n \times \mathfrak{h}_i \) means the direct sum of \( n \) copies of \( \mathfrak{h}_i \).

4.5.2 Extrema in the \( SU(1, 1) \)-Sector

We now wish to find extrema of the potential with respect to the \( SU(1, 1) \)-scalars. The discussion is based on reference [A]. For the \( SU(1, 1) \)-scalars we use the parametrization 4.1.5.

For every factor \( \mathfrak{h}_i \) we denote the structure constants \( f_{RS}^T \), the \( SU(1, 1) \)-angle \( \alpha_i \), the coupling constants \( g_i \) and in a similar way we write \( \Phi_i = e^{i\alpha_i} \phi^1 + e^{-i\alpha_i} \phi^2 \). The potential 4.4.1 can be written as a sum over the factors;

\[
V = \sum_{i,j} V_{ij} R^{(ij)} + W_{ij} I^{(ij)},
\]

(4.5.5)

where

\[
R^{(ij)} = \text{Re}|g_i g_j \Phi_i^* \Phi_j| = \frac{g_i g_j}{1 - r^2} \left[ \cos(\alpha_i - \alpha_j)(1 + r^2) - 2r \cos(\alpha_i + \alpha_j + \chi) \right],
\]

\[
I^{(ij)} = \text{Im}|g_i g_j \Phi_i^* \Phi_j| = -g_i g_j \sin(\alpha_i - \alpha_j).
\]

The \( V_{ij} \) and \( W_{ij} \) contain the structure constants and the fields \( Z_{aR} \) and are symmetric, respectively antisymmetric in the indices \( ij \) and are given by:

\[
\begin{align*}
V_{ij} &= \frac{1}{2} Z_{aR} Z_{SV} (\eta^{TW} + \frac{2}{3} Z^{TW}) f_{RST}^{(i)} f_{UVW}^{(j)}, \\
W_{ij} &= \frac{1}{2} Z_{RSTUVW} f_{RST}^{(i)} f_{UVW}^{(j)}
\end{align*}
\]

(4.5.7)
We define:

\[ C_\pm = \sum_{i,j} g_i g_j \cos(\alpha_i \pm \alpha_j) V_{ij}, \]
\[ S_+ = \sum_{i,j} g_i g_j \sin(\alpha_i + \alpha_j) V_{ij}, \]
\[ T_- = \sum_{i,j} g_i g_j \sin(\alpha_i - \alpha_j) W_{ij}, \]
\[ \Delta = C_+^2 - C_-^2 - S_+^2, \quad \varepsilon = \text{sgn} C_. \]  

(4.5.8)

Equipped with these definitions the potential can be written as

\[ V = C_- \frac{1 + r^2}{1 - r^2} - \frac{2r}{1 - r^2} (C_+ \cos \chi - S_+ \sin \chi) - T_. \]  

(4.5.9)

The potential only has an extremum in the \( SU(1,1) \)-sector if \( \Delta > 0 \) \([A]\). The scalars \( r, \chi \) are at the extremum given by:

\[ \cos \chi_0 = \frac{\varepsilon C_+}{\sqrt{C_+^2 + S_+^2}}, \quad \sin \chi_0 = -\frac{\varepsilon S_+}{\sqrt{C_+^2 + S_+^2}}, \quad r_0 = \frac{|C_-| - \Delta}{\sqrt{C_+^2 + S_+^2}}. \]  

(4.5.10)

and the value of the potential is given by

\[ V_0 = \varepsilon \Delta - T_. \]  

(4.5.11)

The potential at the extremum does not depend on the absolute values of the \( \alpha_i \) but on the differences \( \alpha_i - \alpha_j \). If all \( \alpha_i \) are the same, we have \( \Delta = 0 \) implying \( r_0 = 1 \), which lies outside the range of \( r \), and the potential has no extremum. This result is a generalization of the work Freedman and Schwarz in the pure \( N = 4 \) \( d = 4 \) supergravity \([140]\).

If \( \varepsilon > 0 \) the extremum is a minimum\([A]\). In the limit \( r \to 1 \) the potential becomes singular and at the point \( r = 0 \) the potential and its derivatives are well defined.

To proceed we need to find an extremum of the potential with respect to the \( SO(6,6) \)-scalars, such that the conditions \( \Delta > 0 \) and \( \varepsilon > 0 \) are satisfied. In the following we study for different semisimple gaugings what extremum exists and calculate the value of the potential, \( \varepsilon \) and \( \Delta \) and investigate the stability.

### 4.5.3 Extrema in the \( SO(6,6) \)-sector

If all \( \alpha_i \) are the same, no minimum exists in the \( SU(1,1) \)-sector and therefore we discard all gaugings where the gauge algebra is simple. The gauge algebras \( \mathfrak{so}(3,2) \) and \( \mathfrak{so}(1,4) \) are thus left out of consideration.
To make a distinction between the two different embeddings for every simple factor of the gauge algebra, we have the following notation. If the simple factor is embedded with its compact subalgebra $k_i$ into $h_i^-$, where $\Omega$ is negative definite, we denote the simple factor by $h_i^-$. If the simple factor is embedded with its compact part in $h_i^+$, where $\Omega$ is positive definite, we denote it $h_i^+$.

It was shown in reference [A] that the point where $Z_a^R$ is given by

$$Z_a^R = \delta_a^R, \quad R \leq 6, \quad Z_a^R = 0, \quad 6 < R \leq 12,$$

(4.5.12)
corresponds in all cases that are studied to an extremum, if an extremum exists. Therefore we restrict the analysis to this point. This point corresponds mathematically to the identity of the manifold $SO(6,6)/SO(6) \times SO(6)$ and the representative of this point can be chosen to be the identity element of $SO(6,6)$. Physically this point corresponds to turning off the matter fields. The point is invariant under $SO(6)$-transformations, but not under $SO(6,6)$-transformations that are not $SO(6)$-transformations since these transformations mix matter fields and supergravity fields. We refer to this point as $Z_0$.

We split the indices $R, S, \ldots$ up in capital letter $A, B, \ldots$ from the beginning of the alphabet running from 1 to 6, and middle-alphabet capitals $I, J, \ldots$ running from 7 to 12. With this convention the point $Z_0$ is given by: $Z_a^A = \delta_a^A, \quad Z_a^I = 0$.

The normalization of the generators is chosen such that in the vector representation $|\text{tr}(T_R T_S)| = 2 \delta_{RS}$.

