Foam Dynamics: on Matter and Motions around the Cosmic Foam

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ABSTRACT — The Delaunay Tessellation Field Estimator (DTFE) is a method for reconstructing density or intensity fields from a discrete set of irregularly distributed points sampling this field. In this chapter we show that it can also be used to reconstruct other continuous fields which are sampled at the locations of these points. The main advantage of the DTFE is that the fields are reconstructed locally without the application of an artificial or user-dependent smoothing procedure, resulting in an optimal resolution and the suppression of shot-noise effects. The reconstructed fields are volume-covering and allow for a direct comparison with theoretical predictions. In this chapter we focus on the simultaneous reconstruction of the density and velocity fields corresponding to cosmological N-body simulations. The resulting fields closely adhere to the continuity equation. The DTFE reconstruction results in realistic void density and velocity profiles. We present analytical models for voids in cosmologies with and without a cosmological constant and show that voids can be used to constrain cosmological parameters. We also discuss the reconstruction of the density and velocity fields in filamentary environments and show that the DTFE is able of capturing both the almost caustic behaviour of the density and velocity field as well as the large scale infalling motions towards and around these objects. The DTFE reconstruction of the cosmic velocity field automatically provides a reconstruction of the velocity divergence, shear and vorticity fields, which we shortly discuss. Our results show that the DTFE represents a major step forward for studies of the dynamics and evolution of the cosmic foam.

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6.1 Introduction

The peculiar velocity field provides important information on the formation and evolution of structures in our universe. On cosmological scales it supplies direct information on the dynamics of structure formation, while on smaller scales it provides information on the local environment. In general, the peculiar velocity field is sampled at a discrete set of points. These points correspond to the locations of galaxies in the case of observational data or to particle positions in the case of N-body simulations. In both these cases the sampling density is related to the cosmic density field, which is highly non-uniform. Large scale galaxy redshift surveys have shown that galaxies are not randomly distributed in the universe but reside in an intricate network, the cosmic web (Bond et al. 1996, van de Weygaert 2002). Dense and compact galaxy clusters containing up to thousands of galaxies reside at the nodes of this network. The Virgo and Coma clusters form the most nearby examples. The clusters are connected by filamentary and wall-like structures of intermediate densities, usually consisting of a large number of galaxy groups, each containing tens of galaxies. The most outstanding of these structures are usually identified with superclusters. The Local Supercluster, with the Local Group amongst its members, forms the most nearby example. The extended and elongated superclusters surround large and empty regions, voids. The result is a foamy network which is pervading throughout the whole observable universe.

The complex nature of the large scale matter distribution may be understood in terms of the standard framework of structure formation, gravitational instability theory. In this theory physical objects originate from small primordial fluctuations in the density field. Gravity acts as the driver of structure formation by exerting a larger than average pull towards overdense regions and a smaller than average pull towards underdense regions. The density of overdense regions continues to grow by attracting more and more matter, causing an ever stronger gravitational pull on its surroundings. Once these regions surpass a given density threshold they collapse and virialize, attaining a high density excess. Conversely, regions of low density expand and empty with time, gradually approaching a density contrast of $\delta \approx -1$.

The complex cosmic matter distribution poses a formidable challenge for studies of the cosmic velocity field. The main reason for this is that the cosmic velocity field is sampled by galaxies which predominantly reside in high density environments, such as filaments and clusters. Whereas such regions are compact, the extended and low density voids contain very few galaxies and are very poorly sampled. As a consequence, the peculiar velocity field is sampled in a highly non-uniform fashion. An additional complication concerns the fact that peculiar velocity fields are measured at galaxy positions. Due to the filtering which is applied in reconstruction procedures reconstructed fields are usually mass-weighted, while most analytical relations concern volume-weighted quantities. The a priori unknown bias between dark matter and baryonic matter makes the comparison between measured and theoretical values even more difficult.

Given these issues the central problem becomes how to reconstruct a continuous volume-covering field from a discrete and highly non-uniform sampling without losing relevant information. This is closely related to the reconstruction of a continuous and volume-covering density field from a discrete set of particles, which has been discussed in detail in the previous chapters. The difference is that in the case of the cosmic velocity field one starts with a set of velocity field samples, while in the case of a density field all one has is a set of sampling point
locations, which itself is supposed to represent a fair sample of the underlying density field.

In this thesis we have described the Delaunay Tessellation Field Estimator (DTFE), a method for reconstructing density fields from a discrete set of sampling points. The DTFE forms an elaboration of the formalism first proposed by Bernardeau & van de Weygaert (1996) for the case of assessing the statistical properties of cosmic velocity fields. In the previous chapters we have shown that the main advantage of the DTFE over conventional fixed grid-based and adaptive SPH-like procedures is that it is intrinsically self-adaptive to both the density and the geometry of the point distribution and does not make use of any artificial smoothing. This makes it a superior tool for studying the large scale galaxy distribution. It attains the highest spatial resolution. It can accurately describe the full hierarchy of spatial scales and densities present in the large scale galaxy distribution. Finally, it recovers the complex cellular geometry of the cosmic web involving structures of widely varying densities and anisotropies.

The DTFE consists of three main steps. In the first step the Delaunay tessellation (Delone 1934) of the point sample is constructed. In the second step the properties of this tessellation are employed to obtain an estimate of the density at the location of the sampling points. In the final step these estimates are interpolated to the rest of space, again by making use of the Delaunay tessellation. The resulting density field is volume-covering and continuous and can be used as a starting point for a variety of subsequent analyses.

The central element of the DTFE is the Delaunay tessellation of the point sample. This is a volume-covering tiling of space into tetrahedra whose vertices are formed by four specific points in the dataset. An example of a Delaunay tessellation is shown in Fig. 6.1. The four vertices of each Delaunay tetrahedron are uniquely selected such that their circumscribing sphere does not contain any of the other datapoints. Another important property of the Delaunay tessellation is that among all possible triangulations of a given point distribution the Delaunay tetrahedra are on average objects of minimal size and elongation (Lawson 1977, Okabe et al. 2000). These properties make the network of Delaunay cells the multi-dimensional Delaunay tessellations have been abundantly applied in surface rendering applications such as geographical mapping and various computer imaging algorithms (see Okabe 2000 for a review).

Given the advantages of the DTFE in the reconstruction and analysis of cosmological density fields, as well as the promising results of using Delaunay tessellations for the interpolation of various other physical quantities, it is highly interesting to combine these techniques for a simultaneous reconstruction of the cosmic density and velocity field. In this chapter we
explicitly focus on those environments for which conventional analysis tools fail: low density voids and anisotropic filamentary and wall-like objects of intermediate density.

The outline of this chapter is as follows. In section 6.2 we discuss the reconstruction of a cosmic velocity field by means of a conventional interpolation technique. In section 6.3 we describe the DTFE velocity field reconstruction procedure. In section 6.4 we apply this formalism to a GIF N-body simulation of cosmic structure formation. We discuss the simultaneous reconstruction of the corresponding cosmic density and velocity field and assess the quality of the reconstructions. In section 6.5 we focus on voids, while in section 6.6 we describe filamentary and wall-like environments. In section 6.7 we discuss the reconstruction of the velocity shear and vorticity fields. Finally, in section 6.8 we summarize and discuss our results.

6.2 Case study: a cosmic velocity field

Several methods have been proposed for reconstructing a volume-covering field from a discrete set of data-points. We can divide these methods into three main groups. The first is that of the grid-based schemes, also known as Eulerian, in which the desired field estimates are confined to a set of locations on a pre-defined fixed grid, which in principle does not depend on the point distribution itself (e.g. Hockney & Eastwood 1988; Efstathiou et al. 1985). In cosmology these grids are usually regular but various other options can be applied depending on the system under consideration. Their main limitation is that their spatial resolution is set by the cell-size. This limitation is quite severe in cosmology, since many relevant physical processes are acting over a range of scales, so that once a grid-size has been defined effective smoothing over the Eulerian cell-size usually erases important information on smaller scales.

The second group is formed by the Lagrangian schemes, in which the locations of the interpolation are confined to or defined by the point distribution itself. Of this type, the `SPH-type’ schemes are the best known in astronomy. The latter refers to the abundantly used smooth particle hydrodynamics technique (Lucy 1977, Gingold & Monaghan 1977) to follow the hydrodynamical evolution of astrophysical systems. The main difference between Lagrangian and Eulerian methods is the fact that the former is not restricted to a specific geometry because it does not make use of a mesh. Instead, it follows the trajectories of the displacing matter aggregating into high density regions and ideally with an unlimited spatial resolution. In practice, this is not possible and one has to make use of a ‘softening length’. This length is usually much smaller than the cell-size of an Eulerian scheme, resulting in a significantly higher spatial resolution of Lagrangian schemes. The main difficulty of this technique is that it relies on stochastic arguments which means that it yields only approximate solutions at a given spatial position.

