Chapter 4

Ruled surfaces of degree 4

In this section, we describe the models from Series XIII of Schilling’s collection. The series consists of 10 models that illustrate a classification of the ruled surfaces $S \subseteq \mathbb{P}^3$ given by a homogeneous real irreducible polynomial of degree 4. In the following, we study K. Rohn’s text [Rohn] that describes the mathematics of the models from this Series. In his text, Rohn uses as a definition of a ruled surface the following:

**Definition 5.** Let $S \subseteq \mathbb{P}^3$ be a surface. $S$ is a ruled surface if it is a union of straight lines.

**Remark:** In this text, we do not study the connection with the modern, more abstract definition of ruled surface, as given in [Be] or [Ha]. We only use the given classical definition.

If a point $p \in S$ lies on more than one line it is called a special point of $S$. The collection of all special points forms a curve (see [Ed, Chapter I]), which is the singular locus of $S$. The following theorem from the 19th century classifies the ruled surfaces over $\mathbb{C}$ in terms of this singular locus.

**Theorem 21.** Let $S$ be a ruled surface which is not a cone given by a homogeneous irreducible polynomial of degree 4, with singular locus the curve $\Gamma \subseteq \mathbb{P}^3(\mathbb{C})$. Then, $\Gamma \subseteq \mathbb{P}^3(\mathbb{C})$ is one of the following:

1. $\Gamma$ consists of two skew double lines (including a limit case).
2. $\Gamma$ consists of a double line and a double conic.
3. $\Gamma$ consists of a double space curve of degree 3.

4. $\Gamma$ consists of a triple line.

Proof. See [Ed, Chapter I].

The proof as given in [Ed] starts by intersecting the surface with a general plane through a general line of the surface. This intersection is required to consist of the line together with an irreducible curve of degree 3. This is the reason why cones are excluded: for a cone, the intersection would be a union of lines.

In the following, we study the four cases of the theorem and describe the equations for the surfaces represented by the models. We begin studying the first case of Theorem 21, namely the case when the ruled surface $S$ contains two double lines. This case yields the models 1, 2, 3, 4 and 5 of Series XIII. The surfaces that the models are representing are described in detail by Rohn in [Rohn] and his explicit calculations are replaced by modern proofs in the present text.

### 4.1 Ruled surfaces with two skew double lines

Let $\mathbb{P}^3$ be the projective space with coordinates $x_1, x_2, y_1, y_2$. We recall the following definition:

**Definition 6.** A bi-homogeneous form $G \subseteq \mathbb{P}^3$ of type $(i, j)$ is a polynomial in the variables $x_1, x_2, y_1, y_2$ such that

$$G(\lambda x_1, \lambda x_2, \mu y_1, \mu y_2) = \lambda^i \mu^j G(x_1, x_2, y_1, y_2)$$

for all $\lambda, \mu \in \mathbb{C}$.

The following theorem gives a construction for ruled surfaces using a pair of skew lines:

**Theorem 22.** Consider the rational map:

$$\Phi : \mathbb{P}^3 \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$$

$$(x_1 : x_2 : y_1 : y_2) \mapsto ((x_1 : x_2), (y_1 : y_2)).$$

Let $\Gamma \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a curve. Then, the closure of $\Phi^{-1}(\Gamma)$ is a ruled surface $S \subset \mathbb{P}^3$. Let $\Gamma$ be given by a bi-homogeneous polynomial $F$ of type $(2, 2)$, then $S$ is also given by $F$. 
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Proof. Note that \( \Phi \) is not defined on \( \{L_1 \cup L_2\} \), where \( L_i \) are the two skew lines in \( \mathbb{P}^3 \) given by \( L_1 = \{(x_1 : x_2 : 0 : 0)\} \) and \( L_2 = \{(0 : 0 : y_1 : y_2)\} \). Let \( S \) be the closure of \( \Phi^{-1}(\Gamma) \) and consider the restricted morphism of \( \Phi: S \setminus \{L_1 \cup L_2\} \to \Gamma \). Let us show that \( S \) is a ruled surface. It suffices to show that any point of \( S \setminus \{L_1 \cup L_2\} \) lies on a straight line contained in \( S \).

For every point \( ((a_1 : a_2 : b_1 : b_2)) \in S \setminus \{L_1 \cup L_2\} \), \( \Phi(a_1 : a_2 : b_1 : b_2) = ((a_1 : a_2),(b_1 : b_2)) \in \Gamma \). Hence \( F(a_1,a_2,b_1,b_2) = 0 \). The closure of \( \Phi^{-1}((a_1 : a_2),(b_1 : b_2)) \) is the line \( \{ (\lambda a_1 : \lambda a_2 : \mu b_1 : \mu b_2) \mid (\lambda : \mu) \in \mathbb{P}^1 \} \) which connects the two points \( (a_1 : a_2 : 0 : 0) \) and \( (0 : 0 : b_1 : b_2) \) on \( L_1 \) and \( L_2 \) respectively. Hence, the surface \( S \) is ruled. \( \square \)

For a geometrical meaning of the map \( \Phi \), we refer to Chapter 2.

We have the following result:

**Proposition 11.** Let \( S \subset \mathbb{P}^3 \) be given by \( F = 0 \) with \( F \) homogeneous polynomial of degree 4. Suppose that the two lines \( L_1 \) and \( L_2 \) given above are double lines. Then, \( F \) is bi-homogeneous of type \((2,2)\). Moreover, \( S \) is a ruled surface.