In the point $Z_0$ we have the following simplifications:

$$V_{ij}(Z_0) = \delta_{ij} \left( -\frac{1}{12} \sum_{ABC} f^{(i)}_{ABC} f^{(i)}_{ABC} + \frac{1}{4} \sum_{ABI} f^{(i)}_{ABI} f^{(i)}_{ABI} \right),$$

$$C^-(Z_0) = \sum_i g_i^2 V_{ii}(Z_0),$$

$$\Delta(Z_0) = 2 \sum_{i,j} V_{ii}(Z_0) V_{jj}(Z_0) (g_i g_j \sin(\alpha_i - \alpha_j))^2.$$  

(4.5.13)

The functions $V_{ij}$ are thus nonzero only if $i = j$ at the point $Z_0$ and the diagonal values can be calculated for each simple factor $h_i$. The obtained results are displayed in table 4.5.2 and the details can be found in reference [B].

In order to check that the point $Z_0$ is a minimum one uses a parametrization of the $Z_a^R$ and then differentiate the potential with respect to the parameters and evaluate at $Z_0$. The result then has to vanish in order for the point $Z_0$ to be an extremum. To check for stability, one evaluates the second derivatives at $Z_0$ and the matrix of second derivatives has to be positive definite to be a minimum.

There exist different parameterizations of the $Z_a^R$, for example one can take:
\( \mathcal{N} = 4 \) d = 4 Supergravity and its Scalar Potential

Table 4.5.2: The functions \( V_{ii} \) at the point \( Z_0 \).

<table>
<thead>
<tr>
<th>Simple Factor</th>
<th>( V_{ii}(Z_0) )</th>
<th>Simple Factor</th>
<th>( V_{ii}(Z_0) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>so(3)$_-$</td>
<td>(-\frac{1}{2})</td>
<td>so(3)$_+$</td>
<td>0</td>
</tr>
<tr>
<td>so(3, 1)$_-$</td>
<td>(-\frac{1}{2})</td>
<td>so(3, 1)$_+$</td>
<td>(\frac{3}{2})</td>
</tr>
<tr>
<td>sl(3, IR)$_-$</td>
<td>(-\frac{1}{2})</td>
<td>sl(3, IR)$_+$</td>
<td>(\frac{15}{2})</td>
</tr>
<tr>
<td>su(2, 1)$_-$</td>
<td>(-2)</td>
<td>su(2, 1)$_+$</td>
<td>6</td>
</tr>
<tr>
<td>su(1, 1)$_-$</td>
<td>0</td>
<td>su(1, 1)$_+$</td>
<td>(\frac{1}{2})</td>
</tr>
</tbody>
</table>

\( Z_a^R = (X, Y)_a^R \), where \( X \) and \( Y \) are 6 \times 6\-matrices given by:

\[
X = \frac{1}{2} \left( G + G^{-1} + BG^{-1}B - BG^{-1}B \right),
\]

\[
Y = \frac{1}{2} \left( G - G^{-1} - BG^{-1}B - BG^{-1}B \right),
\]

in which \( G \) is a symmetric 6 \times 6\-matrix and \( B \) is an antisymmetric 6 \times 6\-matrix. The \( X \) and \( Y \) solve the \( SO(6, 6) \)-constraint \( Z_a^R \eta_RZ_b^S = -\delta_{ab} \) since \( XX^T - YY^T = 1 \).

The independent coordinate one can take to be \( P \equiv G + B \). Every arbitrary\(^3\) \( P \) can be decomposed into a symmetric part corresponding to \( G \) and an antisymmetric part corresponding to \( B \).

Working out the derivatives is straightforward but elaborate and not very insightful. Therefore we do not give the full expressions of the first and second derivatives. In reference [B] the expressions can be found and evaluating at \( Z_0 \) is easiest done on a computer.

The results are presented in table 4.5.3. A tachyonic mode corresponds to a mode with imaginary mass; for every negative eigenvalue of the matrix of second derivatives there is a tachyonic mode. If tachyonic modes are present the vacuum is instable. In four cases both the value of the potential and the presence of tachyonic modes in the matter scalars \( Z_a^R \) depends on the choice of the \( SU(1, 1) \)-scalars. In reference [B] it is shown that for these four cases the value of the potential is positive if no matter tachyons are present. However, for these four cases \( C_- < 0 \) and there are tachyons in the \( SU(1, 1) \)-sector. From table 4.5.3 we conclude that the point \( Z_0 \) does not correspond to a stable vacuum, neither de Sitter nor anti-de Sitter nor Minkowski for semisimple gaugings with 6 matter multiplets.

The Vector Fields

We now briefly discuss the masses of the vectors and the signs of the kinetic terms.

\(^3\)The matrix \( P \) is almost arbitrary; the determinant of \( P + P^T \) cannot be zero.
4.5 The Potential with Semisimple Gaugings

Table 4.5.3: Result of the analysis of semisimple gaugings. Only the gaugings with $\Delta_0 > 0$ that give rise to an extremum with respect to the matter scalars at $Z_0$ are listed. The sign of $C_-$ at $Z_0$ is denoted $\varepsilon_0$ and the point $Z_0$ is only a minimum with respect to the $SU(1,1)$-scalars if $\varepsilon_0 = +1$. If the sign of the potential at $Z_0$ depends on the $SU(1,1)$-angles this is denoted $\pm 1$. The third column is used to indicate whether there are tachyonic modes present in the matter sector; a Y means that tachyons are present and N means no tachyons are present. If the presence of the tachyonic modes in the matter sector depends on the choice of the $SU(1,1)$-angles we denote $\text{Y/N}$.