The third class seeks to combine the virtues of Lagrangian and Eulerian methods. These were first introduced by Noh (1964) and referred to as arbitrary Lagrangian-Eulerian (ALE) methods. This technique incorporates the high resolution of the Lagrangian scheme by letting a grid move along with the system, combined with the Eulerian scheme for computing the physical state of the system within each of the resulting distorted grid-cells. Although as yet reluctantly applied in cosmology, there are some noteworthy promising efforts (e.g. Gnedin 1995, Xu 1997, Pen 1998).

For the purpose of merely providing a comparison, we restrict ourselves to a grid-based
TSC scheme as a representative velocity field reconstruction algorithm. All other Eulerian grid-based methods perform in a similar fashion. For a general revision of the performance of several grid and Lagrangian methods we refer the reader to Chapters 4 and 5.

In Fig. 6.2 the outcome of an $N$-body simulation is shown (left-hand frame) together with two TSC velocity field reconstructions (central and right-hand frames) at different resolutions. The region depicted corresponds to a $1h^{-1}$Mpc thick slice through a cosmological simulation box of size $100h^{-1}$Mpc. This box contains several common large scale structures. At the top right-hand and bottom left-hand part of the slice two large voids are present, close to the center a cluster is located and at the bottom right-hand part a filamentary structure is located. For the central frame the resolution of the TSC grid is the standard choice of on average one galaxy per grid-cell for the full galaxy sample, in the right-hand plot it is twice as large.

Clear differences can be distinguished between the two maps. The low density interpolated map assures full volume-coverage, but at the price of blurring out high density regions. Although the main trend of the flows can be distinguished, many of the features visible in the particle velocities are not recovered in the low resolution interpolated map. Even though a trace of related shearing flow is visible at the bottom right-hand part of the map, it is only marginal. The high resolution map of Fig. 6.2 (right-hand frame) illustrates the second point. This map does recover structures that the low resolution map did not. The high density regions can be better noticed in this map. The general flow resembles that visible in the discrete velocity map. Particularly noteworthy are the low density regions, devoid of any objects to base the velocity interpolation on. Here the high resolution TSC procedure is unable to recover any significant information. A possible solution to circumvent this problem is to apply a smoothing procedure in order to extrapolate information to those regions where no information is available. However, the cost of such an operation would be to lose the high sensitivity of the map, degrading it to an equivalent of the low resolution TSC reconstruction.

From the map we may infer that: (1) At high density regions the interpolation smoothes away interesting velocity information because one effectively averages over the velocities within a grid-cell. (2) Highly crucial with respect to the ability to reproduce the one-point velocity distribution function, at low density regions the sampling is so poor and sparse that
there is no strong and significant signal in these regions to be recovered. (3) The field estimates are beset by shot-noise effects. Note that if one has less than one or two particles within the kernel, estimates of the local bulk flow and/or velocity dispersion would not be possible. (4) Cosmic velocity fields are much more quiet than density fields. Hence, features are intrinsically less prominent and therefore require a better sampling to obtain a sufficient contrast.

6.3 DTFE reconstruction of velocity fields

The DTFE was introduced for rendering continuous and fully volume-covering density fields from a discrete set of points sampling this field. It adapts to the density and geometry of the point distribution and does not make use of artificial smoothing procedures. It is an extension of the pioneering work by Bernardeau & van de Weygaert (1996) who used the Delaunay tessellation of a point set as a natural and self-adaptive interpolation frame for recovering the continuous velocity field sampled at those points. The DTFE generalized this formalism to the recovery of a density or intensity field on the assumption that it is fairly sampled by the spatial point distribution.

Mathematically the DTFE interpolation procedure for a field which is sampled at a number of discrete locations can be formulated as follows. Let the value of a field $f$ be known at a set of $N$ irregularly distributed locations $x_i$. Let the corresponding Delaunay tessellation consist of $N_T$ tetrahedra. Let point $x$ lie inside tetrahedron $j$. Let this tetrahedron consist of the 4 vertices $x_0$, $x_1$, $x_2$ and $x_3$. Then the value of the function $f$ at location $x$ is defined as

$$\widehat{f}(x) = f(x_0) + \left[ \nabla f \right]_j \cdot (x - x_0).$$

Here the gradient of $f$ in tetrahedron $j$ is defined by the following set of 3 equations:

$$f(x_i) = f(x_0) + \left[ \nabla f \right]_j \cdot (x_i - x_0), \quad i = 1, 2, 3.$$  

In the case of the velocity field the function $f$ involves a vector field $v = (v_x, v_y, v_z)$ and Eqns 6.1 and 6.2 may be written down for each of the 3 components of the velocity field. Explicitly, the velocity field is given by

$$\widehat{v}_\alpha(x) = v_\alpha(x_0) + \left[ \nabla v_\alpha \right]_j \cdot (x - x_0), \quad \alpha = x, y, z.$$  

Here each of the components $\alpha$ of the gradient is defined by the following set of three equations:

$$v_\alpha(x_i) = v_\alpha(x_0) + \left[ \nabla v_\alpha \right]_j \cdot (x_i - x_0), \quad i = 1, 2, 3; \quad \alpha = x, y, z.$$  

By applying the DTFE interpolation formalism the reconstructed field will have the same mathematical properties as a DTFE reconstructed density field: it is volume-covering and continuous everywhere. Its gradient is constant inside Delaunay tetrahedra, and hence discontinuous at the edges of the tetrahedra. The local resolution (smoothing kernel) is set by the size of the Delaunay tetrahedra. It leads to an optimal resolution of the reconstructed field in the sense that no additional information on the field is present inside the tetrahedra.
6.3.1 Case study: DTFE reconstruction of a velocity field

The DTFE velocity field reconstruction procedure is illustrated in Fig. 6.3. The top left-hand frame represents a discretely sampled velocity field. The length of the arrows is proportional to the velocity amplitude of each object, while the arrows’ head indicates the direction of the motions. The velocity field gives the impression of a velocity flow emerging from the bottom right-hand corner of the box and spanning over the whole box.

From the locations of the sampling points the Delaunay tessellation is constructed (top right-hand frame). The computed velocity gradient for each triangle is depicted in the central frames. The gray scale in both panels is proportional to the amplitude of the determinant of the velocity gradient matrix. In the central left-hand frame a clear dark strip can be recognized, where the velocity field experiences maximal changes. The central right-hand frame represents the corresponding Delaunay surface of the gradient plot. The saddle-like shape of the surface is determined by the velocity distribution. The two maxima are located at the two extremes of the cube, while the minima form the saddle point.

After the gradient matrix has been calculated, the volume-covering velocity field can be calculated. In this case the field has been calculated on a rectangular grid of 8 grid-cells per dimension (bottom left-hand frame). For each grid-point the surrounding triangle has to be identified, indicated by the hatched triangles in the frame. The interpolation of the velocity field is linear inside each triangle, resulting in the velocity field shown in the bottom right-hand frame. This interpolated velocity field clearly displays the same characteristics as the input field: it recovers the two maxima regions located at the opposite extremes as well as the low velocity region at the bottom right-hand corner of the box. Also note that due to the linear interpolation scheme velocity information has been recovered at the top left-hand corner of the box where almost no sampling points were available.

6.3.2 Virtues and limitations

The main virtue of the DTFE over conventional reconstruction procedures is that it is intrinsically self-adaptive and does not involve artificial smoothing. Another important advantage of the DTFE procedure over conventional velocity reconstruction procedures is the fact that it produces a volume-weighted as opposed to a mass-weighted velocity field. Bernardeau & van de Weygaert (1996) have shown that this leads to critical improvements in recovering theoretical predictions. An additional advantage is that the simultaneous use of the DTFE for the reconstruction of both the density and velocity field at the same resolution allows for a coherent dynamical analysis of numerical simulations of cosmic structure formation.

The DTFE also involves a number of limitations. The first is that it assumes a linear variation inside the Delaunay tetrahedra, while in reality the cosmic density and velocity field are smoothly varying. More importantly, the DTFE assumes the field to be continuous. This implies severe limitations for the reconstruction of the velocity field in areas where the cosmic density field is non-linear and orbit crossing has occurred. In such circumstances the velocity field is not uniquely defined and the DTFE interpolation procedure is not applicable. The DTFE also involves a number of technical limitations. The first is that the sampling point distribution should not be degenerate, because in such a case the Delaunay tessellation is not uniquely defined. In practice one rarely encounters degenerate distributions. Secondly, one has to be careful about the boundary conditions. In the case where a sample is spatially
Figure 6.3 — DTFE velocity field reconstruction procedure. The top right-hand frame presents the Delaunay triangulation of the discrete particle positions of the velocity field presented in the top left-hand frame. The central frames show the velocity gradient computed for each triangle. The gray scale corresponds to the amplitude of the determinant of the gradient matrix. The right-hand frame depicts the three-dimensional representation of the gradient surface. The height of each point corresponds to its velocity amplitude. The DTFE velocity field is estimated at the grid-points indicated by the gray-colored grid of the bottom left-hand frame by assuming a linear variation of the velocity field inside each triangle. The bottom right-hand frame presents the outcome of the DTFE velocity field reconstruction procedure.
limited, the boundary points do not have natural neighbours on one side. The outermost Delaunay tetrahedra will then stretch out into infinity. In this chapter we have used periodic boundary conditions for which case this issue is not relevant.