Proof. One can write \( F = F_{2,2} + F_{1,3} + F_{3,1} + F_{0,4} + F_{4,0} \), where \( F_{i,j} \) is bi-homogeneous of type \((i,j)\). Since \( F = 0 \) contains \( L_1 \) and \( L_2 \) one has that \( F_{0,4} = F_{4,0} = 0 \). Using that the points of \( L_1 \) have multiplicity 2, one finds that \( F_{3,1}(a_1,a_2,y_1,y_2) = 0 \) for all \((a_1,a_2) \not= (0,0)\). Hence, \( F_{3,1} = 0 \). Using the line \( L_2 \) one finds that \( F_{1,3} = 0 \). Thus, \( F = F_{2,2} \).

The surface \( S \) is ruled since any point \( (a_1 : a_2 : b_1 : b_2) \in S \) is either on \( L_1 \) or \( L_2 \), or satisfies \((a_1,a_2) \not= (0,0)\) and \((b_1,b_2) \not= (0,0)\). In the latter case, the line \( (\lambda a_1 : \lambda a_2 : \mu b_1 : \mu b_2) \) with \((\lambda : \mu) \in \mathbb{P}^1 \) contains the point and is in \( S \). \( \square \)

Let \( S \) be given by an irreducible bi-homogeneous polynomial \( F \) of type \((2,2)\). Then, we can write \( F \) in the form

\[
A(x_1,x_2)y_1^2 + B(x_1,x_2)y_1y_2 + C(x_1,x_2)y_2^2
\]

with \( A, B \) and \( C \) homogeneous polynomials of degree 2 in \( x_1 \) and \( x_2 \). Since \( F \) is not divisible by a polynomial in \( x_1 \) and \( x_2 \), it follows that \( \gcd(A,B,C) = 1 \). Since \( F \) is not divisible by a polynomial in \( y_1 \) or \( y_2 \), it follows that \( AC \not= 0 \). Furthermore, there is no linear relation \( \alpha A + \beta B + \gamma C = 0 \) with \( \alpha \gamma \not= 0 \) (because \( F \) is not divisible by a linear form in \( y_1 \) and \( y_2 \)).
Proposition 12. Let $\Gamma \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a curve given by a polynomial $F$ bi-homogeneous of type $(2,2)$ of the form:

$$A(x_1, x_2)y_1^2 + B(x_1, x_2)y_1y_2 + C(x_1, x_2)y_2^2.$$ 

Suppose the discriminant $B^2 - 4AC$ has four different zeroes. Then, the curve $\Gamma$ has genus 1.

Proof. Consider the projection of $\Gamma$ to the first $\mathbb{P}^1$, $pr_1 : \Gamma \to \mathbb{P}^1$, which is a 2 : 1 map. By assumption, the discriminant $B^2 - 4AC$ has four different zeroes $p_1, p_2, p_3, p_4$. Hence, $\Gamma$ is non-singular and $pr_1 : \Gamma \to \mathbb{P}^1$ is ramified with ramification index 2 at the four points $p_i$ for $i = 1, \ldots, 4$. By the Riemann-Hurwitz formula, the genus $g$ of $\Gamma$ satisfies $2 - 2g = 2(-2) + 4 = 0$ hence $g = 1$.

From now on, we will assume that the discriminant of the polynomial $F$ defining $\Gamma \subset \mathbb{P}^1 \times \mathbb{P}^1$ has four different zeroes. Hence, the curve $\Gamma$ is non-singular and has genus 1. Let us choose a point $e \in \Gamma$. This makes the curve $\Gamma$ into an elliptic curve with neutral element $e$ for its group structure. The following two lemmas will be needed in the proof of Theorem 23.

Lemma 6. Let $\Gamma$ be an elliptic curve with neutral element $e \in \Gamma$. Consider the automorphisms $\sigma$ and $\tau_a$ with $a \in \Gamma$ of $\Gamma$, given by $\sigma(p) = -p$ and $\tau_a(p) = p + a$. Then, the automorphisms of order two of $\Gamma$ are:

1. $\sigma \tau_a$ with $a \in \Gamma$
2. $\tau_a$ with $a \neq e$ a point of order two on $\Gamma$.

Proof. See [Sil, §3].

Lemma 7. Let $\Gamma$, $\sigma \tau_a$ and $\tau_a$ as in Lemma 6. Then,

1. $\Gamma/\langle \sigma \tau_a \rangle \simeq \mathbb{P}^1$ and
2. $\Gamma/\langle \tau_a \rangle$ with $a \in \Gamma$ of order 2, is isomorphic to an elliptic curve.

Proof.
1. The map \( f : \Gamma \to \Gamma/\langle \sigma \tau_a \rangle \) is a 2 : 1 map. It is ramified at the points \( p \in \Gamma \) with \( \sigma \tau_a(p) = p \), i.e., points \( p \) such that \( 2p = -a \). Considering the isomorphism of the elliptic curve \( \Gamma \cong \mathbb{C}/\Lambda \cong S^1 \times S^1 \) one sees that this equation has four solutions for \( p \). Hence, the map \( f \) is ramified at four points with ramification index 2. By the Riemann-Hurwitz formula, this implies that \( \Gamma/\langle \sigma \tau_a \rangle \) has genus 0 and is isomorphic to \( \mathbb{P}^1 \).

2. Again, the Riemann-Hurwitz formula applied to the covering \( \Gamma \to \Gamma/\langle \tau_a \rangle \) shows that \( \Gamma/\langle \tau_a \rangle \) has genus 1.