<table>
<thead>
<tr>
<th>Gauging</th>
<th>$\varepsilon_0$</th>
<th>sign $V_0'$</th>
<th>Matter Tachyons</th>
</tr>
</thead>
<tbody>
<tr>
<td>$SO(2,1)^2 \times SO(3)$</td>
<td>+1</td>
<td>+1</td>
<td>Y</td>
</tr>
<tr>
<td>$SO(2,1)^2 \times SO(2,1)^2$</td>
<td>+1</td>
<td>+1</td>
<td>Y</td>
</tr>
<tr>
<td>$SO(3,1)<em>+ \times SO(2,1)</em>+ \times SO(2,1)_-$</td>
<td>+1</td>
<td>+1</td>
<td>Y</td>
</tr>
<tr>
<td>$SO(3,1)^2$</td>
<td>+1</td>
<td>+1</td>
<td>Y</td>
</tr>
<tr>
<td>$SO(3)^2 \times SO(3)^2$</td>
<td>$-1$</td>
<td>$\pm 1$</td>
<td>Y/N</td>
</tr>
<tr>
<td>$SO(3)<em>- \times SO(3)</em>+ \times SO(3,1)_-$</td>
<td>$-1$</td>
<td>$\pm 1$</td>
<td>Y/N</td>
</tr>
<tr>
<td>$SO(3,1)^2$</td>
<td>$-1$</td>
<td>$\pm 1$</td>
<td>Y/N</td>
</tr>
<tr>
<td>$SL(3,\mathbb{R})<em>- \times SO(3)</em>-$</td>
<td>$-1$</td>
<td>$\pm 1$</td>
<td>Y/N</td>
</tr>
<tr>
<td>$SU(2,1)<em>+ \times SO(2,1)</em>+$</td>
<td>+1</td>
<td>+1</td>
<td>Y</td>
</tr>
</tbody>
</table>

In reference [B] it was shown that the vectors associated with noncompact groups acquire a mass. This is sensible since the noncompact groups do not leave the points $Z_0$ invariant. Hence the vector fields associated with noncompact gauge groups acquire a mass through a Higgs mechanism.

The vector kinetic term can be found from equation 4.2.56 and the discussion in section 4.3:

$$
\mathcal{L}_{\text{vec-kin}} = -\frac{\eta_{RS} + 2Z_{RS}}{4|\Phi(R)|^2} F_{\mu\nu}^R F^{\mu\nu}.
$$  \hfill (4.5.15)

Evaluating the kinetic term at $Z_0$ gives:

$$
\mathcal{L}_{\text{vec-kin}} = -\frac{\delta_{RS}}{4|\Phi(R)|^2} F_{\mu\nu}^R F^{\mu\nu}.
$$  \hfill (4.5.16)

Hence there are no wrong-sign kinetic terms and the theory has no ghosts.
4.6 The Potential with CSO-Gauings

Most gaugings in field theories involve semisimple gauge groups. The reason is that the trace in any representation of the gauge group is an invariant symmetric bilinear form on the Lie algebra and no components of the field strength get projected out in calculating the kinetic term $\sim \text{Tr} F^2$. But a group manifold reduction of Heterotic supergravity results in a gauged supergravity with a gauge group that is not semisimple [112]. Hence for supergravities even the gauge groups that are not semisimple are interesting, see e.g. [168–174].

A class of groups that are not semisimple is the class of CSO-groups. The CSO-groups are contractions of the special orthogonal groups, the SO-groups. In the following we explain what a CSO-group is and give a theorem that is useful when considering gaugings with CSO-groups.

4.6.1 The CSO-type Algebras: an Introduction

We define the group $\text{CSO}(p,q,r)$ as the connected real Lie group with real Lie algebra $\mathfrak{cso}(p,q,r)$. We now show in detail how the Lie algebra $\mathfrak{cso}(p,q,r)$ is defined.

Consider the Lie algebra $\mathfrak{so}(p,q+r)$, which has the vector representation as a faithful representation. In the vector representation the Lie algebra $\mathfrak{so}(p,q+r)$ admits a set of basis elements $J_{AB} = -J_{BA}$, $1 \leq A, B \leq p+q+r$ satisfying the commutation relation:

$$[J_{AB}, J_{CD}] = \eta_{BC} J_{AD} + \eta_{AD} J_{BC} - \eta_{AC} J_{BD} - \eta_{BD} J_{AC},$$

(4.6.1)

where $\eta_{AB}$ are the components of the diagonal metric $\eta_{p,q+r}$, with $p$ eigenvalues $+1$ and $q+r$ eigenvalues $-1$.

We split the indices $A, B, \ldots$ into indices $I, J, \ldots$ running from 1 to $p+q$ and indices $a, b, \ldots$ running from $p+q+1$ to $p+q+r$. The Lie algebra $\mathfrak{so}(p,q+r)$ splits as a vector space direct sum $\mathfrak{so}(p,q+r) = \mathfrak{so}(p,q) \oplus \mathcal{V} \oplus \mathcal{Z}$, where the elements $J_{IJ}$ span the $\mathfrak{so}(p,q)$ subalgebra, the elements $J_{ia} = -J_{ai}$ span the subspace $\mathcal{V}$ and the elements $J_{ab}$ span the subalgebra $\mathcal{Z}$. The subspace $\mathcal{V}$ consists of $r$ copies of the vector representation of the subalgebra $\mathfrak{so}(p,q)$, whereas the subalgebra $\mathcal{Z}$ consists of singlet representations of $\mathfrak{so}(p,q)$. The commutation relations are schematically given by:

$$[\mathfrak{so}(p,q), \mathcal{V}] \subset \mathcal{V}, \quad [\mathcal{V}, \mathcal{V}] \subset \mathcal{Z} \oplus \mathfrak{so}(p,q),$$

$$[\mathfrak{so}(p,q), \mathcal{Z}] \subset 0, \quad [\mathcal{Z}, \mathcal{V}] \subset \mathcal{V},$$

$$[\mathfrak{so}(p,q), \mathfrak{so}(p,q)] \subset \mathfrak{so}(p,q), \quad [\mathcal{Z}, \mathcal{Z}] \subset \mathcal{Z}. \quad (4.6.2)$$

We define for any real number $\xi$ a linear map $T_\xi : \mathfrak{so}(p,q+r) \rightarrow \mathfrak{so}(p,q+r)$ by
4.6 The Potential with CSO-Gaugings

its action on the subspaces:

\[ x \in \mathfrak{so}(p, q), \quad T_\xi : x \mapsto x, \]
\[ x \in \mathcal{V}, \quad T_\xi : x \mapsto \xi x, \]
\[ x \in \mathcal{Z}, \quad T_\xi : x \mapsto \xi^2 x. \]  (4.6.3)

If \( \xi \neq 0, \infty \) the map \( T_\xi \) is a bijection. The maps \( T_0 \) and \( T_\infty \) give rise to so-called contracted Lie algebras.