Note that the resolution of the DTFE reconstruction procedure is dependent on the number of sampling points. Whereas variations in a reconstructed density field are directly related to variations in the local density of sampling points (we refer the reader to Chapter 3 of this thesis for a discussion of the properties of the DTFE kernel), this is not the case for the reconstructed velocity field.

6.3.3 Divergence, shear and vorticity

The DTFE velocity field reconstruction procedure involves the calculation of the nine elements of the velocity gradient (see section 6.3). From the elements of the gradient matrix it is straightforward to compute any related quantity. The DTFE estimates of the velocity divergence $\nabla \cdot \mathbf{v}$ (the trace of the velocity gradient matrix), the shear $\sigma_{ij}$ (with $i, j = x, y, z$) (the symmetric and traceless part) and the vorticity $\omega_{ij}$ (its antisymmetric part) directly follow:

$$\nabla \cdot \mathbf{v} = \left( \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right),$$

$$\sigma_{ij} = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) - \frac{1}{3} (\nabla \cdot \mathbf{v}) \delta_{ij},$$

$$\omega_{ij} = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right).$$

Here $\delta_{ij}$ represents the Kronecker delta pseudo-tensor and the vorticity $\omega_{ij}$ follows the usual definition $(\nabla \times \mathbf{v})_k = \epsilon_{ijk} \omega_{ij}$, with $\epsilon_{ijk}$ the Levi-Civita tensor.

This results in a fully volume-covering reconstruction of the divergence, shear and vorticity fields. However, notice that while the velocity field itself is continuous, the DTFE reconstruction involves a discontinuous velocity gradient. The velocity gradient is constant within each Delaunay tetrahedron and varies at the edges of the Delaunay tetrahedra. The same holds for the divergence, shear and vorticity fields.

6.4 GIF simulations: DTFE density and velocity field reconstructions

The potential of the DTFE formalism can be fully exploited for the analysis of $N$-body simulations. Here we illustrate this on the basis of a a 256$^3$ particles GIF $N$-body simulation of cosmic structure formation (Kauffmann et al. 1999). Such a simulation offers a challenging variety of objects whose characteristics are widely differing not only in density and geometry, but also with respect to their dynamical properties. The GIF simulation was performed using an adaptive P$^3$M $N$-body code in a cubic box with length $141h^{-1}\text{Mpc}$ and periodic boundary conditions were imposed. A concordance $\Lambda$CDM cosmology was assumed ($\Omega_m = 0.3, \Omega_\Lambda = 0.7, H_0 = 70\text{km/s/Mpc}$).

In the left-hand frame of Fig. 6.4 a slice of thickness $1h^{-1}\text{Mpc}$ through the simulation box is shown. The depicted area has a size of $141 \times 141h^{-1}\text{Mpc}$ and corresponds to a cosmic epoch for which the redshift $z = 3$. This redshift has been chosen to ensure that most structures are still in the linear or semi-linear phases. The right-hand frame shows a two-dimensional
slice through the three-dimensional DTFE reconstructed density field. All structural details present in the particle distribution have a corresponding counterpart in the slice through the reconstructed density field. The foam-like network of structures which the large scale matter distribution forms is clearly visible and more pronounced than in the particle distribution. It consists of such characteristic elements as extended void-like regions, compact regions of high density and filamentary structures of varying degree of anisotropy.

### 6.4.1 Velocity vector fields

In Fig. 6.5 three mutually perpendicular slices through the center of the three-dimensional DTFE reconstructed density field are shown, as well as similar slices through the corresponding DTFE reconstructed velocity field. The squares plotted on top of the velocity reconstructions denote interesting regions of which magnifications are shown in Fig. 6.6. In these figures we have plotted the reconstructed velocity field on a 40³ cubic grid. Note that we may depict the velocity field at arbitrary resolution. The magnifications are therefore real magnifications of the reconstructed field in Fig. 6.5 and not distinct reconstructions. This is in stark contrast with conventional grid-based reconstructions, for which the resolution is arbitrarily set by the user and whose properties are dependent on the adopted resolution of the reconstruction procedure.

Especially when looking at Fig. 6.6 the impressive rendering of both fields becomes apparent, revealing the tight physical relation between these two fields. The top frames of Fig. 6.6 show a proto-cluster region (Kauffmann et al. 1999). The density field reconstruction shows a rather complex configuration with multiple filamentary objects connecting to the central node located at \((X, Y) = (52, 57)\). The velocity field shows the conspicuous matter flows onto the proto-cluster.

The central frames depict an expanding void whose center almost coincides with the geometrical center of its respective box. Theoretical models of voids (Icke 1984, Sheth & van de
Figure 6.5 — DTFE density (left-hand column) and velocity (right-hand column) reconstructions through the GIF N-body simulation shown in Fig. 6.4. The squares plotted on top of the velocity reconstructions denote interesting regions of which magnifications are shown in Fig. 6.6
Figure 6.6 — Magnifications of a number of characteristic regions in the slices through the GIF $N$-body simulation shown in Fig. 6.5.
Weygaert 2004) predict that such empty regions are characterized by an inverse top-hat density profile. Their dynamics correspond to a low $\Omega_m$ universe, represented as ‘super-Hubble’ expanding bubbles with an almost constant velocity divergence. The expansion of the void is clearly recognizable in the velocity field reconstruction. Notice also that substructure is even present in the velocity field of the void. In section 6.5 we analyze the dynamics of these void regions and test how well the analytic predictions are reproduced by the DTFE reconstructions.

The bottom frames show a filamentary feature running almost parallel to the $Y$-axis along the center of its respective box. The velocity field shows that matter is flowing from its surroundings onto the filament. A clear shearing flow can be identified along the filament’s ridge. Notice however also the strong flow towards the clump of matter located at $(X, Y) = (30, 2.5)$. In section 6.6 we discuss the dynamics of filamentary structures in more detail.

A further test of the combined DTFE density and velocity reconstruction is to compare the densities and velocities directly. In Fig. 6.7 the density and velocity profiles are shown along a one-dimensional section through the simulation box. Plotted on top of the particle distribution and the slice through the reconstructed density field is a thick solid line, along which the density and velocity field as a function of distance have been reconstructed with the DTFE procedure. The resulting density and velocity profiles are plotted in the bottom frame of Fig. 6.7. Only the velocity component parallel to the direction of the cut is shown. The density scale is plotted on the left, while the velocity scale is plotted on the right ordinate.

A visual inspection of the density and velocity profiles plotted in Fig. 6.7 shows that the DTFE reconstruction procedure is capable of recovering all the structures present in the point distribution. High density peaks show up at places where the profile intersects with filamentary structures or proto-clusters visible in the particle distribution. There the velocity drops almost instantaneously (notice e.g. the peaks at about $7h^{-1}\text{Mpc}$ and $128h^{-1}\text{Mpc}$ along the cut). The drop in velocity indicates infalling motions onto these structures. Conversely, empty regions are rendered as slowly-varying regions of low density. The velocity appears to increase almost linearly across these regions (notice e.g. the void centered at about $124h^{-1}\text{Mpc}$ along the cut). This linear increase reflects the expansion of voids. The specific properties of voids and filamentary structures are analyzed in sections 6.5 and 6.6. Here we note that the DTFE reconstructed density and velocity fields are indeed tightly correlated and appear to give a realistic image of the large scale structure environment. This result has been obtained without having to invoke any subsequent filtering.