Now consider a projection \( pr_i : \Gamma \subset \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \). This map has degree 2, hence defines \( \Gamma \) as a Galois cover of \( \mathbb{P}^1 \) of degree 2. This implies that a unique automorphism \( \Phi_i \) of \( \Gamma \) of order 2 exists, such that \( pr_i \circ \Phi_i = pr_i \) and \( \Gamma \to \Gamma/\langle \Phi_i \rangle \) where \( c \) is an isomorphism and \( \Phi_i \) is given by \( \sigma \tau_{a_i} \) for \( i = 1, 2 \). Note that \( a_1 \neq a_2 \) since otherwise \( F \) would be reducible. The lemmas will be used to produce a standard form for the curves \( \Gamma \).

**Theorem 23.** Let \( \Gamma \subset \mathbb{P}^1 \times \mathbb{P}^1 \) be an elliptic curve given by a bi-homogeneous polynomial of type \((2, 2)\). Then, there exists an automorphism \( A : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1 \) such that \( A(\Gamma) \) is symmetric in the following sense:

\[
(p, q) \in A(\Gamma) \Rightarrow (q, p) \in A(\Gamma).
\]

In particular, \( A(\Gamma) \) is given by a bi-homogeneous polynomial \( F \) of type \((2, 2)\), such that

\[
F(x_1, x_2, y_1, y_2) = F(y_1, y_2, x_1, x_2).
\]

**Proof.** All automorphisms of \( \mathbb{P}^1 \times \mathbb{P}^1 \) are of the form \((p, q) \mapsto (B_1(p), B_2(q))\) or \((p, q) \mapsto (B_1(q), B_2(p))\). If an automorphism \((p, q) \mapsto (B_1(p), B_2(q))\) has the required property, then also \((p, q) \mapsto (B_3B_1(p), B_3B_2(q))\) for any automorphism \( B_3 \) of \( \mathbb{P}^1 \) works. Similarly, also \((p, q) \mapsto (B_1(q), B_2(p))\) will work. Hence, it is no restriction to consider only automorphisms of \( \mathbb{P}^1 \times \mathbb{P}^1 \) of the form \((p, q) \mapsto (f(p), q)\), with \( f \) an automorphism of the first \( \mathbb{P}^1 \). Suppose that the automorphism
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\[ A : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1 \]
\[ (p, q) \mapsto (f(p), q) \]

has the required property. We now try to describe such \( f \). Now, \((p, q) \in \Gamma \Rightarrow (f(p), q) \in A(\Gamma) \Rightarrow (q, f(p)) \in A(\Gamma) \Rightarrow (f^{-1}(q), f(p)) \in \Gamma \). We therefore obtain an automorphism \( C \) of \( \Gamma \) given by:

\[ C : \Gamma \to \Gamma \]
\[ (p, q) \mapsto (f^{-1}(q), f(p)) \]

Clearly, \( C \) is an automorphism of order 2. By Lemma 6, \( C \) has one of the following forms:

1. \( \sigma \tau a \) with \( a \in \Gamma \) or
2. \( \tau a \) with \( a \neq e \) a point of order two on \( \Gamma \).

We claim that \( C\sigma \tau a_1 = \sigma \tau a_2 C \), where \( a_i \) is as defined below Lemma 7. To see that, let \( \gamma = (p, q) \in \Gamma \). Then, \( pr_2(C(\gamma)) = pr_2(f^{-1}(q), f(p)) = f(p) = f(pr_1(\gamma)) = f(pr_1(\sigma \tau a_1(\gamma))) = pr_2(C(\sigma \tau a_1(\gamma))) \). By the discussion following Lemma 7, this implies that \( C(\sigma \tau a_1(\gamma)) \) is either \( C(\gamma) \) or \( \sigma \tau a_2(C(\gamma)) \). The first equality implies that \( 2\gamma = -a_1 \) which has only four solutions. Hence, the second equality holds for almost all \( \gamma \in \Gamma \) and thus holds for all \( \gamma \in \Gamma \). We conclude that \( C\sigma \tau a_1 = \sigma \tau a_2 C \).

Using the last equality and Lemma 6, we have to consider the following possibilities.

1. Suppose that \( C = \sigma \tau c \) for some \( c \in \Gamma \). Then, one has that \( \sigma \tau c \sigma \tau a_1 = \sigma \tau a_2 \sigma \tau c \). This is equivalent to \( a_1 + a_2 = 2c \). There are 4 points \( c \in \Gamma \) satisfying this equation.

2. Suppose that \( C = \tau c \) with \( c \in \Gamma \) an element of order 2. Then, \( \tau c \sigma \tau a_1 = \sigma \tau a_2 \tau c \) implies \( a_1 = a_2 \), which yields a contradiction.

We have therefore proved that the automorphism \( C \) of order 2 of \( \Gamma \) is given by \( C = \sigma \tau c \), with \( 2c = a_1 + a_2 \).

Vice versa, suppose \( c \in \Gamma \) satisfies \( 2c = a_1 + a_2 \). Define \( f : \mathbb{P}^1 \to \mathbb{P}^1 \) by

\[ f(pr_1(\gamma)) := pr_2(\sigma \tau c(\gamma)) \]
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for all \( \gamma \in \Gamma \).

