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ON THE PARAMETRIZATION OF ALL REGULARLY IMPLEMENTING AND STABILIZING CONTROLLERS*

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Abstract. In this paper we deal with problems of controller parametrization in the context of behavioral systems. Given a full plant behavior, a subbehavior of the manifest plant behavior is called *regularly implementable* if it can be achieved as the controlled behavior resulting from the interconnection of the full plant behavior with a suitable controller behavior, in such a way that the controller does not impose restrictions that are already present in the plant. We establish a parametrization of *all* controllers that regularly implement a given behavior. We also obtain a parametrization of all stabilizing controllers.

Key words. linear differential system behaviors, implementability, stabilization, parametrization of controllers, Youla parametrization

AMS subject classifications. 93B05, 93B07, 93B50, 93B52, 93C05, 93C15, 93D15, 93D20.

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1. Introduction.

An important issue in the behavioral approach to control is that of *implementability*. The concept of implementability has been successfully applied to resolve a number of important synthesis problems in the behavioral approach, in particular the synthesis of dissipative systems [11] and the behavioral versions of the problems of pole placement and stabilization [2]. The concept was also studied in [3], for nD behaviors in [7], and for general behaviors in [8]. A nice overview can also be found in [1]. Implementability deals with the issue which system behaviors can be achieved ("implemented") by interconnecting a given system with a controller, and is thus concerned with the limits of performance of a given plant. In the behavioral framework this is made precise as follows. Given is a system behavior (plant) with two types of variables: the variable w to be controlled, and the variable c (the control variable) on which we are allowed to put restrictions. A controller for our plant behavior is an additional system behavior, called controller behavior. Interconnecting the plant with the controller simply means requiring c to be an element of the controller behavior. The space of all w trajectories that are possible after interconnecting the plant behavior with the controller behavior forms the so-called *manifest controlled behavior*. A behavior is called implementable (w.r.t. the given plant behavior) if it can be achieved as manifest controlled behavior in this way. In the contexts of synthesis of dissipative systems, pole placement, and stabilization, an important role is also played by *regular implementability*. A given behavior is called regularly implementable if it can be achieved by a controller behavior that does not impose restrictions on the control variable that are already present in the plant; equivalently, the number of outputs of the associated full controlled behavior is equal to the sum of the number of outputs of the plant and the number of outputs of the controller. In [11], for a given plant behavior a characterization was given of all implementable behaviors, and

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in [2] a characterization was given of all regularly implementable behaviors. Once a given behavior is (regularly) implementable, it is important to know which controller behaviors implement it. In this paper we establish a parametrization of all controller behaviors that regularly implement a given behavior.

A controller is called a stabilizing controller if it regularly implements a stable behavior. In [2], conditions on the plant behavior were given for the existence of a stabilizing controller. Once a stabilizing controller exists, it is important to have a parametrization of all stabilizing controllers. In fact, a result of paramount importance in feedback control is the celebrated Youla parametrization of all stabilizing controllers; see [12] and [9]. In this paper we find a parametrization of all stabilizing controllers in the behavioral framework. In this framework, the parametrization problem was considered before in [4] for the so-called full interconnection case. Here, we resolve the general, partial interconnection case.

The outline of this paper is as follows. In section 2 we review the basic material on linear differential systems needed in this paper and some less known material on left and right minimal annihilators. In section 3 we formulate the main problems that we treat in this paper, the problems of parametrizing all regularly implementing controllers and all stabilizing controllers. We also give some motivating examples there. Section 4 deals with the special case of full interconnection. We review the basic facts on implementability by full interconnection and present a new condition for regular implementability. Next, we solve the problems of parametrizing all regularly implementing controllers and all stabilizing controllers, both for the full interconnection case. Section 5 solves the parametrization problem for the case that in the plant behavior the control variable is observable from the manifest variable. Next, in section 6, we reduce the general, nonobservable, case to the observable case by describing two reduction steps. Then, in section 7 we give a parametrization of all stabilizing controllers, first in the observable case, and next by reducing the general case to the observable case. Finally, in section 8, we provide a number of worked out examples to illustrate the theory of this paper.

In several places in this paper we denote by $\mathrm{col}(A, B)$ the matrix obtained by stacking A over B. A polynomial p is called Hurwitz if all its zeros are contained in the open left half complex plane $\mathbb{C}^- := \{\lambda \in \mathbb{C} \mid \mathrm{Re}(\lambda) < 0\}$. A square polynomial matrix P is called Hurwitz if $\det(P)$ is Hurwitz. Finally, we denote $\mathbb{C}^+ := \{\lambda \in \mathbb{C} \mid \mathrm{Re}(\lambda) \geq 0\}$.