We define the limits

\[ \mathfrak{so}(p, q, r) = \mathfrak{so}(p, q) \oplus \mathfrak{r} \oplus \mathfrak{z} \]

and the commutation rules are of the form

\[ [\mathfrak{s}, \mathfrak{s}] \subset \mathfrak{s}, \quad [\mathfrak{r}, \mathfrak{r}] \subset \mathfrak{r}, \quad [\mathfrak{r}, \mathfrak{z}] = [\mathfrak{s}, \mathfrak{z}] = [\mathfrak{z}, \mathfrak{z}] = 0. \]  (4.6.4)

We mention some special cases and properties. If \( r = 0 \) the construction is trivial and therefore we take \( r > 0 \). If \( p + q = 1 \) we have \( \mathfrak{s} = 0 \) and if \( p + q = r = 1 \) also \( \mathfrak{z} = 0 \) and we have \( \mathfrak{cso}(1, 0, 1) \cong \mathfrak{cso}(0, 1, 1) \cong \mathfrak{u}(1) \). If \( p + q = 2 \) the Lie algebra \( \mathfrak{s} \) is abelian and if \( p + q > 2 \) the Lie algebra \( \mathfrak{s} \) is semisimple and the vector representation is irreducible. Hence if \( p + q > 2 \) we have \( [\mathfrak{s}, \mathfrak{r}] \cong \mathfrak{r} \). If \( r = 1 \) we have \( \mathfrak{z} = 0 \) and the Lie algebra is an In"on"u–Wigner contraction. If \( r > 1 \) the subalgebra \( \mathfrak{z} \) is nontrivial and is contained in the center of \( \mathfrak{cso}(p, q, r) \). If the center of a Lie algebra is nonzero, the adjoint representation is not faithful.

From the construction follows a convenient set of basis elements of \( \mathfrak{cso}(p, q, r) \). The elements \( S_{IJ} = -S_{JI} \) are the basis elements of the subalgebra \( \mathfrak{s} \), the elements \( v_{Ka} \) are the basis elements of \( \mathfrak{r} \) and the elements \( z_{ab} = -z_{ba} \) are the basis elements of \( \mathfrak{z} \). The only nonzero commutation relations are:

\[ [S_{IJ}, S_{KL}] = \eta_{IK} S_{JL} - \eta_{IL} S_{JK} + \eta_{JK} S_{IL}, \]
\[ [S_{IJ}, v_{Ka}] = \eta_{JK} v_{Ja} - \delta_{IK} v_{Ja}, \]
\[ [v_{Ja}, v_{Jb}] = \eta_{IJ} Z_{ab}. \]  (4.6.5)

The numbers \( \eta_{IJ} \) are the elements of the metric \( \eta_{p,q} \). The commutation relations 4.6.5 can also be taken as the definition of the Lie algebra \( \mathfrak{cso}(p, q, r) \).

4.6.2 Gaugings with CSO-algebras

Having outlined the construction of CSO-algebras, we now analyze their gaugings. We first work out the constraint 4.3.2 and investigate how the \( SU(1,1) \)-angles can be chosen.

As explained in section 4.5.1, the constraint 4.3.2 implies the existence of an invariant metric \( \Omega \) on the gauge algebra. Hence for a CSO-type algebra to be a gauge algebra it has to admit an invariant metric. It turns out that demanding
invariance and nondegeneracy of a bilinear form $\Omega$ on $\mathfrak{so}(p,q,r)$ is in all but a few cases impossible. We state the result in a theorem:

**Theorem 4.6.1.** The Lie algebra $\mathfrak{so}(p,q,r)$ with $r > 0$ admits an invariant none-degenerate symmetric bilinear form (i.e. an invariant metric) only if (1) $p + q + r = 2$ or (2) $p + q + r = 4$.

The proof of this theorem is postponed to section 4.8. In the proof we have listed the most general invariant metrics up to a multiplicative factor for each of the Lie algebras $\mathfrak{so}(p,q,r)$ with $p + q + r$.

We now turn to the $SU(1,1)$-angles. As in section 4.5.1 we investigate the gauge orbits. For the algebras $\mathfrak{so}(2,0,2)$, $\mathfrak{so}(1,1,2)$ the gauge orbit of $\mathfrak{s}$, which is one-dimensional, is $\mathfrak{s} \oplus \mathfrak{r}$ and the gauge-orbit of $\mathfrak{r}$ is $\mathfrak{r} \oplus \mathfrak{j}$. For the algebras $\mathfrak{so}(3,0,1)$ and $\mathfrak{so}(2,1,1)$ the gauge orbit of every element of $\mathfrak{s}$ is the whole Lie algebra. Finally, for $\mathfrak{so}(1,0,3)$ the gauge orbit of each element $\mathfrak{r}$ is contained in $\mathfrak{r} \oplus \mathfrak{j}$ and all gauge orbits overlap. Hence for all CSO-type algebras under consideration the $SU(1,1)$-angles have to be constant over the whole Lie algebra.

Analogous to the semisimple gaugings, we need at least two factors to have an extremum in the $SU(1,1)$-sector.

To embed a Lie algebra $\mathfrak{g}$ into the gauge algebra $\mathfrak{h}$, the metric $\Omega$ has to be diagonalized and brought into a form with eigenvalues $\pm \lambda$ and the Lie algebra is split into the eigenspaces of $\Omega$: $\mathfrak{g} = \mathfrak{g}_\lambda \oplus \mathfrak{g}_{-\lambda}$. There are two inequivalent embeddings: either $\mathfrak{g}_\pm \subset \mathfrak{h}_\pm$ or $\mathfrak{g}_\pm \subset \mathfrak{h}_\mp$. In the first case we denote the embedding of $\mathfrak{g}$ into the gauge algebra by $\mathfrak{g}_-$ and in the second case we write $\mathfrak{g}_+$. Recall that the subspace $\mathfrak{h}_\pm$ is the subspace of the gauge algebra on which the metric $\eta_{RS}$ is $\pm 1$.

In the basis of the Lie algebra where the invariant metric is diagonal with eigenvalues $\pm \lambda$, the structure constants are calculated, which are then used to evaluate the potential and its derivatives at $Z_0$. We demonstrate this procedure by an example below.

### 4.6.3 Analysis of $\mathfrak{so}(3,0,1) \oplus \mathfrak{so}(3,0,1)$-Gaugings

The invariant metric and commutation relations of $\mathfrak{so}(3,0,1)$ is given by equations 4.8.13 and 4.8.12 in section 4.8. We use the same notation as in section 4.8. The metric 4.8.13 has eigenvalues $\lambda_{\pm}(a) = \frac{1}{2}(a \pm \sqrt{a^2 + 4})$, each with multiplicity three. The eigenvectors with eigenvalues $\lambda_{\pm}$ are $\lambda_{\pm} t_i + v_i$. The eigenvectors with eigenvalues $\lambda_{-}$ are $\lambda_{-} t_i + v_i$.