We show that the map \( f \) is well defined: suppose that \( pr_1(\gamma) = pr_1(\gamma') \) and \( \gamma \neq \gamma' \). Then, \( \gamma' = \sigma \tau_a \gamma \) and \( pr_2(C\gamma') = pr_2(C\sigma \tau_a \gamma) = pr_2(\sigma \tau_a C \sigma) = pr_2(C\gamma) \). It is easily verified that \( f \) is an isomorphism and has the required property.

The previous theorem shows that the bi-homogeneous polynomial \( F \) that defines \( \Gamma \subset \mathbb{P}^1 \times \mathbb{P}^1 \) can be assumed to be symmetric. In order to simplify further the equation defining \( \Gamma \), Rohn introduces the following definition:

**Definition 7.** Let \( \Gamma \subset \mathbb{P}^1 \times \mathbb{P}^1 \) be given by \( F = Ay_1^2 + By_1 y_2 + Cy_2^2 \) with \( A, B, C \) homogeneous in \( x_1 \) and \( x_2 \) of degree 2. The discriminant \( B^2 - 4AC \) is said to be *normalized* if it is a scalar multiple of \( x_1^4 + bx_1^2 x_2 + x_2^4 \).

The polynomial \( F \) is in its *normal form* if \( F \) is symmetric (in the sense of Theorem 23), and if its discriminant is normalized.

**Lemma 8 (Rohn).** Suppose \( F = Ay_1^2 + By_1 y_2 + Cy_2^2 \) is symmetric and its discriminant has 4 distinct zeroes. Then an automorphism \( s \) of \( \mathbb{P}^1 \) exists such that \( F(s(x), s(y)) \) is in normal form.

**Proof.** One can normalize the discriminant \( B^2 - 4AC \) as follows: recall that the discriminant has four simple zeroes \( p_1, p_2, p_3 \) and \( p_4 \) on the first \( \mathbb{P}^1 \). By a change of coordinates, we can suppose that \( p_1 = 0, p_2 = \infty, p_3 = 1 \) and \( p_4 = \lambda \). There exists an automorphism \( s \) of \( \mathbb{P}^1 \) permuting each pair \( \{0, \infty\} \) and \( \{1, \lambda\} \), namely the automorphism \( s \) given by

\[
s : \mathbb{P}^1 \rightarrow \mathbb{P}^1, \quad z \mapsto \frac{\lambda}{z}.
\]

The automorphism \( s \) has order two, and fixes the two points \( \pm \sqrt{\lambda} \). Again by a change of coordinates, we can suppose that these two points are 0 and \( \infty \), which transforms the automorphism \( s \) into \( s(z) = -z \). In terms of this, the four elements \( p_i \) that are pairwise interchanged are given by \( \pm d \) and \( \pm e \). The discriminant is therefore (written in affine coordinates) \( az^4 + bz^2 + c \) with \( ac \neq 0 \) which becomes, after shifting \( z \) to some \( \delta \mu, \mu^4 + b\mu^2 + 1 \). Rewriting in terms of homogeneous coordinates, one obtains the required form of Definition 7.

One can calculate that the discriminant of \( F(s(x_1, x_2), s(y_1, y_2)) \) is a scalar multiple of the discriminant of \( F((x_1, x_2), (y_1, y_2)) \). Since \( F(s(x), s(y)) \) is clearly symmetric, the last implies that it is in normal form. \( \square \)
We now determine the shape of an $F$ which is in normal form. Put $\mu = \frac{x_1}{x_2}$ and $\lambda = \frac{y_1}{y_2}$. Then, one has that a symmetric polynomial $F$ has the following general form (using Rohn’s notation):

$$a_{11}\lambda^2\mu^2 + a_{22}(\lambda + \mu)^2 + a_{33} + 2a_{23}(\lambda + \mu) + 2a_{13}\lambda\mu + 2a_{12}\lambda\mu(\lambda + \mu),$$

with $a_{ij} \in \mathbb{C}$ and with discriminant

$$\mu^4(a_{12}^2 - a_{11}a_{22}) + 2\mu^3(a_{12}a_{13} - a_{11}a_{23}) + \mu^2(a_{13}^2 + 2a_{13}a_{22} - 2a_{12}a_{23} - a_{11}a_{33}) + 2\mu(a_{13}a_{23} - a_{12}a_{33}) + a_{23}^2 - a_{22}a_{33}.$$

The discriminant is in its normal form if and only if

$$a_{12}a_{13} - a_{11}a_{23} = 0, \quad a_{13}a_{23} - a_{12}a_{33} = 0$$

$$a_{12} - a_{11}a_{22} = a_{23}^2 - a_{22}a_{33}.$$

One may suppose, by scaling, that $a_{11} = a_{33} = 1$. The last equation implies $a_{12} = \pm a_{23}$. Suppose that $a_{12} \neq 0$. Then $a_{23} \neq 0$ and $a_{13} = \pm 1$. One can check in this case that $F$ becomes reducible. Hence, we may suppose that $a_{12} = a_{23} = 0$ and $a_{11} = a_{33} = 1$. This gives the final normal form for $F$:

$$a_{11}(x_1^2y_1^2 + x_2^2y_2^2) + a_{22}(x_1^2y_2^2 + x_2^2y_1^2) + 2a_{13}x_1x_2y_1y_2$$

with $a_{ij} \in \mathbb{C}$, and discriminant a scalar multiple of

$$T^4 + \frac{a_{11}^2 + a_{22}^2 - a_{13}^2}{a_{11}a_{22}}T^2 + 1.$$

The real case:

We now want to find the equations of the surfaces represented in models 1 to 5 from Series XIII. The above calculations are valid over $\mathbb{C}$. In order to study the models, we need to see what happens when the curve $\Gamma$ is defined over $\mathbb{R}$, i.e., when the polynomial $F$ has real coefficients. Furthermore, we assume that $\Gamma(\mathbb{R})$ is not empty, since otherwise the real model would not have points. As in the complex case, we make $\Gamma$ into an elliptic curve by fixing a neutral element $e \in \Gamma(\mathbb{R})$. It is well known (see [Sil] and also Section 3.1.3 of Chapter 3) that the group $\Gamma(\mathbb{R})$ is either isomorphic to $S^1$ (we will call this the connected case) or to $S^1 \times \mathbb{Z}/2\mathbb{Z}$ (the disconnected case). It follows that in the connected case, $\Gamma(\mathbb{R})$ has two elements of order dividing 2, and in the disconnected case there are 4 such elements. From Lemma 6 one easily obtains the following result.
Lemma 9. Let $\Gamma(\mathbb{R})$ be an elliptic curve with neutral element $e \in \Gamma(\mathbb{R})$. Consider the automorphisms $\sigma$ and $\tau_a$ with $a \in \Gamma(\mathbb{C})$ as in the complex case. Then,

The real automorphisms of order two of $\Gamma$ are:

1. $\sigma \tau_a$ with $a \in \Gamma(\mathbb{R})$

2. $\tau_a$ with $a \in \Gamma(\mathbb{R})$ of order 2.

We now reconsider the proof of Theorem 23. Since in our case the projections $pr_i : \Gamma \rightarrow \mathbb{P}^1$ are defined over $\mathbb{R}$, it follows that the points $a_i \in \Gamma$ such that $pr_i \sigma \tau_a = pr_i$ are real points of $\Gamma$. We now study whether a real automorphism $A$ of $\mathbb{P}^1 \times \mathbb{P}^1$ exists. This is the case if and only if a $c \in \Gamma(\mathbb{R})$ exists such that $2c = a_1 + a_2$. We study the two cases separately:

The connected case:

Since $\Gamma(\mathbb{R}) \simeq S^1$, it is clear that there are two such real solutions to the equation $2c = a_1 + a_2$. Hence Theorem 23 holds with a real automorphism $A$ and a real polynomial $F$ in this case.

We next study which of the four complex ramification points of the two projections $pr_i : \Gamma \rightarrow \mathbb{P}^1$ are real. The ramification points of $pr_1$ are the fixed points $b \in \Gamma$ of $\sigma \tau_{a_1}$, with $a_1 \in \Gamma(\mathbb{R})$. Rohn calls these points pinch points. One has that $\sigma \tau_{a_1}(p) = p \iff -a_1 = 2b$. This has two real solutions $b$ and two complex conjugated. Hence, two of the ramification points for $pr_1 : \Gamma \rightarrow \mathbb{P}^1$ are real and the other two are complex conjugated. The same holds for the ramification points of $pr_2 : \Gamma \rightarrow \mathbb{P}^1$. After a real change of variables of $\mathbb{P}^1$, we may assume that these ramification points are given by $\pm \alpha$ and $\pm \beta i$. In these new variables, the discriminant is a scalar multiple of $T^4 + bT^2 - 1$. An analogous calculation to the one below the proof of Lemma 8 shows that $\Gamma$ may be assumed to be given as

$$a_{11}(-x_1^2 y_1^2 + x_2^2 y_2^2) + a_{22}(x_1^2 y_2^2 + x_2^2 y_1^2) + 2a_{13}x_1 x_2 y_1 y_2$$

with $a_{ij} \in \mathbb{R}$, and whose discriminant has the form

$$T^4 + \frac{a_{11}^2 - a_{22}^2 - a_{13}^2}{a_{11} a_{22}} T^2 - 1.$$

There are no models in the series that illustrate this case. We have represented the ruled surface given by the previous equation in Figure 4.1.
Figure 4.1: Ruled surface with two real double lines and two real pinch points.

Remark: Note that the real pinch points are situated on $L_1$ and $L_2$, respectively.

The disconnected case:
We study the real pinch points and the real solutions to the equation $2c = a_1 + a_2$. We distinguish four cases:

1. $a_1$ and $a_2$ lie on the component of the identity of $\Gamma(\mathbb{R})$. Then also $a_1 + a_2$ is on this component. Hence, the equation $2c = a_1 + a_2$ has also four real solutions, so again Theorem 23 holds with a real automorphism $A$ and a real polynomial $F$. Moreover, the equation $2b = -a_i$ has four real solutions $b$, so the maps $pr_i$ have all four ramification points real.

Therefore, one obtains for this case the same equation as in the complex case, namely:

$$a_{11}(x_1^2y_1^2 + x_2^2y_2^2) + a_{22}(x_1^2y_2^2 + x_2^2y_1^2) + 2a_{13}x_1x_2y_1y_2$$

with $a_{ij} \in \mathbb{R}$. The discriminant has up to a real scalar the form

$$T^4 + \frac{a_{11}^2 + a_{22}^2 - a_{13}^2}{a_{11}a_{22}}T^2 + 1.$$

Remark: Note that the polynomial $T^4 + bT^2 + 1$ has four distinct zeroes precisely when $b \neq 2$. These zeroes are all real when $b < -2,$
and all complex in the remaining cases. So in our case, we deduce that
\[
\frac{a_{11}^2 + a_{22}^2 - a_{13}^2}{a_{11}a_{22}} < -2.
\]
Model nr. 1 in Series XIII represents the case where all zeroes are real (see Figure 4.2).