2. Linear differential systems. In the behavioral approach to linear systems, a dynamical system is given by a triple $\Sigma = (\mathbb{R}, \mathbb{R}^q, \mathfrak{B})$, where $\mathbb{R}$ is the time axis, $\mathbb{R}^q$ is the signal space, and the behavior $\mathfrak{B}$ is a subset of $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q)$ (the space of all infinitely often differentiable functions from $\mathbb{R}$ to $\mathbb{R}^q$) consisting of all solutions of a set of higher order, linear, constant coefficient differential equations. More precisely, there exists a real polynomial matrix R with q columns such that $\mathfrak{B} = \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q) \mid R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0\}$. Any such dynamical system Σ is called a *linear differential system*. The set of all linear differential systems with q variables is denoted by $\mathcal{L}^q$. Since the behavior $\mathfrak{B}$ of the system Σ is the central item, we will speak mostly about the system $\mathfrak{B} \in \mathcal{L}^q$ (instead of $\Sigma \in \mathcal{L}^q$). Henceforth, in this paper we will suppress the notation $\frac{\mathrm{d}}{\mathrm{d}t}$ and write Rw instead of $R(\frac{\mathrm{d}}{\mathrm{d}t})w$.

The behavioral approach makes a distinction between the behavior as the space of all solutions to a set of (differential) equations and the set of equations itself. A set of equations in terms of which the behavior is defined is called a *representation* of the behavior. If a behavior $\mathfrak{B}$ is represented by $Rw = 0$, then we call this a kernel representation of $\mathfrak{B}$ and often write $\mathfrak{B} = \ker(R)$.

Suppose R has p rows. Then the kernel representation is said to be *minimal* if every other kernel representation of $\mathfrak{B}$ has at least p rows. A given kernel representation $\mathfrak{B} = \ker(R)$ is minimal if and only if the polynomial matrix R has full row rank (see [5, Theorem 3.6.4]). The number of rows in any minimal kernel representation of $\mathfrak{B}$ is denoted by $\mathtt{p}(\mathfrak{B})$. This number is called the *output cardinality* of $\mathfrak{B}$. It corresponds to the number of outputs in any input/output representation of $\mathfrak{B}$ (see [5, section 3.3]).

A linear differential system is *defined* as the solution space $\mathfrak{B}$ of a differential equation of the form $Rw = 0$. However, such a system can have other representations as well. One of these is the image representation. Let M be a real polynomial matrix with q rows and, say, l columns. If

$$\mathfrak{B} = \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q) \mid \text{ there exists } \ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^l) \text{ such that } w = M\ell\},$$

then we call $w = M\ell$ an image representation of the system behavior $\mathfrak{B}$ and often write $\mathfrak{B} = \text{im}(M)$. The linear differential system $\mathfrak{B}$ has an image representation if and only if it is *controllable* (see [5, Theorem 6.6.1]). If $\mathfrak{B} = \ker(R)$, then $\mathfrak{B}$ is controllable if and only if the rank of the complex matrix $R(\lambda)$ is independent of λ for $\lambda \in \mathbb{C}$. If $\mathfrak{B}$ is a linear differential system, then we denote by $\mathfrak{B}_{\text{cont}}$ the largest controllable subbehavior of $\mathfrak{B}$ (see [5, Theorem 5.2.14]). The system $\mathfrak{B}$ is *stabilizable* if and only if the rank of $R(\lambda)$ is independent of λ for $\lambda \in \mathbb{C}^+$ (see [5, Theorem 5.2.30]).

We now recall the concepts of minimal (left and right) annihilator. If M is a polynomial matrix, then the polynomial matrix R is called *a minimal left annihilator* (MLA) of M if $\text{im}(M) = \ker(R)$. Since, for any M, $\text{im}(M)$ is a linear differential behavior (see [5, section 6.6]), and since every linear differential behavior has a kernel representation $\ker(R)$, every M has an MLA. For a given polynomial matrix R, the polynomial matrix M is called *a minimal right annihilator* (MRA) of R if $\text{im}(M) = (\ker(R))_{\text{cont}}$. Given R, the controllable part $(\ker(R))_{\text{cont}}$ of $\ker(R)$ admits an image representation $\text{im}(M)$ (see [5, Theorem 6.6.1]). Thus every R has an MRA.