### 6.4.2 The continuity equation

The velocity divergence and the density contrast are related through the continuity equation (Peebles 1980). Within linear theory this relationship is given by

$$\nabla \cdot \mathbf{v}(x) = -H f(\Omega_m, \Omega_{\Lambda}, z) \delta(x).$$

(6.8)

Here $H = H(z)$ is the Hubble parameter, $\Omega_m$ the cosmic density parameter at the current epoch and $\Omega_{\Lambda}$ the cosmic vacuum energy density parameter at the current epoch. The function $f$ may be approximated by $f(z) \approx \Omega_m^{0.6}(z)$ (Lahav et al. 1991). In terms of the cosmological parameters at the present epoch this approximation can be written as

$$f(z) \approx \left[ \frac{\Omega_m(1 + z)^3}{\Omega_m(1 + z)^3 - (\Omega_m + \Omega_{\Lambda} - 1)(1 + z)^2 + \Omega_{\Lambda}} \right]^{0.6}$$

(6.9)
Figure 6.7 — Slice through a GIF N-body simulation (top), the DTFE reconstructed density field (center) and density and velocity profiles along the central line (bottom).
A lot of theoretical effort has been devoted to obtain accurate density-velocity divergence relations beyond the linear regime (see the review of Bernardeau et al. 2002). On the basis of perturbation theory Bernardeau (1992) derived a second-order relationship for the quasi-linear regime,

$$\frac{\nabla \cdot \mathbf{v}(x)}{H} = \frac{3}{2} f(\Omega_m, \Omega_\Lambda, z)[1 - (1 + \delta(x))^{2/3}] .$$

(6.10)

Other authors have derived even higher-order corrections (e.g. Bernardeau et al. 1999, Kudlicki et al. 2000, Bernardeau et al. 2002). As the DTFE is capable of simultaneously recovering the cosmic density and velocity fields at the same resolution, the one-to-one relationship between the velocity divergence and the density contrast provides a unique opportunity of testing the capacity and consistency of the DTFE procedure with respect to probing the dynamics in cosmic structure formation scenarios.

Fig. 6.8 gives a first impression of the strong relationship between the density and velocity divergence fields as reconstructed by the DTFE. The figure shows the velocity divergence for the slice presented in Fig. 6.4. For a better appreciation of the correspondence between high and low density regions, we have splitted the divergence field into two parts, positive and negative divergence regions. No additional smoothing has been applied to the density and velocity reconstructions. In the right-hand frame regions with negative divergence are shown. They clearly correspond to regions where infall motions are present, such as matter accretion onto clusters and filaments. In the left-hand frame regions with positive divergence are shown. These regions are expanding and may be identified with low density voids. The substructures which are visible in the density fields inside voids (see Fig. 6.4) have conspicuous counterparts in the velocity divergence field, where they present themselves as regions of positive divergence.
Fig. 6.9 shows a scatter plot of the relationship between the velocity divergence and the density contrast for the GIF simulation shown in Fig. 6.4. Each dot corresponds to the DTFE reconstructed density contrast and velocity divergence at the center of gravity of a given Delaunay tetrahedron. The thick continuous line represents the prediction from linear theory (Eqn. 6.8), while the thin continuous line represent the prediction from a second-order approximation (Eqn. 6.10). The DTFE reconstructed fields have not been smoothed.

The agreement of the DTFE reconstructions with the predictions from theory is remarkably good, especially when considering the fact that we have not smoothed the fields at all. It appears that the second-order approximation is a better fit to the data than linear theory, which is not surprising given the relatively advanced dynamical state of the cosmic network and the fact that we have not smoothed the data. Nevertheless there is a large amount of scatter present in the data. This scatter is present for several reasons:

1. Physical deviations from (semi-)linearity. Clearly both the low density voids as the intermediate and high density filaments and clusters have entered the non-linear regime. In this regime the one-to-one relation between the density and the velocity relation is not valid.

2. Inconsistency of a combined DTFE density and velocity reconstruction. Strictly speaking such a combined reconstruction is not self-consistent because the DTFE implies a linear density and velocity field, whereas the continuity equation demands a constant density field for a linearly varying velocity field.

3. Poisson scattering of density field values. A Poisson sampling of the density field will lead to the presence of sampling noise in the density field (which is larger for higher density values). The velocity field is not beset by this effect because the values at the location of the sampling points are exact.
Given the presence of these effects, it is remarkable how well the data fits the theoretical predictions. This clearly indicates that the combined DTFE density and velocity reconstruction works very well. Note that each of these effects may be diminished by smoothing over the data. Smoothing of the data does indeed lead to a tighter correlation at the price of a more linear relation (e.g. Bernardeau & van de Weygaert 1996, Romano-Díaz 2004).

6.5 Voids

By nature void-like regions contain very few galaxies or simulation particles. This poses specific problems for the analysis of their density and velocity fields in observations or numerical simulations. Conventional methods are in general unable of accurately describing the density and velocity field inside voids over the whole range of relevant scales. The rigid non-adaptive nature of grid-based schemes leads to a loss of information in the reconstructed field on scales smaller than the size of the effective kernel or smoothing radius. Anisotropic and caustic features will therefore not be recovered if their scale is smaller than the size of the grid-cells. Also, if the resolution of the grid is set too high, the sampling particle density will on average be less than one particle per cell. In such a case the interpolated fields will be dominated by shot-noise. In the case of the velocity field the problems are even more severe than for the density field, because the sampling is based on variations in the density field. This means that in low density environments the velocity field is severely undersampled and previous studies had to rely on heavy smoothing or other specifically designed techniques in order to obtain any useful information in such environments see (e.g. van de Weygaert & van Kampen 1993, Dubinski et al. 1993). Although SPH-like procedures whose resolution adapt to the local density of sampling points certainly do represent a significant improvement over the traditional methods, SPH-kernels are almost exclusively spherically symmetric (see however Shapiro et al. 1996 and Owen et al. 1998) and do not adapt to the local geometry of the point distribution. In Chapters 4 and 5 we have studied the differences between DTFE and SPH-like reconstruction methods and showed that SPH-like methods are not well-suited for studying anisotropic structures like the filaments and wall-like structures present in the large scale galaxy distribution. The SPH kernel has a somewhat larger effective smoothing kernel and thus a lower resolution. The spherical smoothing tends to blur out anisotropic structures into their surrounding environment, affecting the outer parts of voids. We have shown that DTFE reconstructed fields are not affected by such problems.

6.5.1 Structure and dynamics

In Fig. 6.10 a typical void-like region is shown, together with the DTFE density and velocity field reconstructions. The solid line running from the bottom to the top of these fields indicates the one-dimensional section along which the density and velocity field are shown in the bottom right-hand frame of the figure. The DTFE procedure clearly manages to render the void as a realistically slowly varying region of low density. Notice the clear distinction between the empty (dark) interior regions of the void and its edges. In the interior of the void several smaller `sub-voids’ may be recognized, whose boundaries consist of low density filamentary or wall-like structures. The presence of a hierarchy of voids, with large voids composed of the merging of smaller ones is in agreement with theories of void evolution (Regös & Geller 1991, Dubinski et al. 1993, van de Weygaert & van Kampen 1993, Sheth &
Figure 6.10 — A typical void-like region in the simulation shown in Fig. 6.7. Top left-hand frame: particle distribution in a thin slice through the simulation box. Top right-hand frame: two-dimensional slice through the three-dimensional DTFE density field reconstruction. Bottom left-hand frame: two-dimensional slice through the three-dimensional DTFE velocity field reconstruction. Bottom right-hand frame: density and velocity reconstructions along the thick line shown in the other frames.
van de Weygaert 2004). The velocity field shows that the void is expanding. This expansion is rather uniform as can be observed in the one-dimensional sections through the density and velocity reconstruction shown in the bottom right-hand frame of Fig. 6.10.

The reconstructed velocity field shows a consistent behavior over the entire void region. The velocity increases approximatively linearly along the one-dimensional profile, while it suddenly drops after crossing the edge of relatively high density. This linear ‘super-Hubble’ expansion of voids is well understood in terms of gravitational dynamics. According to Birkhoff’s theorem the internal dynamics of a spherically symmetric system is independent of the dynamics of the outside universe (Birkhoff 1923). This theorem has been applied in the development of spherically symmetric infall models for galaxies (e.g. Gunn & Gott 1973, Schechter 1980), large-scale inhomogeneities in general (Silk 1974) and cluster infall regions (Regós & Geller 1989). According to Birkhoff’s theorem voids can be approximated as expanding, isolated universes unto themselves that do not accrete matter from the universe at large (e.g. van de Weygaert & van Kampen 1993, Goldberg & Vogeley 2004). Because voids are emptier than the rest of the universe they will expand faster than the rest of the universe with a net velocity divergence equal to

$$\frac{1}{3} \nabla \cdot \mathbf{v} = H_{\text{void}} - H.$$  \hspace{1cm} (6.11)

When we define the normalized velocity divergence $\theta = \nabla \cdot \mathbf{v}/H$ and the ratio of the Hubble expansion of the void universe and the Hubble expansion of the universe $\alpha = H_{\text{void}}/H$ this equation may simply be written as

$$\theta = 3(\alpha - 1).$$ \hspace{1cm} (6.12)

The super-Hubble expansion of voids has been observed in numerical simulations of void-like regions (e.g. van de Weygaert & van Kampen 1993, Dubinksi et al. 1993), be it that a large amount of artificial smoothing or other specifically designed techniques had to be imposed to derive credible results. The DTFE reconstruction yields natural and continuous void density and velocity profiles without the application of smoothing procedures and thus allows a study of its physical structure and dynamics.