Figure 4.2: String model representing a ruled surface with two real double lines and four real pinch points on each line. Series XIII nr. 1, University of Groningen.

2. Both \(a_1\) and \(a_2\) do not lie on the component of the identity of \(\Gamma(\mathbb{R})\). Then, \(a_1 + a_2\) is in the component of the identity. Hence as before, the equation \(2c = a_1 + a_2\) has four real solutions, so again Theorem 23 holds with a real automorphism \(A\) and a real polynomial \(F\). Since also \(-a_1\) and \(-a_2\) are not in the component of the identity of \(\Gamma(\mathbb{R})\), there are no real solutions to \(2b = -a_1\). Hence, the maps \(pr_i\) have all four ramification points imaginary. It can be calculated that the discriminant is in this case up to scalar of the form \(T^4 + bT^2 + 1\) with \(b > -2\) and \(b \neq 2\). Therefore, the corresponding equation reads:

\[
a_{11}(x_1^2y_1^2 + x_2^2y_2^2) + a_{22}(x_1^2y_2^2 + x_2^2y_1^2) + 2a_{13}x_1x_2y_1y_2
\]

with \(a_{ij} \in \mathbb{R}\) and discriminant up to real scalar of the form

\[
T^4 + \frac{a_{11}^2 + a_{22}^2 - a_{13}^2}{a_{11}a_{22}}T^2 + 1,
\]

with \(\frac{a_{11}^2 + a_{22}^2 - a_{13}^2}{a_{11}a_{22}} > -2\) and \(\frac{a_{11}^2 + a_{22}^2 - a_{13}^2}{a_{11}a_{22}} \neq 2\).

This type of equation yields model nr. 2 from Series XIII which is shown in Figure 4.3.
3. If $a_1$ lies in the component of the identity of $\Gamma(\mathbb{R})$ and $a_2$ does not, then, there are 4 real ramification points for $pr_1$ and none for $pr_2$. The equation $2c = a_1 + a_2$ has no real solutions for $c$. Rohn uses a trick to arrive at a normal form for this case. Namely, instead of using the flip as in the proof of Theorem 23, now the automorphism $((x_1 : x_2), (y_1 : y_2)) \mapsto ((iy_1 : y_2), (-ix_1 : x_2))$ of order 2 is used. The invariant forms of type $(2,2)$ for this automorphism are all

$$a_{11}\lambda^2\mu^2 - a_{22}(\lambda^2 - \mu^2) - a_{11} + 2a_{13}\lambda\mu$$

with $a_{ij} \in \mathbb{C}$. It can be shown that in the present case a real transformation exists which puts the curve in this form with $a_{ij} \in \mathbb{R}$:

$$a_{11}(x_1^2 y_2^2 - x_2^2 y_1^2) - a_{22}(x_1^2 y_2^2 - x_2^2 y_1^2) + 2a_{13}x_1 x_2 y_1 y_2$$

The two discriminants with respect to the first and second $\mathbb{P}^1$’s are a scalar multiple of

$$T^4 + \frac{a_{11}^2 + a_{22}^2 + a_{13}^2 T^2}{a_{11}a_{22}} + 1 \quad \text{and} \quad T^4 - \frac{a_{11}^2 + a_{22}^2 + a_{13}^2 T^2}{a_{11}a_{22}} + 1,$$

respectively. This equation yields model nr. 3 from Series XIII which is shown in Figure 4.4.

4. If $a_2$ lies in the component of the identity of $\Gamma(\mathbb{R})$ and $a_1$ does not, then, there are 4 real ramification points for $pr_2$ and none for $pr_1$. The
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\[ 2c = a_1 + a_2 \] has no real solutions for \( c \). This case is similar to the previous case.

Rohn also studies the situation where the ruled surface has a pair of complex conjugated lines as double lines. After a transformation \( \lambda = \alpha + i\beta \) and \( \mu = \alpha - i\beta \), his equation reads

\[
(\alpha^2 + \beta^2)^2 + 1 + 2b\alpha^2 + 2c\beta^2 = 0,
\]

with \( b < -1 \) and \( c \) arbitrary (or \( c < -1 \) and \( b \) arbitrary).

This equation gives rise to model 4 of Series XIII, shown in Figure 4.5.

Remark: Note that this is related to the blow-down morphism for real cubic surfaces containing a pair of complex conjugate skew lines (see Chapter 2).

4.1.1 The limit case

Rohn considers a limit case of the previous case which consists of moving the double line \( L_1 \) to the line \( L_2 \) until, in Rohn’s terminology, \( L_1 \) is infinitely near to \( L_2 \). The result is a ruled surface, now with the singular line \( L_2 \) which Rohn calls a Selbstberührungsgerade (self intersecting line). \( L_2 \) is more singular than a double line: any plane passing through \( L_2 \) intersects the surface in \( L_2 \).
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Figure 4.5: String model representing a ruled surface with two complex conjugated double lines. Series XIII nr. 4, University of Groningen.

and in two more lines intersecting on $L_2$. On $L_2$ there are four pinch points. This yields model number 5 of the series (see Figure 4.6), with equation:

$$a(x_1^4 + x_2^4) + 2b_1x_1^2x_2^2 + b_2(x_1y_2 - x_2y_1)^2 = 0.$$  

The condition $b_1/a < -1$ is needed here to make sure that these pinch points are real. Note that the equation is no longer bi-homogeneous but is still symmetric.

