On t-Motifs
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Chapter 2

Duality and $t$-Motifs

In the present chapter the category of effective $t$-motifs will be extended to a slightly larger category which has internal homs. The objects in the resulting category will be called $t$-motifs.

2.1 Internal Hom

2.1.1. Let $M_1$ and $M_2$ be effective $t$-motifs over $K$. Inspired by the theory of linear representations of groups we could try to assign to $M_1$ and $M_2$ an effective $t$-motif of internal homomorphisms as

\[ \text{Hom}(M_1, M_2) \overset{\text{def}}{=} \text{Hom}_{K[t]}(M_1, M_2) \text{ with } \sigma(f) \overset{\text{def}}{=} \sigma \circ f \circ \sigma^{-1}, \]

where $\sigma \circ f \circ \sigma^{-1}$ is to be read as $\sigma_2 \circ f \circ \sigma_1^{-1}$. This does, however, not make sense, since $\sigma_1$ need not be invertible. First of all, $K$ need not be perfect, and secondly—more seriously—the determinant of $\sigma_1$ is $(t - \theta)^d$ up to a constant, and hence not invertible if $d > 0$.

2.1.2. This can be partially resolved. Write $K^d$ for some algebraic closure of $K$. Note that after extension of scalars from $K[t]$ to $K^d(t)$ the induced action of $\sigma$ on $M_1 \otimes_{K[t]} K^d(t)$ is invertible.

**Proposition.** For $n$ sufficiently large, the subgroup

\[ \text{Hom}_{K[t]}(M_1, M_2 \otimes C^n) \subset \text{Hom}_{K^d(t)}(M_1 \otimes K^d(t), M_2 \otimes C^n \otimes K^d(t)) \]
is stable under $f \mapsto \sigma \circ f \circ \sigma^{-1}$.

Proof. Choose bases and express $\sigma$ on $M_1$ and $M_2$ by matrices $S_1$ and $S_2$ respectively. Then $M_2 \otimes C^n$ has a basis on which $\sigma$ is expressed by the matrix $(t-\theta)^n S_2$. The map $f \mapsto \sigma \circ f \circ \sigma^{-1}$ translates to a map

$$M(r_2 \times r_1, K^a(t)) \to M(r_2 \times r_1, K^a(t)) : F \mapsto F'$$

with

$$F' = (t-\theta)^n S_2 \tau(F \tau^{-1}(S_1^{-1})) = (t-\theta)^n S_2 \tau(F) S_1^{-1}$$

(2.1)

The Proposition claims that $M(r_2 \times r_1, K[t])$ is mapped into itself. But since the determinant of $S_1$ is a power of $(t-\theta)$, the matrix $(t-\theta)^n S_1^{-1}$ has entries in $K[t]$ when $n$ is sufficiently large. This immediately implies that $M(r_2 \times r_1, K[t])$ is mapped into itself.

2.1.3. It follows from the explicit formula (2.1) that $\sigma(f) \overset{\text{def}}{=} \sigma \circ f \circ \sigma^{-1}$ induces the structure of an effective $t$-motif on $\text{Hom}_{K[t]}(M_1, M_2 \otimes C^n)$ for large $n$. We shall denote it by $\mathcal{H}om(M_1, M_2 \otimes C^n)$. These internal homs are stable for growing $n$ in the sense that there are natural isomorphisms

$$\mathcal{H}om(M_1, M_2 \otimes C^n) \otimes C \to \mathcal{H}om(M_1, M_2 \otimes C^{n+1})$$

(2.2)

relating them.

2.2 $t$-Motifs

2.2.1. The previous section hints that the obstruction to having internal homs will be lifted as soon as the Carlitz $t$-motif is made invertible.\(^{(1)}\)

This can be done quite easily, because of the following:

\(^{(1)}\)Very reminiscent of the inversion of the Lefschetz motif in the construction of the category of pure motifs: If $X$ is a smooth and projective variety of dimension $d$ then $\ell$-adic Poincaré duality defines a perfect pairing

$$H^i_{\text{ét}}(X_{K^s}, \mathbb{Q}_\ell) \times H^{2d-i}_{\text{ét}}(X_{K^s}, \mathbb{Q}_\ell) \to \mathbb{Q}_\ell(-d)$$

which suggests that the motif $h^i(X, \mathbb{Q})$ is dual to $h^{2d-i}(X, \mathbb{Q})$ shifted by the $d$-th power of the Lefschetz motif. See §4.1 of [André 2004].
Lemma. If $M_1$ and $M_2$ are effective t-motifs, then the natural map

$$\text{Hom}_\sigma(M_1, M_2) \to \text{Hom}_\sigma(M_1 \otimes C^n, M_2 \otimes C^n)$$

that takes $f$ to $f \otimes \text{id}$ is an isomorphism.