### 6.5.2 Constraining cosmological parameters using void dynamics

The expansion velocity of voids is directly related to the cosmology of the background universe, which determines the expansion ratio $\alpha$. The velocity divergence inside voids follows from Eqn. 6.12. This relation thus allows the possibility of using the largest voids to constrain the cosmic density parameter (van de Weygaert & van Kampen 1993, Dekel & Rees 1994). Bernardeau et al. (1997) used the cut-off and the peak of the probability distribution function (PDF) of the velocity divergence field to measure the normalized velocity divergence of the largest voids to constrain the cosmic density parameter (see also Bernardeau 1994, Bernardeau et al. 1995).
6.5.2.1 Empty voids in cosmologies with no cosmological constant

The largest expansion ratios will occur for the emptiest voids. For cosmologies with no cosmological constant the age of the universe may be approximated by (see e.g. Peacock 1999)

\[ t \approx H^{-1} \left( 1 + \frac{1}{2} \Omega_m^{0.6} \right)^{-1}. \]  

(6.13)

According to Birkhoff’s theorem the largest voids behave like empty FRW universes and their age is therefore equal to

\[ t_{\text{void}} = H_{\text{void}}^{-1}. \]  

(6.14)

Setting the age of voids equal to the age of the universe one finds after some algebraic manipulations for the expansion ratio of empty voids,

\[ \alpha = 1 + \frac{1}{2} \Omega_m^{0.6}. \]  

(6.15)

6.5.2.2 Empty voids in cosmologies with a non-zero cosmological constant

For cosmologies with a non-zero cosmological constant Eqn. 6.15 is not valid. Not only is the age of the universe different from the value given by Eqn. 6.13, which evidently is true for voids too. The dynamics of voids is also affected by the cosmological constant and their age will differ from the value given by Eqn. 6.14. Romano-Díaz (2004) nevertheless applied Eqn. 6.15 to measure the cosmological density parameter in a ΛCDM cosmology, which may explain part of the differences between their measurements and the imposed value. The age of the universe with a non-zero cosmological constant can be approximated by (Carroll, Press & Turner 1992)

\[ t \approx \frac{2}{3} H^{-1} \left( 0.7 \Omega_m - 0.3 \Omega_\Lambda + 0.3 \right)^{-0.3}. \]  

(6.16)

Similarly, the age of voids can be approximated by

\[ t_{\text{void}} \approx \frac{2}{3} H_{\text{void}}^{-1} \left( -0.3 \Omega_{\Lambda,\text{void}} + 0.3 \right)^{-0.3}. \]  

(6.17)

Here \( \Omega_{\Lambda,\text{void}} \) is the vacuum energy density parameter inside voids. Although the cosmological constant \( \Lambda \) clearly is the same inside voids as inside the background universe, the corresponding vacuum energy density parameters are not. The reason for this is that voids are expanding faster than the background universe and the Hubble parameter is correspondingly larger. It follows that the value of the vacuum energy density parameter inside empty voids is given by

\[ \Omega_{\Lambda,\text{void}} = \frac{\Lambda}{3H_{\text{void}}^2} = \Omega_\Lambda \frac{H^2}{H_{\text{void}}^2} = \Omega_\Lambda \alpha^{-2}. \]  

(6.18)

Equating the age of voids with the age of the universe leads to the following expression:

\[ \alpha(-\Omega_\Lambda \alpha^{-2} + 1)^{0.3} = (2.33\Omega_m - \Omega_\Lambda + 1)^{0.3}. \]  

(6.19)

This equation has to be solved numerically for the expansion ratio \( \alpha \). The velocity divergence in the interior of voids can subsequently be found by applying Eqn. 6.12.
Figure 6.11 — Probability distribution functions (PDFs) of the normalized DTFE velocity divergence \( \theta \) at actual time for three different GIF simulations (see text for description). The velocity divergence field has been smoothed with a Gaussian kernel with \( R_G = 5h^{-1}\text{Mpc} \).

We have tested this formalism for a number of cosmologies by means of cosmological N-body simulations. Following Bernardeau et al. (1997) we have determined the velocity divergence inside empty voids by measuring the position of the cut-off of the probability distribution function (PDF) of the velocity divergence field. In Fig. 6.11 the PDFs of the normalized velocity divergence \( \theta \) at the present epoch is shown for three GIF-simulations, each corresponding to a different cosmology (see Table 6.1 for their parameters). They correspond to a standard cold dark matter model (SCDM), an open model (OCDM) and a concordance model (\( \Lambda \)CDM). The velocity divergence field has been computed by means of the DTFE procedure. Negative values of the velocity divergence correspond to overdense regions where inflows occur, while positive values correspond to underdense regions where outflows occur. The strongest outflows occur for the emptiest voids.

For all three simulations the PDF clearly deviates from Gaussianity and shows a pronounced negative skewness. This is because in the non-linear regime the inflows to high density regions are faster than the outflows from low density regions. Notice that the shapes of the OCDM and the \( \Lambda \)CDM PDFs are very similar, while the SCDM PDF appears very different. In particular the SCDM PDF is non-zero for higher values of the velocity divergence, while negative values of the velocity divergence are much more prevalent. This difference is due to the fact that the SCDM model contains more mass than both the OCDM and \( \Lambda \)CDM models, resulting in stronger infall and outflow motions. The predicted sharp cut-off at positive values of \( \theta \) is clearly present in all three models. To obtain this sharp cut-off we have smoothed the velocity divergence field with a Gaussian filter of radius \( R_G = 5h^{-1}\text{Mpc} \). This smoothing has been applied to obtain a uniform scale for the measured divergence values and because the non-linear velocity field inside high density environments such as clusters and filaments cannot be represented by the DTFE interpolation procedure, which assumes a linear variation of the field between neighbouring sampling points. Because of virialization and orbit cross-

<table>
<thead>
<tr>
<th>Model</th>
<th>( \Omega_m )</th>
<th>( \Omega_\Lambda )</th>
<th>( h )</th>
<th>( L (h^{-1}\text{Mpc}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>SCDM</td>
<td>1.0</td>
<td>0.0</td>
<td>0.5</td>
<td>84.5</td>
</tr>
<tr>
<td>OCDM</td>
<td>0.3</td>
<td>0.0</td>
<td>0.7</td>
<td>141.3</td>
</tr>
<tr>
<td>( \Lambda )CDM</td>
<td>0.3</td>
<td>0.7</td>
<td>0.7</td>
<td>141.3</td>
</tr>
</tbody>
</table>

We have tested this formalism for a number of cosmologies by means of cosmological N-body simulations. Following Bernardeau et al. (1997) we have determined the velocity divergence inside empty voids by measuring the position of the cut-off of the probability distribution function (PDF) of the velocity divergence field. In Fig. 6.11 the PDFs of the normalized velocity divergence \( \theta \) at the present epoch is shown for three GIF-simulations, each corresponding to a different cosmology (see Table 6.1 for their parameters). They correspond to a standard cold dark matter model (SCDM), an open model (OCDM) and a concordance model (\( \Lambda \)CDM). The velocity divergence field has been computed by means of the DTFE procedure. Negative values of the velocity divergence correspond to overdense regions where inflows occur, while positive values correspond to underdense regions where outflows occur. The strongest outflows occur for the emptiest voids.

For all three simulations the PDF clearly deviates from Gaussianity and shows a pronounced negative skewness. This is because in the non-linear regime the inflows to high density regions are faster than the outflows from low density regions. Notice that the shapes of the OCDM and the \( \Lambda \)CDM PDFs are very similar, while the SCDM PDF appears very different. In particular the SCDM PDF is non-zero for higher values of the velocity divergence, while negative values of the velocity divergence are much more prevalent. This difference is due to the fact that the SCDM model contains more mass than both the OCDM and \( \Lambda \)CDM models, resulting in stronger infall and outflow motions. The predicted sharp cut-off at positive values of \( \theta \) is clearly present in all three models. To obtain this sharp cut-off we have smoothed the velocity divergence field with a Gaussian filter of radius \( R_G = 5h^{-1}\text{Mpc} \). This smoothing has been applied to obtain a uniform scale for the measured divergence values and because the non-linear velocity field inside high density environments such as clusters and filaments cannot be represented by the DTFE interpolation procedure, which assumes a linear variation of the field between neighbouring sampling points. Because of virialization and orbit cross-
Table 6.2 — Comparison of measured and theoretical expansion ratios. Expansion ratios are given both for voids modeled as empty universes (\(\alpha_{\text{theory, empty}}\)) and as low density universes (\(\alpha_{\text{theory, low density}}\)).