4.2 Ruled surfaces with a double line and a double conic

In this case, the singular locus of the ruled surface $S$ is assumed to be the union of a straight line $L$ and a conic $Q$, where the intersection $L \cap Q$ is one point and $L$ and $Q$ are not in the same plane. Rohn fixes homogeneous coordinates $x_1, x_2, y_1$ and $y_2$ on $P^3$ such that $L$ is given by $y_1 = y_2 = 0$ and $Q$ is given by $x_1 = 0 = y_2^2 - x_2y_1$. The line $L \subset P^3$ and the conic $Q \subset P^3$ can be parameterized respectively as $\{(\lambda : 1 : 0 : 0) \mid \lambda \in \mathbb{C}\}$ and $\{(0 : 1 : \mu^2 : \mu) \mid \mu \in \mathbb{C}\}$.

Rohn now considers bi-homogeneous curves $\Gamma \subset P^1 \times P^1 \simeq L \times Q$ of type $(2, 2)$ with the property that $L \cap Q$ defines a singular point of $\Gamma$ and such that
the projections $\Gamma \to \mathbb{P}^1$ ramify over points $\pm \lambda_0$ respectively $\pm \mu_0$. Explicitly, such a curve $\Gamma$ is given by an equation

$$a_{11}\lambda^2\mu^2 + a_{22}(\mu^2 \pm \lambda^2) + 2a_{13}\lambda\mu = 0.$$ 

As before, we write $pr_1$ and $pr_2$ for the two projections of $\Gamma$ to $L$ and $Q$. For any point $\gamma \in \Gamma$ one considers the straight line connecting $pr_1(\gamma)$ and $pr_2(\gamma)$. The union of all these lines $\{(0 : 1 : \mu^2 : \mu) + \rho(\lambda : 1 : 0 : 0)\}$ forms a surface in $\mathbb{P}^3$. From the equation $(\rho\lambda : \rho + 1 : \mu^2 : \mu) = (x_1 : x_2 : y_1 : y_2)$ and the equations defining $L$ and $Q$ one obtains the parameters

$$\mu = \frac{y_1}{y_2} \quad \text{and} \quad \lambda = \frac{x_1y_1}{x_2y_1 - y_2^2}.$$ 

Substituting these values in the equation of $\Gamma$,

$$a_{11}\left(\frac{x_1y_1}{x_2y_1 - y_2^2}\right)^2 \cdot \frac{y_1^2}{y_2^2} + a_{22}\left(\frac{y_1^2}{y_2^2} \pm \frac{(x_1y_1)^2}{(x_2y_1 - y_2^2)^2}\right) + 2a_{13}\frac{x_1y_1}{x_2y_1 - y_2^2} \cdot \frac{y_1}{y_2} = 0,$$

and multiplying the expression by $(x_2y_1 - y_2^2)^2 \cdot y_2^2$ one obtains:

$$a_{11}(x_1y_1)^2 \cdot y_1^2 + a_{22}y_1^2(x_2y_1 - y_2^2)^2 \pm (x_1y_1)^2 y_2^2 + 2a_{13}(x_2y_1 - y_2^2)y_2x_1y_1^2 = 0.$$ 

Dividing the expression by $y_1^2$ we obtain a component, given by an equation of degree 4:

$$a_{11}(x_1y_1)^2 + a_{22}(x_2y_1 - y_2^2)^2 \pm x_1^2y_2^2 + 2a_{13}(x_2y_1 - y_2^2)y_2x_1 = 0.$$
Note that the equation is no longer bi-homogeneous of type $(2,2)$ nor symmetric.

Remark: The condition that $L \cap Q$ is one point $p$ which moreover defines a singular point $(p,p) \in \Gamma$ is necessary in order to be able to divide out the equation by $y_1^2$. If $L \cap Q = \emptyset$, or if $(p,p) \notin \Gamma$, then the equation of the surface would be irreducible of degree 6.

The case with the $`+`$ sign and the condition $\frac{a_{11}a_{22}}{a_{11}a_{22}} > 0$ corresponds to the case when all the 8 pinch points are real. This case is illustrated in model nr. 8 of the series (shown in Figure 4.7).

![Figure 4.7: String model representing a ruled surface with a double line and a double conic intersecting eachother. Series XIII nr. 8, University of Padova.](image)

4.3 Ruled surfaces with a double space curve of degree 3

In this case, the singular locus of the ruled surface $S$ is a curve $N \subset \mathbb{P}^3$ called normal curve given by the embedding

$$\mathbb{P}^1 \rightarrow \mathbb{P}^3$$

$$(x_1 : x_2) \mapsto (x_1^3 : x_1^2x_2 : x_1x_2^2 : x_2^3).$$

Again, one considers a curve $\Gamma \subset N \times N \simeq \mathbb{P}^1 \times \mathbb{P}^1$, non-singular and bi-homogeneous of type $(2,2)$. Let $pr_1$ and $pr_2$ denote the two projections of $\Gamma$. 

4.4 WITH A TRIPLE LINE

to $N$. For a general point $\gamma \in \Gamma$ one considers the straight line connecting
$pr_1(\gamma)$ and $pr_2(\gamma)$. The closure of the union of all these lines forms the surface
$S$. The classification of the curve $\Gamma \subset N \times N$ is the same as in the previous
cases. The equation for the surface is

$$a_{11}(X^2 \pm Z^2) + a_{22}Y^2 + 2(a_{13} - a_{22})XZ,$$

where $X = (x_1y_1 - x_2^2)$, $Y = (x_1y_2 - x_2y_1)$, $Z = (x_2y_2 - y_1^2)$. The case
when the surface has four real pinch points corresponds to model nr. 9 of
the series, and the case where all pinch points are complex yields model 10
(shown in Figure 4.8).