In many places in this paper, we will use, for a given M, an MLA of R *with full row rank*. If the given M has full row rank, say q, then for consistency we *define* such a full row rank MLA as the "void" matrix R with 0 rows and q columns. Likewise we often use, for a given R, an MRA with the property that $M(\lambda)$ has full column rank for all λ. If R has full column rank q, we define such an M to be the void matrix with q rows and 0 columns. In that case, if K is a given matrix with q columns, then KM is again void. Finally, we use the convention that if R is void, with zero rows and q columns, then a full column rank MRA is given by the $q \times q$ identity matrix I_q.

Note that if R is a full row rank MLA, say of M, then automatically $R(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}$. Indeed, if $\ker(R) = \text{im}(M)$, then $\ker(R)$ is controllable so its rank is independent of λ. As an immediate consequence of this, if R is a full row rank MLA, then there exists a polynomial matrix R' such that $\text{col}(R, R')$ is unimodular.

Next, we review some facts on observability. Suppose $\mathfrak{B} \in \mathfrak{L}^q$ with system variable $w = (w_1, w_2)$, where w_1 and w_2 take values in $\mathbb{R}^{q_1}$ and $\mathbb{R}^{q_2}$, respectively, $q = q_1 + q_2$. We call w_2 *observable from* w_1 if $(w_1, w_2), (w_1, w_2') \in \mathfrak{B}$ implies $w_2 = w_2'$. We call w_2 *detectable from* w_1 if $(w_1, w_2), (w_1, w_2') \in \mathfrak{B}$ implies $\lim_{t\to\infty}(w_2(t) - w_2'(t)) = 0$. If $\mathfrak{B}$ is represented by $R_1 w_1 + R_2 w_2 = 0$, then w_2 is observable from w_1 if and only if $R_2(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$ ([5, Theorem 5.3.3]). Also, w_2 is detectable from w_1 if and only if $R_2(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}^+$ ([5, Theorem 5.3.17]).

We now review some facts on elimination. Again, let $\mathfrak{B} \in \mathfrak{L}^q$ with system variable $w = (w_1, w_2)$. Let P_{w_1} denote the projection onto the w_1-component. Then the set $P_{w_1}\mathfrak{B}$ of all w_1 for which there exists w_2 such that $(w_1, w_2) \in \mathfrak{B}$ is again a linear differential system. In this paper we denote $P_{w_1}\mathfrak{B}$ by $\mathfrak{B}_{w_1}$. We call $\mathfrak{B}_{w_1}$ the

system obtained by eliminating w_2 from $\mathfrak{B}$ ([5, section 6.2.2]). If $\mathfrak{B} = \ker(R_1 \ \ R_2)$, then a representation of $\mathfrak{B}_{w_1}$ is obtained as follows: choose a unimodular matrix V such that $VR_2 = \mathrm{col}(R_{12}, 0)$, with R_{12} full row rank, and conformably partition $VR_1 = \mathrm{col}(R_{11}, R_{21})$. Then $\mathfrak{B}_{w_1} = \ker(R_{21})$ (see [5, section 6.2.2]). This is due to the fact that R_{12} is a full row rank polynomial matrix, which therefore induces a surjective differential operator from $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{q_2})$ to $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^r)$, with $r := \mathrm{rowdim}(R_{12})$.

Sometimes, system behaviors are represented by latent variable representations of the form $Rw = M\ell$, with *latent variable* ℓ. Of course, this equation represents the full behavior of all (w, ℓ) that satisfy the differential equation. The w-behavior $\mathfrak{B}$ obtained by eliminating ℓ from this full behavior is called the *manifest behavior* associated with this latent variable representation. On several occasions in this paper we need to compute the output cardinality $\mathtt{p}(\mathfrak{B})$ of this behavior in terms of the polynomial matrices R and M. It was shown in [2, Lemma 8], that $\mathtt{p}(\mathfrak{B}) = \mathrm{rank}(R, M) - \mathrm{rank}(M)$.

Finally, we recall some facts on autonomous systems. If the behavior $\mathfrak{B}$ has the property that $\mathtt{p}(\mathfrak{B}) = q$ (the number of variables; thus all variables are output), then we call $\mathfrak{B}$ *autonomous.* An autonomous system is called *stable* if $\lim_{t\to\infty} w(t) = 0$ for all $w \in \mathfrak{B}$ (see [5, section 7.2]).