<table>
<thead>
<tr>
<th>Model</th>
<th>(\theta_{\text{max}})</th>
<th>(\alpha_{\text{theory, empty}})</th>
<th>(\alpha_{\text{measured}})</th>
<th>(\delta_{\text{min}})</th>
<th>(\alpha_{\text{theory, low density}})</th>
</tr>
</thead>
<tbody>
<tr>
<td>SCDM</td>
<td>0.90</td>
<td>1.50</td>
<td>1.30</td>
<td>0.19</td>
<td>1.32</td>
</tr>
<tr>
<td>OCDM</td>
<td>0.55</td>
<td>1.24</td>
<td>1.18</td>
<td>0.11</td>
<td>1.18</td>
</tr>
<tr>
<td>ΛCDM</td>
<td>0.59</td>
<td>1.21</td>
<td>1.20</td>
<td>0.065</td>
<td>1.20</td>
</tr>
</tbody>
</table>

ing the velocity field is not uniquely defined in these regions, which leads to non-physical estimates of the velocity divergence.

We have measured the position of the maximum velocity divergence by determining the position of the cut-off in the PDFs shown in Fig. 6.11). The results are listed in Table 6.2. By means of Eqn. 6.12 we have calculated the expansion ratios \(\alpha_{\text{measured}}\) of these empty voids with respect to the corresponding background cosmologies. We have compared these measurements with the analytical predictions \(\alpha_{\text{theory}}\) calculated from Eqn. 6.15 (SCDM and OCDM models) and Eqn. 6.19 (ΛCDM model). The results are also listed in Table 6.2.

The measured expansion ratios are substantially lower than predicted for the SCDM and the OCDM models, while for the ΛCDM model the measured value coincides with the theoretical prediction. The fact that we find a too low expansion ratio for the SCDM and OCDM models may be due to the fact that the measured expansion ratios should be interpreted as lower bounds, as we have considered empty voids and in reality voids are not empty. The simulations do not contain completely empty voids, which instead should be approximated by low density universes.

### 6.5.2.3 Non-empty voids

The density of non-empty voids can be parametrized with a non-zero cosmic density parameter \(\Omega_{m,\text{void}}\) given by

\[
\Omega_{m,\text{void}} = \Omega_m \frac{H^2}{H_{\text{void}}^2} (\bar{\delta}_{\text{void}} + 1) = \Omega_m \alpha^{-2} (\bar{\delta}_{\text{void}} + 1).
\]  

(6.20)

Here \(\bar{\delta}_{\text{void}} + 1\) is the mean underdensity inside the void. The age of a non-empty void in a universe without a cosmological parameter is given by Eqn. 6.13 with \(\Omega_m\) replaced by \(\Omega_{m,\text{void}}\). The age of a non-empty void in a universe with a non-zero cosmological parameter is given by Eqn. 6.16, again with \(\Omega_m\) replaced by \(\Omega_{m,\text{void}}\).

To check if this explains the differences between the measured and predicted expansion ratios, we have measured the mean underdensity inside the most empty void for the three GIF simulations. These values are listed in Table 6.2. The SCDM universe has the highest minimal density, in correspondence with the fact that this model has the highest matter density. The OCDM and ΛCDM universe have an equal matter density, but the density of the emptiest void is substantially lower in the ΛCDM universe. This is due to the extra expansion caused by a non-zero cosmological parameter. We have calculated the expansion ratio for the three GIF simulations. These are also listed in Table 6.2. The agreement with the analytically predicted values is now very good for all three models. This shows that voids can be modeled...
Table 6.3 — Evolution of measured and theoretical expansion ratios for a characteristic void from \( z = 5 \) to \( z = 0 \). The void is modeled as a low density universe.

<table>
<thead>
<tr>
<th>( z )</th>
<th>( H(z) )</th>
<th>( \Omega_{m0}(z) )</th>
<th>( \Omega_{\Lambda}(z) )</th>
<th>( \delta_{\text{void}} )</th>
<th>( \alpha_{\text{theory}} )</th>
<th>( \nabla \cdot \mathbf{v} ) [( 10^2 ) km/s]</th>
<th>( \alpha_{\text{measured}} \equiv \nabla \cdot \mathbf{vH}(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.0</td>
<td>( 5.7 \times 10^2 )</td>
<td>0.99</td>
<td>0.01</td>
<td>0.38</td>
<td>1.25</td>
<td>4.4</td>
<td>1.26</td>
</tr>
<tr>
<td>3.0</td>
<td>( 3.1 \times 10^2 )</td>
<td>0.96</td>
<td>0.04</td>
<td>0.29</td>
<td>1.29</td>
<td>3.0</td>
<td>1.33</td>
</tr>
<tr>
<td>2.0</td>
<td>( 2.1 \times 10^2 )</td>
<td>0.92</td>
<td>0.08</td>
<td>0.24</td>
<td>1.31</td>
<td>2.2</td>
<td>1.36</td>
</tr>
<tr>
<td>1.0</td>
<td>( 1.2 \times 10^2 )</td>
<td>0.77</td>
<td>0.23</td>
<td>0.16</td>
<td>1.31</td>
<td>1.4</td>
<td>1.38</td>
</tr>
<tr>
<td>0.5</td>
<td>92</td>
<td>0.59</td>
<td>0.41</td>
<td>0.12</td>
<td>1.29</td>
<td>0.96</td>
<td>1.35</td>
</tr>
<tr>
<td>0.3</td>
<td>82</td>
<td>0.49</td>
<td>0.51</td>
<td>0.10</td>
<td>1.26</td>
<td>0.77</td>
<td>1.32</td>
</tr>
<tr>
<td>0.0</td>
<td>70</td>
<td>0.30</td>
<td>0.70</td>
<td>0.09</td>
<td>1.19</td>
<td>0.53</td>
<td>1.25</td>
</tr>
</tbody>
</table>

as spherically symmetric low density universes. Moreover, the DTFE combined density and velocity reconstruction procedure clearly provides a consistent description of voids.

### 6.5.3 Evolution

The description of voids as spherically symmetric low density universes is valid not just for most empty voids at the present epoch, but should in principle apply to any void at any cosmic time. To explore and test this notion we have followed the evolution of one particular void. In Fig. 6.12 we show the same typical void-like region as depicted in Fig. 6.10 at a number of cosmic times ranging from \( z = 5 \) to \( z = 0 \). At each cosmic time the particle distribution in a thin slice is shown together with the DTFE density and velocity field reconstructions, as well as one-dimensional sections through the density and velocity fields. The figure shows the evolution of the void from a region which is relatively underdense to a region which is almost empty. The expansion of the void is clearly visible as it gets almost twice as big (in comoving coordinates) between \( z = 5 \) and \( z = 0 \). The expansion velocity decreases as the void expands. This can be seen in the one-dimensional sections through the velocity field.

At each cosmic time we have measured the average density inside the void, which together with the cosmological parameters provide a theoretical prediction of the expansion ratio \( \alpha_{\text{theory}} \) of this void with respect to the background cosmology. We have also measured the average velocity divergence inside the void, which translates into a measurement of the expansion ratio \( \alpha_{\text{measured}} \) of the void. The predicted and measured expansion ratios are listed in Table 6.3. Given the unknown accuracy of the assumption of spherical symmetry of the void the theoretical and measured values match strikingly well. On the other hand, the measured expansion ratio is systematically about 5% larger than the predicted value. This may be due to the tidal forces of the large scale environment outside the void, which may induce a positive velocity divergence. This explanation could be tested by subtracting the model velocity field for the void from the DTFE reconstructed velocity field and checking whether the resulting residual field can be explained by features present in the density field. Such an analysis is beyond the scope of this chapter.

Given the close agreement between the measured and predicted values it should in principle be possible to constrain the values of the cosmological parameters from the density and
Figure 6.12 — Evolution of a void from $z = 5$ to $z = 0$. The void shown is the same as depicted in Fig. 6.10. At each cosmic time the particle distribution is shown in a thin slice together with a two-dimensional slice through the full three-dimensional DTFE reconstructed density and velocity fields and one-dimensional sections through these fields. The sections are taken along the same directions as shown in Fig. 6.10.
velocity fields in voids. The DTFE technique is clearly able of consistently representing these fields even though relatively few sampling particles are present in these low density environments.

We have modelled voids as uniform low density universes. A more realistic model would take into account that voids do not have a uniform density. Instead, they may be modeled as a series of evolving concentric low density shells. These shells remain concentric and are assumed to be perfectly uniform, without any substructure. The resulting solution of the equation of motion for each shell covers the full non-linear evolution of the void, as long as shell crossing does not occur. This ‘spherical model’ has been described and worked out by Gunn & Gott (1972) and Lilje & Lahav (1991).

In Fig. 6.13 (source: Sheth & van de Weygaert 2004) the evolution of a void in the spherical model is illustrated. It shows the time evolution of the density deficit profile. The evolving density profile bears out the characteristic tendency of voids to expand, with mass streaming out from the interior, and hence for the density to decrease continuously in value (and approach emptiness, $\delta = -1.0$). Notice that this model provides the most straightforward illustration of the formation of a ridge. Looking from the inside out, one sees the interior shells expanding outwards more rapidly than the outer shells. With time the inner matter catches up with the outer shells, leading to a steepening of the density profile in the outer realms and the formation of a ridge. Meanwhile, over a growing area of the void interior, the density distribution is rapidly flattening. This is a direct consequence of the outward expansion of the inner void layers: the ‘flat’ part of the density distribution in the immediate vicinity of the dip gets ‘inflated’ along with the void expansion (Sheth & van de Weygaert 2004).