We define

$$T_i^+ = \lambda_{+} t_i + v_i, \quad T_i^- = \frac{\lambda_{+}}{\lambda_{-}}(\lambda_{+} t_i + v_i).$$

(4.6.6)
from which we find $\Omega(T_{i}^{\pm}, T_{j}^{\pm}) = 0$ so $\Omega$ is in the required form. If we use the indices $1 \leq a, b, \ldots \leq 6$ for the generators $T_{i}^{\pm}$, $T_{1}^{\pm}, T_{2}^{\pm}$ and the indices $1 \leq \alpha, \beta, \ldots \leq 3$ for the generators $T_{1}^{0}, T_{2}^{0}, T_{3}^{0}$, we find after a suitable rescaling the structure constants:

$$
\begin{align*}
    f_{\alpha \beta}^{\gamma} &= (\lambda_{+}^{2} + 2)\epsilon_{\alpha \beta \gamma}, &
    f_{\alpha \beta}^{\gamma} &= \epsilon_{\alpha \beta \gamma}, \\
    f_{\alpha \beta \gamma} &= \lambda_{+}^{2}\epsilon_{\alpha \beta \gamma}, &
    f_{\alpha \beta \gamma} &= -(2\lambda_{+}^{2} + 1)\epsilon_{\alpha \beta \gamma},
\end{align*}
$$

(4.6.7)

where $\epsilon_{xyz}$ is the three-dimensional totally antisymmetric alternating symbol.

The embedding of the generators $T_{i}^{\pm}$ into the gauge algebra, with generators $T_{i}$, can be done as follows. For the gauging $CSO(3, 0, 1)_{-} \otimes CSO(3, 0, 1)_{-}$ we take $T_{i}^{-} = T_{i}$ and $T_{i}^{+} = T_{i+6}$ with for the first subgroup and $T_{i}^{-} = T_{3+i}$ and $T_{i}^{+} = T_{i+9}$ with $i = 1, 2, 3$ for the second subgroup. For the gauging $CSO(3, 0, 1)_{+} \otimes CSO(3, 0, 1)_{+}$ we take $T_{i}^{+} = T_{i}$ and $T_{i}^{-} = T_{i+6}$ for the first subgroup and $T_{i}^{+} = T_{i}$ and $T_{i}^{-} = T_{i+9}$ for the second subgroup. Finally, for the gauging $CSO(3, 0, 1)_{+} \otimes CSO(3, 0, 1)_{-}$ we take $T_{i}^{+} = T_{i}$ and $T_{i}^{-} = T_{i+6}$ for the $CSO(3, 0, 1)_{+}$ subgroup and $T_{i}^{+} = T_{i}$ and $T_{i}^{-} = T_{i+9}$ for the $CSO(3, 0, 1)_{-}$ subgroup. One checks that with the chosen embeddings the tensors $f_{RS}^{T} = f_{RST}$ are totally antisymmetric in $R$, $S$, and $U$. Hence the structure constants satisfy 4.3.2. For example, for the gauging $CSO(3, 0, 1)_{+} \otimes CSO(3, 0, 1)_{+}$ one finds the tensors:

$$
\begin{align*}
    f_{ABC}^{(1)} &= -\delta_{+}(a_{1})^{2} + 2)\delta_{ABC}^{123}, &
    f_{ABC}^{(2)} &= -(\lambda_{+}(a_{2})^{2} + 2)\delta_{ABC}^{456}, \\
    f_{AB}^{(1)} &= \lambda_{+}(a_{1})^{2}\delta_{AB}^{123}, &
    f_{AB}^{(2)} &= \lambda_{+}(a_{2})^{2}\delta_{AB}^{456}, \\
    f_{IJ}^{(1)} &= \delta_{(A+6)IJ}, &
    f_{IJ}^{(2)} &= \delta_{(A+6)IJ}^{10,11,12}, \\
    f_{IJK}^{(1)} &= -(2\lambda_{+}(a_{1})^{2} + 6)\delta_{IJK}^{123}, &
    f_{IJK}^{(2)} &= -(2\lambda_{+}(a_{2})^{2} + 1)\delta_{IJK}^{456},
\end{align*}
$$

(4.6.8)

where $1 \leq A, B, \ldots \leq 6$ are the indices for $h_{-}$ and $7 \leq I, J, \ldots \leq 12$ are the indices for $h_{+}$ and $\delta_{ABC}^{123}$ is the totally antisymmetric Kronecker delta, see appendix A.

The value of $V(Z_{0})$ is the same for both $CSO(3, 0, 1)_{+}$ and $CSO(3, 0, 1)_{-}$ and given by:

$$
V(Z_{0})_{ij} = -\frac{1}{2}\delta_{ij}(a_{1}^{2} + 6)\lambda_{+}(a_{1})^{2}.
$$

(4.6.9)

From 4.6.9 follows that for the gaugings $CSO(3, 0, 1) \otimes CSO(3, 0, 1)$ we have $C_{-}(Z_{0}) < 0$ and $\Delta(Z_{0}) > 0$. Hence an extremum in the $SU(1, 1)$-sector exists and is a maximum, that is, unstable.
marized as:
\[
\frac{\partial V}{\partial P_{ab}}(Z_0) = \frac{2}{\sqrt{\Delta(Z_0)}} \sum_{ij} \frac{\partial V_{ij}}{\partial P_{ab}}(Z_0) V(Z_0)_{jj} a_{ij}^2 - \sum_{ij} a_{ij} \frac{\partial W_{ij}}{\partial P_{ab}}(Z_0),
\]
\[
\frac{\partial V_i}{\partial P_{ab}}(Z_0) = \sum_{BJ} f_{(a+6)BJ}^i f_{BJ}^i,
\]
\[
\frac{\partial W_{ij}}{\partial P_{ab}}(Z_0) = \frac{1}{12} \epsilon^{bCDEF} \left[ f_{(a+6)BC}^i f_{DEF}^j - (i \leftrightarrow j) \right],
\]
where \(a_{ij} = g_i g_j \sin(\alpha_i - \alpha_j)\). Applying the formulas 4.6.10 we find that not all derivatives of the potential vanish at \(Z_0\) unless \(a_{ij} = 0\). Hence the point \(Z_0\) is not an extremum with respect to the \(SO(6,6)\)-scalars when an extremum exists for the \(SU(1,1)\)-scalars.