Proof. Note that $f \mapsto f \otimes \text{id}$ defines a natural isomorphism

$$\text{Hom}_{K[t]}(M_1, M_2) \to \text{Hom}_{K[t]}(M_1 \otimes C^n, M_2 \otimes C^n)$$

of $K[t]$-modules. An element $g = f \otimes 1$ of the latter is a morphism of effective t-motifs if and only if it satisfies

$$(t - \theta)^n \sigma_2 \circ f = f \circ (t - \theta)^n \sigma_1$$

which is equivalent with $\sigma_2 \circ f = f \circ \sigma_1$, that is, with $f$ being an element of $\text{Hom}_\sigma(M_1, M_2)$. \qed

2.2.2. Now we are ready to make the following definition.

Definition. A t-motif is a pair $(M, i)$ consisting of an effective t-motif $M$ and an integer $i \in \mathbb{Z}$. Morphisms between t-motifs are defined by

$$\text{Hom}_\sigma((M_1, i_1), (M_2, i_2)) \overset{\text{def}}{=} \text{Hom}_\sigma(M_1 \otimes C^{n+i_1}, M_2 \otimes C^{n+i_2}),$$

for $n$ sufficiently large. The resulting category is denoted by $t\mathcal{M}(K)$ or simply by $t\mathcal{M}$.

It suffices to take $n \geq \max(-i_1, -i_2)$ in the definition. The module of morphisms is independent of $n$ by the preceding Lemma.

The functor $M \mapsto (M, 0)$ is fully faithful and we will identify $t\mathcal{M}_{\text{eff}}$ with its image in $t\mathcal{M}$.

2.2.3. The natural isomorphism between $M \otimes C^{n+1}$ and $M \otimes C^n \otimes C$ defines a distinguished isomorphism of t-motifs

$$(M, i + 1) = (M \otimes C, i). \quad (2.3)$$

In particular, we can identify $C^i$ with $(1, i)$. But note that $(1, i)$ is an object in $t\mathcal{M}$ even when $i$ is negative.
2.2.4. The operations ⊕ and ⊗ and \( \mathcal{H}om \) extend from the category of effective \( t \)-motifs—or parts thereof—to the full category of \( t \)-motifs:

\[
(M_1, i_1) \oplus (M_2, i_2) \overset{\text{def}}{=} (M_1 \otimes C_{i_1}^{n+i_1} \oplus M_2 \otimes C_{i_2}^{n+i_2}, -n)
\]

\[
(M_1, i_1) \otimes (M_2, i_2) \overset{\text{def}}{=} (M_1 \otimes M_2, i_1 + i_2)
\]

\[
\mathcal{H}om((M_1,i_1),(M_2,i_2)) \overset{\text{def}}{=} (\mathcal{H}om(M_1,M_2 \otimes C_{i_1-i_2+n}^{1-n}), -n)
\]

The occurrences of \( n \) in these definitions should be read ‘with \( n \) sufficiently large’. Using the isomorphisms (2.2) and (2.3), one verifies that these are independent of \( n \) and coincide with the operations on effective \( t \)-motifs, whenever defined.

2.2.5. From now on we will often drop the integer \( i \) from the notation and write \( M \) for a \( t \)-motif, effective or not.

As usual, we define the dual of a \( t \)-motif \( M \) to be \( M^\vee \overset{\text{def}}{=} \mathcal{H}om(M, 1) \). The operations of direct sum, tensor product, duality and internal hom satisfy the expected relations—those familiar from the theory of linear representations of groups. In particular, there is an \textbf{adjunction formula}

\[
\mathcal{H}om(M_1 \otimes M_2, M_3) = \mathcal{H}om(M_1, \mathcal{H}om(M_2, M_3)).
\]  

(2.4)

Also, taking duals is \textbf{reflexive}: the natural morphism

\[
M \rightarrow (M^\vee)^\vee
\]

(2.5)

is an isomorphism. And finally, \( \mathcal{H}om \) is \textbf{distributive} over \( \otimes \) in the sense that the natural morphism

\[
\mathcal{H}om(M_1, M_3) \otimes \mathcal{H}om(M_2, M_4) \rightarrow \mathcal{H}om(M_1 \otimes M_2, M_3 \otimes M_4),
\]  

(2.6)

is an isomorphism. Proofs of these three statements are given in the coming paragraphs. Assuming them for now, we can show:

**Theorem.** \( t\mathcal{M} \) \textbf{is a rigid} \( k[t] \)-linear \textbf{pre-abelian tensor category}.\(^{(2)}\)

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\(^{(2)}\)That is: a rigid \( k[t] \)-linear \( \otimes \)-category that is also pre-abelian. Appendix c reviews terminology on \( \otimes \)-categories.
Proof. The category $t\mathcal{M}$ is evidently a $k[t]$-linear tensor category.