These characteristics are also visible in Fig. 6.12. At high redshifts the void boundaries are not very conspicuous and only slightly more dense than their surroundings. As matter streams outwards and starts to collect at the boundaries of the voids a ridge is formed, which is clearly
visible at low redshifts. At high redshifts, the void interior still contains a considerable amount of matter and structure, while it empties and flattens almost completely at low redshifts. Note that the void density profile in Fig. 6.12 is not as smooth as the spherical model in Fig. 6.13. The latter is a spherically symmetric model, while in reality voids are not.

6.6 Filaments and walls

Filamentary and wall-like structures are difficult to describe using conventional analysis techniques. These structures are highly anisotropic in one or two spatial dimensions and conventionally used spherically symmetric smoothing kernels are therefore inappropriate. In Chapters 4 and 5 of this thesis we have shown that such kernels tend to blur out anisotropic structures and as a result they obtain a larger volume and become intrinsically less anisotropic. The DTFE kernel is not beset by these problems. It automatically adapts to both the local density and geometry of the distribution of sampling points, resulting in a much more faithful representation of filaments and walls.

6.6.1 Structure and dynamics

In Fig. 6.14 a typical filamentary structure is shown, together with the DTFE density and velocity field reconstructions. The solid line running from the bottom left-hand to the top right-hand part of these fields represents the one-dimensional section along which the density and velocity field are shown in the bottom right-hand frame of the figure. The DTFE procedure renders the filament as a distinct feature in both the density and velocity field. Clearly visible is the shearing flow along the filaments ridge. The one-dimensional section slices roughly perpendicularly through the filament, which shows up as a sharp double peak in the density reconstruction. At the same time the value of the velocity field drops about 900 km/s at the location of the filament. This decrease corresponds to infalling motions towards the filament due to its gravitational pull on its surroundings. Clearly noticeable to the left and right of the filament is the linear super-Hubble expansion of the surrounding voids.

It is known that anisotropic collapse of matter in gravitational instability scenarios leads to the formation of sheets and filament, which can already be predicted from the Zel’’dovich approximation (Zel’dovich 1970, see also Shandarin & Zel’dovich 1989). In Fig. 6.15 the density and velocity profile of a Zel’dovich pancake is shown (source: Shandarin & Zel’dovich 1989). The density and velocity profiles look very similar to those plotted in Fig. 6.14.

A fully non-linear model of the formation and evolution of cosmic walls and filaments is that of isolated homogeneous ellipsoidal overdensities. In particular, the early work by Icke (1972, 1973) elucidated transparently the crucial characteristics of their development and morphology. He showed that any slight asphericity in the initial density field will inevitably amplify during its subsequent evolution and collapse. The work of Icke was followed up upon by White & Silk (1979) who specifically studied the formation of isolated ellipsoidal structures in general background cosmologies (with no cosmological constant). A lot of work has been devoted to generalizing their results to non-isolated and/or non-homogeneous configurations. It was soon recognized that the shear field is closely related to the dynamical evolution of filamentary and wall-like structures and that the dynamical evolution of aspherical density perturbations is rather complicated and determined by the initial ellipticity, the external matter distribution and the background cosmology.
Figure 6.14 — A typical filamentary region in the simulation shown in Fig. 6.7. Top left-hand frame: particle distribution in a thin slice through the simulation box. Top right-hand frame: two-dimensional slice through the three-dimensional DTFE density field reconstruction. Bottom left-hand frame: two-dimensional slice through the three-dimensional DTFE velocity field reconstruction. Bottom right-hand frame: density and velocity reconstructions along the thick line shown in the other frames.
6.6.2 Evolution

The potential of the DTFE is further illustrated by Fig. 6.16, in which the same filamentary region is shown as depicted in Fig. 6.14 at a number of cosmic times, ranging from \( z = 5 \) to \( z = 0 \). At each cosmological time the particle distribution in a thin slice is shown together with the DTFE density and velocity field reconstructions, as well as one-dimensional sections through these fields. The figure shows the evolution of the filamentary region from a relatively extended overdense region to a compact and collapsed filament. The velocity profile evolves from a rough sinoid to a distinct Z when the filament has collapsed. Using high resolution DTFE reconstructions to study the evolution of the density and velocity field of a large sample of representative filaments and walls may significantly increase our understanding of these complex structures and the relation with their environment.

6.7 Shear and vorticity

So far we have only considered the velocity field itself and its divergence. In section 6.3 we have shown that the DTFE velocity field reconstruction technique also provides the shear and vorticity of the velocity field. Shear and vorticity patterns are expected to be prominent near high density regions, where distortions in the velocity flows are relatively strong. In general, vorticity measures the speed of rotation of a fluid element, while shear measures the anisotropy in its expansion rate.

Shear is a dominant factor in the shaping of large scale structure (e.g. Hoffman 1986, van de Weygaert & Babul 1994, Bond et al. 1996, Bond & Myers 1996a, b, c). The shear in the velocity field can be due to the intrinsic asphericity of evolving structures (due to anisotropic collapse) and/or tidal forces exerted on the local matter distribution by the surrounding large scale matter distribution. Shear is therefore expected to be present in linear, quasi-linear and non-linear regions as a result of gravitational interactions and the collapse of matter. In fact, the rate of growth of the density field gets amplified by the presence of shear, which increases

Figure 6.15 — The formation of a Zel’dovich pancake begins with the development of a singularity in the density distribution (top frame). At the same time the velocity field develops a vertical tangent at the position of the singularity (bottom frame). Source: Shandarin & Zel’dovich 1989.
Figure 6.16 — Evolution of a filamentary region from \( z = 5 \) to \( z = 0 \). The filament shown is the same as depicted in Fig. 6.14. At each cosmic time the particle distribution is shown in a thin slice together with a two-dimensional slice through the full three-dimensional DTFE reconstructed density and velocity fields and one-dimensional sections through these fields. The sections are taken along the same directions as shown in Fig. 6.14.
the rate of growth of fluid convergence \( -\nabla \cdot \mathbf{v} \) following a given fluid element (Hoffman 1986, Bertschinger & Jain 1994). This convergence takes place around collapsing structures, such as filaments and clusters. Bond et al. (1996) pointed out that the cosmic web is in fact a consequence of the distribution and spatial coherence of the shear field in the medium (van de Weygaert 2002). The density field and the velocity shear are therefore expected to be correlated as a function of time. There is however no simple relation between the density field and the velocity shear. In the linear regime the velocity shear is related to the density excess growth factor \( D(t) \) via (van de Weygaert & Bertschinger 1996)

\[
\sigma_{ij} \propto D(t) H(t) f(\Omega_m). \tag{6.21}
\]

In the linear regime the velocity shear therefore does have a direct relation with the density field since \( \delta(t) \propto D(t) \).

According to linear theory of gravitational instability (Peebles 1980) the large scale velocity field is irrotational, with \( \nabla \times \mathbf{v} = 0 \). Any vorticity mode would have decayed away during the linear growth of density fluctuations, while the only growing modes are curl-free. Based on Kelvin’s circulation theorem, the flow remains vorticity free as long as it is laminar, i.e., until orbit crossing occurs (Bertschinger & Dekel 1989, Dekel et al. 1990). This condition is expected to hold even in the weakly non-linear regime (Bertschinger & Dekel 1989). It is therefore expected that vorticity is only present around non-linear features in the density field.

Figure 6.17 shows the shear (top left-hand frame) and vorticity (top right-hand frame) amplitudes, \( \sigma = (\Sigma \sigma_{ij} \sigma_{ij})^{1/2}, \omega = (\Sigma \omega_{ij} \omega_{ij})^{1/2} \) in a 100 \( \times \) 100 \( h^{-1} \) Mpc slice at actual time through a cosmic N-body simulation. This simulation concerns a standard ΛCDM simulation and the slice depicted is the same as shown in Fig. 6.2. In the bottom frame the density contrast and velocity field in this slice are also shown. All fields have been convolved with a Gaussian kernel of \( R_G = 1h^{-1} \) Mpc. The color scales of the shear and vorticity are inverted with respect to the density contrast in order to get a better visual impression of the shear and vorticity patterns. As can be noticed, both the shear and the vorticity trace the matter distribution rather well. As expected, the shear gives a better impression of the large scale matter distribution than the vorticity which is mainly visible around non-linear features in the density field.