From the above we can draw conclusions. The free parameters in the invariant metric give rise to inequivalent embeddings and different potentials, potentially with different properties.

In reference [E] the question whether other \(CSO\)-gauging gives rise to a stable de Sitter vacuum is treated in more generality.

### 4.7 Extensions

The search for stable de Sitter vacua as presented in the previous section can be extended in a few directions. We briefly comment on the possible extensions.

The number of matter multiplets can be chosen to be different. From the dimensional reduction of Heterotic supergravity over a torus one obtains a theory with 28 abelian vector fields, since only the abelian subalgebra of the Yang–Mills gauge group survives. Therefore it seems natural to take 22 matter multiplets. This extension allows for gauging other semisimple groups, for example \(G_2\)-gaugings.

The analysis of section 4.5 ignored the existence of a two-parameter family of invariant metrics on \(so(1,3)\). The two parameters can be captured in an overall scaling parameter and an angle \(\theta\). The overall scaling is unimportant, while the angle \(\theta\) gives physical effects. The value of \(C_-(Z_0)\) and \(\Delta(Z_0)\) can be tuned with the \(\theta\)'s of all \(so(1,3)\) subalgebras of the gauge algebra but even whether a potential has an extremum in \(Z_0\) depends on \(\theta\). It needs to be investigated what all consequences are of the angle \(\theta\).

The analysis of section 4.6 is incomplete since not all gaugings using \(CSO\)-gaugings are investigated. Performing a group manifold reduction of Heterotic supergravity one obtains a gauged supergravity with a gauge group that is not semisimple [112]; from a reduction over an \(SU(2) \times SU(2)\)-manifold the gauge group is a \(CSO\)-group[C]. This motivates the study of \(CSO\)-gaugings, which is performed in [E].
In this line of thinking, one can even try using other nonsemisimple groups than the \( CSO \)-groups. The nonsemisimple groups are not classified, which impedes a systematic analysis. However, the nonsemisimple Lie algebras have a simple structure; due to a theorem of Levi (see e.g. [76]) every Lie algebra \( \mathfrak{g} \) is the direct vector space sum of a semisimple subalgebra \( \mathfrak{l} \) and a solvable ideal \( \mathfrak{s} \). It follows that the adjoint action of \( \mathfrak{l} \) on \( \mathfrak{s} \) is a representation of the semisimple Lie algebra \( \mathfrak{l} \). The representations of semisimple Lie algebras are direct sums of irreducible representation, which are classified for the semisimple Lie algebras. This suggests that fixing a number \( N \), one can construct using representation theory of semisimple Lie algebras all nonsemisimple Lie algebras with dimension up to \( N \). The drawback of this programme is that the number of possibilities grows rapidly with \( N \). The nonsemisimple gaugings can provide interesting phenomena, but a complete discussion seems not feasible.

An extension in another direction is given by the recent developments following from the work of de Wit, Samtleben and Trigiante [175–177]. They found a scheme that enables a more systematic treatment of maximal gauged supergravities, on which the work of Schön and Weidner on gauged \( \mathcal{N} = 4 \) supergravity is based. The idea can be explained as follows. In constructing a gauged supergravity one first finds the ungauged version of the theory, writes down the action and symmetry-variations of the fields and tries to gauge a subgroup of the global-symmetry group, to get local interactions. However, one generality is lost due to the writing down of the action: one needs to choose which gauge fields are electric and which are magnetic. In five dimensions the situation is clear, since any massless two-form can be traded for a massless gauge vector by dualization. Hence by choosing a symplectic gauge one looses generality. This loss can be restored by first introducing the local interaction by gauging and afterwards choosing a symplectic gauge. One therefore introduces for every gauge field a magnetic dual and auxiliary tensor fields and then gauges a subgroup of the global-symmetry group. The equations of motion of the auxiliary tensors and extra gauge fields can be solved for to achieve a gauged supergravity theory.

For \( \mathcal{N} = 4 \) supergravity this programme has been worked out by Schön and Weidner [146] and the most general gauging is then described by parameters \( f_{\alpha KLM} \) and \( \xi_{\alpha M} \), where \( \alpha = 1, 2 \) and \( 1 \leq K, L, M \leq 6 + n \), which have to satisfy a set of linear and quadratic equations. The gaugings in this thesis correspond to the gaugings with \( \xi_{\alpha M} = 0 \). The gaugings with \( \xi_{\alpha M} \neq 0 \) are physically inequivalent to those with \( \xi_{\alpha M} = 0 \). Unfortunately, both the authors of [146] and this thesis have not yet managed to solve the equations for \( \xi_{\alpha M} \) and \( f_{\alpha KLM} \) if \( \xi_{\alpha M} \neq 0 \) in the general case.
4.8 Proof of Theorem 4.6.1

The proof consists of two parts. In the first part we prove for all but the CSO-algebras listed in 4.6.1 that no invariant metric exists. We do this by assuming a bilinear form $\Omega$ is invariant and then prove it is degenerate. In the second part we give for CSO-algebras listed in 4.6.1 the invariant metrics.

The first part uses the concepts of isotropic subspaces and Witt-indices. For a bilinear symmetric form $B$ on a real vector space $V$, an isotropic subspace is a subspace $W$ of $V$ on which $B$ vanishes. A maximal isotropic subspace is an isotropic subspace with a maximal dimension; any other subspace with a larger dimension is not isotropic. One can show that all isotropic subspaces are related by a nonsingular linear transformation; one therefore speaks of the maximal isotropic subspace. The dimension of the maximal isotropic subspace is the Witt-index of the pair $(B, V)$ and is denoted $m_W$.

If $B$ is nondegenerate and the dimension of $V$ is $n$, one can always choose a basis in which $B$ has the matrix form

$$
B = \begin{pmatrix}
1_{p \times p} & 0 \\
0 & 0 & 1_{r \times r} \\
0 & 1_{r \times r} & 0
\end{pmatrix}, \text{ for } p, r \text{ with } p + 2r = n. \tag{4.8.1}
$$

This clearly shows that the Witt-index is $r$. Hence we have the inequality: $m_W \leq [n/2]$.