Figure 4.8: String model representing a ruled surface with a double space curve
of degree 3. Series XIII nr. 10, University of Groningen.

4.4 Ruled surfaces with a triple line

This case is related to the case in Section 4.1. One has a similar result as
that of Proposition 11.

**Proposition 13.** Let $S \subset \mathbb{P}^3$ be an irreducible surface given by $F = 0$ with $F$
homogeneous polynomial of degree 4. Suppose that the line $L = \{(x_1 : x_2 : 0 : 0)\}$ is triple. Then, $F$ has the form

$$x_1A(y_1, y_2) + x_2B(y_1, y_2) + C(y_1, y_2)$$
with $A$, $B$ and $C$ homogeneous polynomials in $y_1$ and $y_2$ of degrees 3, 3 and 4 respectively. Moreover, $S$ is a ruled surface.

**Proof.** Decompose $F$ as $F = F_{2,2} + F_{1,3} + F_{3,1} + F_{0,4} + F_{4,0}$, where $F_{i,j}$ is bi-homogeneous of type $(i,j)$. Since $F$ is zero on $L$, one has $F_{4,0} = 0$. Consider the point $(0 : 1 : 0 : 0)$ and affine coordinates $u := \frac{x_1}{x_2}$, $v := \frac{x_2}{x_3}$, $w := \frac{x_3}{x_4}$. The affine equation $F_{2,2}(u, 1, v, w) + F_{1,3}(u, 1, v, w) + F_{3,1}(u, 1, v, w) + F_{0,4}(u, \frac{1}{x_4}, v, w)$ has the form $(u^2 A_2(v, w) + uB_2(v, w) + C_2(v, w)) + (uA_3(v, w) + B_3(v, w)) + (u^3 A_1(v, w) + u^2 B_1(v, w) + uC_1(v, w) + D_1(v, w)) + (A_4(v, w))$, where $A_i$, $B_i$, $C_i$ and $D_i$ are homogeneous polynomials of degree $i$. The intersection of the surface with a plane $y_2 = \alpha y_1$ yields a curve, which should contain the line $L$ as a triple line. In other words, the polynomial $u^2 A_2(v, \alpha v) + uB_2(v, \alpha v) + C_2(v, \alpha v) + uA_3(v, \alpha v) + B_3(v, \alpha v) + u^3 A_1(v, \alpha v) + u^2 B_1(v, \alpha v) + uC_1(v, \alpha v) + D_1(v, \alpha v) + A_4(v, \alpha v)$ contains a factor $v^3$. This implies the form of the equation for $S$.

We note that the g.c.d $(A, B, C) = 1$ since $S$ is irreducible. Consider a point $(a_1 : a_2 : b_1 : b_2) \in S$ with $(b_1, b_2) \neq (0, 0)$. If $A(b_1, b_2) = B(b_1, b_2) = 0$, then also $C(b_1, b_2) = 0$. This contradicts g.c.d $(A, B, C) = 1$. Hence, there exists a pair $(c_1, c_2) \neq (0, 0)$ (unique up to a multiple) such that $c_1 A(b_1, b_2) + c_2 B(b_1, b_2) = 0$. Then, the line through $(a_1, a_2, b_1, b_2)$ and $(c_1, c_2, 0, 0)$ lies on $S$ which implies that $S$ is a ruled surface. \hfill \Box

Let $(a_1 : a_2 : 0 : 0) \in L$ be a point on the triple line $L$. In general, there are 3 tangent planes at the point. The union of these tangent planes is given by the cubic equation $a_1 A(y_1, y_2) + a_2 B(y_1, y_2) = 0$. In general, there are 4 points on $L$ such that two of the three tangent planes coincide. These points are called again by Rohn pinch points. The reality of these four pinch points and the possible coincidence of them gives rise to various models. Only two of them are in the series. Namely model nr. 6 with four real pinch points (shown in Figure 4.9), and model nr. 7, where $A$ and $B$ are assumed to have a common linear factor (see Figure 4.10).

**Remark:** The historical part of [Be, III] and the exercises 9 and 10 give some hints for the classification of ruled surfaces of degree 4 in $\mathbb{P}^3$, as given by Rohn. A complete modern treatment seems to be missing in the literature.

We note that the above classification is somewhat vague and moreover cones in $\mathbb{P}^3$ have been excluded (see Theorem 21). An example of a cone is the following: Let $x_1, x_2, x_3, x_4$ denote homogeneous coordinates for $\mathbb{P}^3$. Let $F = F(x_1, x_2, x_3)$ be homogeneous of degree 4, defining a non-singular curve
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Figure 4.9: String model representing a ruled surface with a triple line and four real pinch points. Series XIII nr. 6, University of Padova.

Figure 4.10: String model representing a ruled surface with a triple line. Series XIII nr. 7, University of Groningen.
$C \subset \mathbb{P}^2$. The cone of $C$ is the surface $S \subset \mathbb{P}^3$ given by the same equation $F = 0$. This surface consists of the lines connecting the point $(0 : 0 : 0 : 1)$ with the points of $C \subset \mathbb{P}^2 \subset \mathbb{P}^3$. 
Bibliography