### 6.7.1 Shear tensor and eigenvalues

Above we have considered the amplitude of the shear. The velocity shear is a symmetric and traceless tensor which contains much more information than just its amplitude. The shear tensor describes how the matter distribution is dynamically affected by its surroundings, i.e. if the matter distribution is being stretched or squeezed along a given direction. This information is contained within the eigenvalues and eigenvectors of the diagonalized matrix representation of the velocity shear tensor. The eigenvalues indicate the strength of the stretching or compression, while the direction of stretching or compression is given by the corresponding eigenvectors. Below we have ordered the eigenvalues \( \lambda_i \) (and corresponding eigenvectors) according to their amplitudes, with \( \lambda_1 > \lambda_2 > \lambda_3 \). Because the shear tensor is traceless, only two of its three eigenmodes will be linearly independent, i.e. \( \sum \lambda_i = 0 \). By construction the eigenvectors form a unitary and orthogonal system. The amplitude of this shear reference frame will be given by the eigenvalues, and since in general these have different values, they form
Figure 6.17 — DTFE recovered velocity shear and vorticity amplitudes (top frames) and density and velocity fields (bottom frame) for the same slice through the cosmological $N$-body simulation as shown in Fig. 6.2. The color scales of the shear and vorticity are inverted with respect to the density contrast in order to get a better visual impression of the shear and vorticity patterns.

This an ellipsoid which is known as the *velocity ellipsoid*. The first eigenmode ($\lambda_1$) is positively defined and indicates the direction and intensity of the maximum stretching exerted over a given region. The third eigenvalue ($\lambda_3$) is always negative and corresponds to a compression along a direction perpendicular to the stretching one. The second eigenmode ($\lambda_2$) can be positive and negative. The configuration of the tidal field is determined by this eigenmode, with an extra stretching or compression perpendicular to the first eigenmode. If $\lambda_1 > \lambda_2 > 0$ the
velocity ellipsoid describes a planar structure, while if $\lambda_3 < \lambda_2 < 0$ it describes a filamentary configuration.

Using the DTFE procedure these eigenmodes can be computed for each Delaunay tetrahedron. However, for reasons of efficiency we did so from the grid-interpolated velocity gradient field. By diagonalizing the shear matrix at each pixel of the image, we find its eigenvectors and eigenvalues. Note that since the velocity-gradient matrix is not continuous, the same holds for the computed eigenvalues and eigenvectors.

Fig. 6.18 depicts the amplitude of the two linearly independent eigenvalues ($\lambda_1$ and $\lambda_2$) for the same slice as shown in Fig. 6.17. The stretching mode ($\lambda_1$, left-hand frame) traces the filamentary structure present in the map (see bottom panel of Fig. 6.17). Stretching is stronger at the core of high density regions than along filaments. As expected, no stretching is found inside void regions. For the second eigenvalue (right-hand frame) the brightest regions correspond to a positive mode ($\lambda_2 > 0$) where stretching also occurs. Filamentary structures show up as light colored regions. Most regions in the map correspond to $\lambda_2 \approx 0$ (gray areas), predominantly voids where almost no shear is present.

The amplitudes of the eigenvectors are very illustrative in displaying the characteristics of the large scale matter distribution. They are however only a part of the total information contained within the shear tensor. In Fig. 6.19 we present a two-dimensional section through the velocity shear and its projected eigenvalues and eigenvectors. This is equivalent to the intersection of the velocity ellipsoid with the two-dimensional plane. This intersection is an ellipse whose major axis corresponds to the stretching mode, and whose minor axis corresponds to the compression along the bisecting plane. Both modes are presented in Fig. 6.19.
Figure 6.19 — Decomposition of the velocity shear components along the same slice as shown in Fig. 6.18. The dilational (stretching) component is shown in the left-hand frame, while the compressional component is shown in the right-hand frame. The bars are proportional to the intensity of the components and are aligned along the corresponding eigenvectors’ direction. The particle distribution is also depicted.

as ‘shear bars’. The left-hand frame represents the stretching mode, the right-hand frame the compressional mode. The bars are proportional to the amplitude of each eigenvalue and oriented according to their corresponding eigenvectors. The point distribution corresponds to the matter distribution in a $5h^{-1}\text{ Mpc}$ thick slice. One may observe that there is a correlation between the compressional bars and the filamentary structure of the point distribution. This correlation is stronger for the gravitational tidal field (see van de Weygaert 2002). The compressional bars are strong (large) and oriented almost perpendicular to the filamentary structures. The dilational bars tend to be more aligned with the filament. More massive structures like clusters are delineated by the compressional bars as a result of infalling motions around these clusters. In this representation clusters form the ‘nodes’ of the cosmic web. Underdense regions appear very quiet in both representations. They are clearly recognizable as those regions where bars are not perturbed, but coherently aligned instead.

With the shear-bar analysis we have shown the strong interplay between shear and the filamentary structure of the universe. This confirms the theoretical basis of the Zel’dovich formalism. By using a similar approximation Hoffman (1986) noticed that objects with large initial shear collapse sooner than one would predict. This result can also be derived from first principles from the Raychaudhuri equation which connects the velocity divergence, the shear and the vorticity with the density field. According to the Zel’dovich formalism (Zel’dovich 1970), objects collapse first along the eigenmode with the largest eigenvalue to form Zel’dovich pancakes, then along the next eigenmode to form filaments, and finally along the last eigenmode to collapse into clusters. If the primordial Gaussian velocity field would be shear-less and ir-
rotational, the velocity-gradient matrix would be isotropic, allowing for spherical collapse as predicted by the spherical model.

6.8 Summary and discussion

In this chapter we have described the formalism for simultaneously reconstructing the density and velocity fields corresponding to a discrete set of irregularly distributed sampling points. In principle the same technique can be applied to any dynamical field which has been sampled at the locations of these points. The formalism is an essential ingredient of the Delaunay Tessellation Field Estimator (DTFE), which itself is an extension of the work by Bernardeau & van de Weygaert (1996). The DTFE reconstruction procedure yields continuous and volume-covering fields. Its main advantage is that it is intrinsically self-adaptive to the density and geometry of the distribution of sampling points and does not make use of any pre-specified smoothing kernel.

We have shown by the specific example of the simultaneous reconstruction of a density and velocity field corresponding to an $N$-body simulation of cosmic structure formation that the characteristic elements of the large scale matter distribution are realistically rendered. The reconstructed density and velocity fields adhere closely to analytically predicted density-velocity divergence relation.

We have explicitly discussed the dynamical modeling of voids and filaments, which both are problematic structures for conventional reconstruction methods. Void-like regions contain very few galaxies or simulation particles, which poses specific problems for the analysis of their density and velocity fields in both observations and numerical simulations. Conventional density field reconstructions tend to be dominated by shot-noise effects. The same holds for velocity field reconstructions, which are even more problematic as they are severely undersampled. The DTFE reconstruction of the velocity field in underdense regions is not beset by this problems. Instead, the density field is realistically rendered as a slowly varying low density valley. At the same time, the void’s velocity profile resembles the super-Hubble linear expansion which is predicted by analytical models. We have described these models and explicitly formulated the equations governing the evolution of a void in a $\Lambda$CDM universe.

A comparison of the model predictions with numerical simulations showed that they are in good agreement. We have also discussed the possibility of using these analytic void models to constrain cosmological parameters.

Next we have discussed the reconstruction of filamentary and wall-like structures. Because these are highly anisotropic in one or two of their spatial dimensions, conventional methods which usually make use of spherically symmetric smoothing kernels fail to give an accurate description of such objects (see Chapters 4 and 5). The DTFE reconstructions of a filamentary structure and its evolution shows that the method is capable of tracing with high resolution both the collapsing structure as well as the infalling motions towards and around these structures. The resulting density and velocity profiles are in good agreement with the Zel’dovich approximation.

Because of the linear interpolation inside each of the Delaunay tetrahedra it is rather straightforward to compute velocity-related quantities such as the velocity divergence, shear and vorticity. We have found that both the shear and vorticity trace the matter distribution very well, which is in accordance with theoretical predictions. The same holds for the eigenvalues
of the velocity shear, which describe whether the matter distribution is squeezed or stretched along a given direction. In particular, we have shown that the compressional mode is strongly related to the filamentary structure of the large scale matter distribution, again in accordance with theoretical predictions (see e.g. van de Weygaert 2002).

The main restriction of the DTFE is that it is a linear reconstruction procedure which assumes the reconstructed fields to vary linearly in between the sampling points. In high density regions shell crossing has occurred and the velocity field is not uniquely defined. In such regions the DTFE interpolation is not valid for the velocity field. However, the reconstructed field may be smoothed in order to obtain physically reliable field estimates.

In conclusion, we have shown that the application of the DTFE to the reconstruction of cosmic velocity fields has important advantages over traditional reconstruction procedures. We have also shown that the dynamics of the cosmic web can be used to constrain cosmological parameters. Applying the DTFE reconstruction procedure to the analysis of large simulations of cosmic structure formation will therefore certainly help us to gain understanding of the formation and evolution of the cosmic web and its characteristic constituent elements.

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