If the center $z$ is nonzero we have $[v, v] = z$, that is, for every $z \in \mathfrak{z}$ there are $v, w \in \mathfrak{r}$ such that $[v, w] = z$. Hence if $z, z'$, with $z = [v, w]$ and $v, w \in \mathfrak{r}$, we have $\Omega(z, z') = \Omega([v, w], z') = \Omega(v, [v, w]) = 0$ and hence the center $\mathfrak{z}$ is contained in the maximal isotropic subspace. Hence if the dimension of $\mathfrak{z}$ exceeds half the dimension of the Lie algebra, any invariant symmetric bilinear form is necessarily degenerate.

**Part I**

We split part I in six different cases. For every case we assume an invariant symmetric bilinear form $\Omega$ exists and prove degeneracy. We use the same decomposition as in section 4.6.1, $\mathfrak{g} = s \oplus \mathfrak{r} \oplus \mathfrak{z}$, with $\mathfrak{g}$ a CSO-type Lie algebra, and the commutation relations 4.6.5, which we for convenience list again:

$$
\begin{align*}
[S_{IJ}, S_{KL}] &= \eta_{JK}S_{IL} - \eta_{IK}S_{JL} - \eta_{JL}S_{IK} + \eta_{IL}S_{JK} \\
[S_{IJ}, v_{Ka}] &= \eta_{JK}v_{Ia} - \delta_{IK}v_Ja \\
[v_{Ia}, v_{Jb}] &= \eta_{IJ}Z_{ab}.
\end{align*} \tag{4.8.2}
$$

CSO$(p, q, r)$ with $p + q > 2$ and $r > 1$
We have \([s, s] = s, [r, r] = r\) and \([s, r] = r\). We prove that \(\mathfrak{z}\) is perpendicular to the whole algebra with respect to \(\Omega\), which implies that \(\Omega\) is degenerate.

The center \(\mathfrak{z}\) is perpendicular to itself since it is nonzero and thus defines an isotropic subspace. For every \(v \in \mathfrak{r}\) there are \(j \in \mathfrak{s}\) and \(w \in \mathfrak{r}\) such that \([j, w] = v\). Hence for such \(v\) and \(z \in \mathfrak{z}\) we have \(\Omega(v, z) = \Omega([j, w], z) = \Omega(j, [w, z]) = 0\) and \(\Omega\) is zero on \(\mathfrak{z} \times \mathfrak{r}\). Since \(\mathfrak{s}\) is semisimple a similar argument shows that \(\Omega\) is zero on \(\mathfrak{z} \times \mathfrak{s}\) and then \(\mathfrak{z}\) is orthogonal to the whole Lie algebra with respect to \(\Omega\).

\[\text{cso}(p, q, r) \text{ with } p + q = 1 \text{ and } r > 3\]

We have \(s = 0\) and \(\dim \mathfrak{r} = r\) and \(\dim \mathfrak{s} = r(r-1)/2\). The dimension of the center, which is contained in the maximal isotropic subspace, becomes too large for \(\Omega\) to be nondegenerate if \(r(r-1)/2 > r(r+1)/4\). It follows that if \(r > 3\) there is no invariant metric.

\[\text{cso}(p, q, r) \text{ with } p + q = 1 \text{ and } r = 2\]

From the commutation relations 4.8.2 we see that we can choose a basis \(e, f, z\) such that the only nonzero commutator is \([e, f] = z\). We have \(\Omega(z, z) = 0\), but also \(\Omega(e, z) = \Omega(e, [e, f]) = \Omega(e, e, f) = 0\). Similarly \(\Omega(z, f) = 0\) and thus \(z\) is perpendicular to the whole algebra and \(\Omega\) is degenerate.

\[\text{cso}(p, q, r) \text{ with } p + q = 2 \text{ and } r = 1\]

The Lie algebras \(\text{cso}(1, 1, 1)\) and \(\text{cso}(2, 0, 1)\) have zero center and hence \([\mathfrak{r}, \mathfrak{r}] = 0\). For every \(x \in \mathfrak{r}\) there are \(y \in \mathfrak{r}\) and \(A \in \mathfrak{s}\) such that \(x = [A, y]\). Therefore we have for such \(x, y, A\) and \(v \in \mathfrak{r}\) : \(\Omega(x, v) = \Omega([A, y], v) = \Omega(A, [y, v]) = 0\). Thus \(\mathfrak{r}\) is an isotropic subspace of dimension 2, whereas the dimension of the Lie algebra is 3.

\[\text{cso}(p, q, r) \text{ with } p + q = 2 \text{ and } r > 2\]

We choose a basis \(\{j, e_a, f_a, z_{ab}\}\), where \(j \in \mathfrak{s}\), \(e_a, f_a \in \mathfrak{r}\) and \(z_{ab} = -z_{ba} \in \mathfrak{z}\) and \(1 \leq a, b \leq r\). In terms of the basis elements in 4.8.2 we have \(j = J_{12}, e_a = v_{1a}, f = v_{2a}\). The only nonzero commutation relations are

\[
[j, e_a] = f_a, \quad [j, f_a] = \sigma e_a, \quad [f_a, f_b] = \sigma z_{ab} \quad [e_a, e_b] = z_{ab}, \quad (4.8.3)
\]

where \(\sigma = +1\) for \(\text{cso}(1, 1, r)\) and \(\sigma = -1\) for \(\text{cso}(2, 0, r)\).

From the commutation relations 4.8.3 one deduces that the subspace spanned by the elements \(e_a\) and \(z_{ab}\) defines an isotropic subspace of dimension \(r(r+1)/2\). The dimension of this isotropic subspace exceeds half the dimension of the Lie algebra if \(r > 2\).

\[\text{cso}(p, q, r) \text{ with } p + q > 3 \text{ and } r = 1\]
The Lie algebras in this class have zero center and hence $[\mathfrak{r}, \mathfrak{r}] = 0$. We have $\mathfrak{r} = [\mathfrak{s}, \mathfrak{r}]$, $\mathfrak{s} = [\mathfrak{s}, \mathfrak{s}]$ and $\mathfrak{s}$ is semisimple. It follows that $\Omega$ is zero on $\mathfrak{r} \times \mathfrak{r}$ and $\Omega$ coincides with the Cartan–Killing metric of $\mathfrak{s}$ on $\mathfrak{s} \times \mathfrak{s}$. Hence we are interested in $\Omega$ on $\mathfrak{r} \times \mathfrak{s}$.