For $t\mathcal{M}$ to be pre-abelian it needs to have kernels and cokernels. All morphisms in $t\mathcal{M}$ become morphisms of effective $t$-motifs after an appropriate shift with a tensor power of the Carlitz motif. It is thus sufficient to show that $t\mathcal{M}_{\text{eff}}$ has kernels and cokernels.

Let $M_1 \to M_2$ be a morphism of effective $t$-motifs. Its group-theoretic kernel is automatically a $t$-motif and a kernel in the category $t\mathcal{M}_{\text{eff}}$. The cokernel of $f$ in the pre-abelian category of free $K[t]$-modules—the ordinary cokernel modulo torsion—inherits an action of $\sigma$ and one verifies that this defines an effective $t$-motif and a cokernel of $f$ in $t\mathcal{M}_{\text{eff}}$. Hence $t\mathcal{M}$ is pre-abelian.

That it is rigid means by definition that there is a bifunctor $\text{Hom}$ for which the stated adjunction formula, the reflexivity and the distributivity hold.

2.2.6. Proof of the adjunction formula. After a shift by powers of the Carlitz motif, we may assume that the $M_1$, $M_2$, and $M_3$ occurring in (2.4) are effective $t$-motifs and that the $\text{Hom}$ that occurs in the adjunction formula is well defined in the sense of 2.1.2.

There is certainly a natural isomorphism of $K[t]$-modules

$$\text{Hom}_{K[t]}(M_1 \otimes M_2, M_3) = \text{Hom}_{K[t]}(M_1, \text{Hom}(M_2, M_3)), \quad (2.7)$$

mapping an element $f$ of the left hand side to

$$g : m_1 \mapsto (m_2 \mapsto f(m_1 \otimes m_2)).$$

The map $f$ is a morphism of $t$-motifs when it satisfies

$$f \circ (\sigma_1 \otimes \sigma_2) = \sigma_3 \circ f \quad (2.8)$$

while $g$ is a morphism of $t$-motifs when it satisfies

$$g(\sigma_1(m_1)) \circ \sigma_2 = \sigma_3 \circ g(m_1). \quad (2.9)$$

Observe that (2.8) is verified if and only if (2.9) is, and hence that the bijection (2.7) restricts to the claimed adjunction formula (2.4).

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2.2.7. **Proof of the reflexivity and distributivity.** First of all, the existence of the maps (2.5) and (2.6) is a formal consequence of the adjunction formula.\(^5\) To see that they are isomorphisms, it suffices to note that on the level of \(K[t]\)-modules, these are just the ordinary reflexivity and distributivity homomorphisms and hence they are isomorphisms of \(K[t]\)-modules. The claims then follow at once since a morphism of \(t\)-motifs is an isomorphism if and only if it is an isomorphism on the underlying \(K[t]\)-modules. \(\square\)

### 2.3 Isogenies

2.3.1. An isogeny between two effective \(t\)-motifs \(M_1\) and \(M_2\) is by definition a morphism \(f \in \text{Hom}_\sigma(M_1, M_2)\) such that there exists a \(g \in \text{Hom}_\sigma(M_2, M_1)\) and a nonzero \(h\) in \(k[t]\) with \(fg = h\ id = gf\).

The category whose objects are effective \(t\)-motifs over \(K\) and whose hom-sets are the modules \(\text{Hom}_\sigma(\cdot, \cdot) \otimes_{k[t]} k(t)\) is denoted by \(tM^\circ_{\text{eff}}(K)\). Sometimes we will refer to its objects as effective \(t\)-motifs up to isogeny.

2.3.2. Denote by \(M(t)\) the \(K(t)\)-module \(M \otimes_{K[t]} K(t)\). The action of \(\sigma\) on \(M\) extends naturally and makes \(M(t)\) into a \(K(t)[\sigma]\)-module.

**Proposition.** The natural map

\[
\text{Hom}_\sigma(M_1, M_2) \otimes_{k[t]} k(t) \to \text{Hom}_{K(t)[\sigma]}(M_1(t), M_2(t))
\]

is an isomorphism.