From 4.8.2 we see that we can choose a basis $\{S_{IJ}, v_I\}$, where $1 \leq I, J \leq p + q$, and the only nonzero commutation relations are:

\[
[S_{IJ}, S_{KL}] = \eta_{JK} S_{IL} - \eta_{IK} S_{JL} - \eta_{JL} S_{IK} + \eta_{IL} S_{JK},
\]

\[
[S_{IJ}, v_K] = \eta_{JK} v_I - \delta_{IK} v_J.
\]

(4.8.4)

We define $\Omega_{IJK} = \Omega(v_I, S_{JK}) = -\Omega_{IKJ}$. Invariance requires $\Omega([S_{IJ}, v_K], S_{LM}) = -\Omega(v_K, [S_{IJ}, S_{LM}])$, from which we obtain:

\[
\eta_{JK} \Omega_{ILM} - \eta_{IK} \Omega_{KLM} = -\eta_{IL} \Omega_{JKM} + \eta_{IL} \Omega_{KJM} + \eta_{JM} \Omega_{KIL}.
\]

(4.8.5)

Contracting equation 4.8.5 with $\eta^{IL}$ we obtain:

\[
\eta^{IL} \Omega_{IJK} = 0, \forall K.
\]

(4.8.6)

Contracting 4.8.5 with $\eta^{IK}$ and using 4.8.6 we find:

\[
-(p + q) \Omega_{IJK} = \Omega_{IJK} + \Omega_{KIJ} + \Omega_{JIK}.
\]

(4.8.7)

Writing out 4.8.7 three times with the indices cyclically permuted and adding the three expressions we find the result:

\[
(p + q - 3) (\Omega_{IJK} + \Omega_{JKI} + \Omega_{KIJ}) = 0.
\]

(4.8.8)

Since we assumed $p + q > 3$ the cyclic sum of $\Omega_{IJK}$ has to vanish.

Using the relation $[S_{IJ}, v^J] = v_I$, where no sum is taken over the repeated index $J$ and where $v^J = \eta^{JK} v_K$, and requiring $\Omega([S_{IJ}, v^J], S_{KL}) = -\Omega(v^J, [S_{IJ}, S_{KL}])$ we obtain:

\[
\Omega_{IJK} + \Omega_{JIK} + \Omega_{KJI} = 0.
\]

(4.8.9)

Combining 4.8.9 and the vanishing of the cyclic sum we see that $\Omega_{IJK} = 0$. Hence the subspace $\mathfrak{r}$ is orthogonal to the whole Lie algebra with respect to $\Omega$ and $\Omega$ is degenerate. This concludes part I.

**Part II**

We now give for the Lie algebras listed in 4.6.1 the most general invariant metric up to a multiplicative constant.

The Lie algebra $\mathfrak{so}(1, 0, 1)$ is abelian and hence any metric is invariant.
For the Lie algebras $\mathfrak{cso}(2,0,2)$ and $\mathfrak{cso}(1,1,2)$ we use the ordered basis $\beta = \{j, e_1, e_2, f_1, f_2, z\}$ with the only nonzero commutation relations

$$[j, e_a] = f_a, \quad [j, f_a] = \sigma e_a, \quad [f_a, f_b] = \sigma z, \quad [e_a, e_b] = z,$$  

(4.8.10)

where $\sigma = +1$ for $\mathfrak{cso}(1,1,2)$ and $\sigma = -1$ for $\mathfrak{cso}(2,0,2)$.

In the basis $\beta$ the invariant metric can be written in matrix form as:

$$\Omega = \begin{pmatrix} a & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad a \in \mathbb{R},$$  

(4.8.11)

for both $\mathfrak{cso}(1,1,2)$ and $\mathfrak{cso}(2,0,2)$. The eigenvalues are $-1, -1, +1, +1, \frac{1}{2}(a+\sqrt{a^2+4})$, $\frac{1}{2}(a-\sqrt{a^2+4})$ and the signature is $++-++-$. 

For the Lie algebras $\mathfrak{cso}(2,1,1)$ and $\mathfrak{cso}(3,0,1)$ we use the ordered basis $\beta = \{t_1, t_2, t_3, v_1, v_2, v_3\}$ such that the commutation relations are

$$[t_i, t_j] = \epsilon_{ijk} \eta^{kl} t_l, \quad [t_i, v_j] = \epsilon_{ijk} \eta^{kl} v_l, \quad [v_i, v_j] = 0,$$  

(4.8.12)

where $\epsilon_{ijk}$ is the three-dimensional alternating symbol and $\eta^{ij}$ is the inverse of $\eta_{1,2}$ for $\mathfrak{cso}(2,1,1)$ and the inverse of $\eta_{3,0}$ for $\mathfrak{cso}(3,0,1)$. The matrix $\eta_{p,q}$ is diagonal with $p$ eigenvalues $+1$ and $q$ eigenvalues $-1$.

With respect to the ordered basis $\beta$ the invariant metric is given by

$$\Omega = \begin{pmatrix} a \eta & \eta \\ \eta & 0 \end{pmatrix},$$  

(4.8.13)

where each entry is a $3 \times 3$-matrix. The eigenvalues are $\lambda_{\pm} = \frac{1}{2}(a \pm \sqrt{a^2+4})$, both with multiplicity three and the signature is $-++-++$. 

For the Lie algebra $\mathfrak{cso}(1,0,3)$ we use the ordered basis $\beta = \{v_1, v_2, v_3, z_1, z_2, z_3\}$ such that the commutation relations are

$$[v_i, v_j] = \frac{1}{2} \epsilon_{ijk} z_k, \quad [v_i, z_j] = [z_i, z_j] = 0,$$  

(4.8.14)

where a summation is understood for every repeated index. The invariant metric is given in matrix form with respect to the basis $\beta$ by:

$$\Omega = \begin{pmatrix} A_{3 \times 3} & \mathbb{I}_{3 \times 3} \\ \mathbb{I}_{3 \times 3} & 0 \end{pmatrix},$$  

(4.8.15)
where $A_{3 \times 3}$ is an undetermined $3 \times 3$-matrix. Since $\det \Omega = -1$ there are no null vectors. We use an LU-decomposition to find

$$
\det (\Omega - \lambda I_{6 \times 6}) = -\lambda^3 \det \left( A - (\lambda - \frac{1}{\lambda}) I \right).
$$

Hence if $\mu_1, \mu_2, \mu_3$ are the eigenvalues of $A$, then $\lambda_i = \frac{1}{2} \left( \mu_i \pm \sqrt{\mu_i^2 + 4} \right)$, are the eigenvalues of $\Omega$. Hence the signature is $+++-$. 