Hence the functor \(M \mapsto M(t)\) is fully faithful on \(tM^\circ_{\text{eff}}\). We shall identify \(tM^\circ_{\text{eff}}\) with its image in the category of \(K(t)[\sigma]\)-modules. If we take \(M_1\) and \(M_2\) in the Proposition to be the unit \(t\)-motif \(1\), we obtain that the field of invariants \(K(t)^\sigma\) equals \(k(t)\).

**Proof of the Proposition.**\(^4\) Note that the map is \(k(t)\)-linear. Injectivity is clear.

\(^5\)See §1 of [Deligne and Milne 1982].

\(^4\)See also the pre-print [Papanikolas 2005].
To show surjectivity, choose $K[t]$-bases for $M_1$ and $M_2$ and express the action of $\sigma$ on them through matrices $S_1$ and $S_2$ (as in 1.2.5.) Expressed on the induced bases for $M_1(t)$ and $M_2(t)$, a $K(t)[\sigma]$-homomorphism from $M_1(t)$ to $M_2(t)$ is a matrix $F$ over $K(t)$ that satisfies

$$S_2^{-1}FS_1 = \tau(F). \quad (2.10)$$

Let $h$ be the minimal common denominator of the entries of $F$, that is, the minimal monic polynomial in $K[t]$ with the property that $hF$ has entries in $K[t]$. The minimal common denominator of the entries of the right-hand-side $\tau(F)$ is $\tau(h)$ and the minimal common denominator of the left hand side is $(t - \theta)^r h$ for some $r$. Equating them yields $r = 0$ and $\tau(h) = h$, hence the Proposition.

2.3.3. This Proposition has an important consequence:

**Corollary.** $tM_\text{eff}^\circ$ is an abelian $k(t)$-linear tensor category. $tM^\circ$ is a rigid abelian $k(t)$-linear tensor category.

Note that $tM_\text{eff}$ is certainly not abelian. Indeed, take for example the multiplication-by-$t$ map from 1 to 1. Even though both its cokernel and kernel are trivial in $tM_\text{eff}$, it is not an isomorphism.

**Proof of the Corollary.** The kernels and cokernels in $tM_\text{eff}^\circ$ are just the ordinary group-theoretic kernels and cokernels in the category of left $K(t)[\sigma]$-modules, and it is clear that a morphism whose kernel and cokernel vanish is an isomorphism. Hence $tM_\text{eff}^\circ$ is abelian.

That $tM^\circ$ is abelian is implied by the abelianness of $tM_\text{eff}^\circ$ and that it is rigid is implied by the rigidity of $tM$, the required properties of $\text{Hom}$ are preserved under extension of scalars from $k[t]$ to $k(t)$.  

2.3.4. Let $K$ be algebraically closed. Clearly the category $\mathcal{C}$ of finite dimensional $K(t)$-modules equipped with a surjective semi-linear endomorphism $\sigma$ is a rigid $k(t)$-linear $\otimes$-category. By Proposition 2.3.2 the functor

$$M \mapsto M(t)$$

is fully faithful on $tM_\text{eff}^\circ$. Thus we could have defined $tM^\circ$ to be the rigid $\otimes$-subcategory of $\mathcal{C}$ generated by $tM_\text{eff}^\circ$.  

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In [PAPANIKOLAS 2005] a rigid \( \otimes \)-category of \( t \)-motifs is defined as the rigid subcategory of \( C \) generated by the effective \( t \)-motifs that are finitely generated over \( K[\sigma] \). The resulting category embeds naturally in \( tM^\circ \). I do not know if this embedding is an equivalence.\(^{(5)}\)

2.4 Summary

2.4.1. We have a ‘commutative square’ of categories

\[
\begin{array}{ccc}
  tM & \xrightarrow{-\otimes k(t)} & tM^\circ \\
\uparrow & & \uparrow \\
  tM_{\text{eff}} & \xrightarrow{-\otimes k(t)} & tM_{\text{eff}}^\circ 
\end{array}
\]

The vertical arrows are fully faithful embeddings and the horizontal arrows denote extension of scalars on the Hom modules. These categories have the following properties:

- \( tM_{\text{eff}} \) is a pre-abelian \( k[t] \)-linear tensor category,
- \( tM \) is a pre-abelian rigid \( k[t] \)-linear tensor category,
- \( tM_{\text{eff}}^\circ \) is an abelian \( k(t) \)-linear tensor category,
- \( tM^\circ \) is a rigid abelian \( k(t) \)-linear tensor category.

\(^{(5)}\)Added in proof: It seems that this is indeed an equivalence.