3. Problem formulation. In this section we introduce the main problems that are considered in this paper. We first briefly review the relevant definitions on interconnection and implementability. For an extensive treatment, see [2]. Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{q+k}$ be a linear differential system, with system variable (w, c), where w takes its values in $\mathbb{R}^q$ and c in $\mathbb{R}^k$. The variables w should be interpreted as the variables to be controlled, and the variables c are those through which we can interconnect the plant to a controller and are called *the control variables.* Let $\mathcal{C} \in \mathfrak{L}^k$ be a controller behavior, with variable c.

DEFINITION 1. *The interconnection of* $\mathcal{P}_{\text{full}}$ *and* $\mathcal{C}$ *through* c *is defined as the system behavior* $\mathcal{K}_{\text{full}}(\mathcal{C}) \in \mathfrak{L}^{q+k}$, *given by* $\mathcal{K}_{\text{full}}(\mathcal{C}) = \{(w, c) \mid (w, c) \in \mathcal{P}_{\text{full}}$ *and* $c \in \mathcal{C}\}$. *This behavior is called* the full controlled behavior.

DEFINITION 2. *The interconnection of* $\mathcal{P}_{\text{full}}$ *and* $\mathcal{C}$ *through* c *is called* regular *if the output cardinality of the full controlled behavior is the sum of the output cardinalities of the plant and the controller, i.e.,* $\mathtt{p}(\mathcal{K}_{\text{full}}(\mathcal{C})) = \mathtt{p}(\mathcal{P}_{\text{full}}) + \mathtt{p}(\mathcal{C})$.

This condition is equivalent to the following: $\mathcal{C}$ does not reimpose restrictions on $\mathcal{K}_{\text{full}}(\mathcal{C})$ that are already present in $\mathcal{P}_{\text{full}}$ (see also [7, Definition 3.1]).

DEFINITION 3. *The behavior* $(\mathcal{K}_{\text{full}}(\mathcal{C}))_w \in \mathfrak{L}^q$ *that is obtained by eliminating* c *from* $\mathcal{K}_{\text{full}}(\mathcal{C})$ *is called the* manifest controlled behavior.

Let $\mathcal{K} \in \mathfrak{L}^q$ be a given behavior, which should be interpreted as a "desired" behavior. A fundamental question is whether this $\mathcal{K}$ can be achieved as controlled behavior as stated in the following definition.

DEFINITION 4. *If there exists* $\mathcal{C} \in \mathfrak{L}^k$ *such that* $\mathcal{K} = (\mathcal{K}_{\text{full}}(\mathcal{C}))_w$, *then* $\mathcal{K}$ *is called* implementable by partial interconnection (through c, w.r.t. $\mathcal{P}_{\text{full}}$). *If there exists* $\mathcal{C} \in \mathfrak{L}^k$ *such that* $\mathcal{K} = (\mathcal{K}_{\text{full}}(\mathcal{C}))_w$ *and* $\mathtt{p}(\mathcal{K}_{\text{full}}(\mathcal{C})) = \mathtt{p}(\mathcal{P}_{\text{full}}) + \mathtt{p}(\mathcal{C})$, *then we call* $\mathcal{K}$ regularly implementable by partial interconnection (through c, w.r.t. $\mathcal{P}_{\text{full}}$).

Necessary and sufficient conditions for a given $\mathcal{K} \in \mathfrak{L}^q$ to be (regularly) implementable by partial interconnection have been obtained in [11] and [2]. We will review these conditions in section 5.

We now formulate the first main problem that we deal with in this paper. Let $\mathcal{P}_{\text{full}} = \ker(R_1 \ \ R_2)$ be a minimal representation of the plant. Let $\mathcal{K} = \ker(K)$ be a minimal representation of the desired behavior. Then the problem is as follows: *give a*

parametrization, in terms of the polynomial matrices R_1, R_2 and K, of all *polynomial matrices C such that the controller* $\ker(C)$ *regularly implements $\mathcal{K}$.*

Example 5. Consider the plant behavior $\mathcal{P}_{\text{full}}$ with manifest variable $w = (w_1, w_2)$ and control variable $c = (c_1, c_2)$ represented by

$$\begin{aligned} w_1 + \dot{w}_2 + \dot{c}_1 + c_2 &= 0, \\ c_1 + c_2 &= 0. \end{aligned}$$

Clearly, $(\mathcal{P}_{\text{full}})_w = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^2)$. For the desired behavior $\mathcal{K}$ we take $\mathcal{K} = \{(w_1, w_2) \mid w_1 + \dot{w}_2 = 0\}$. The following controller regularly implements $\mathcal{K}$ through c: $\mathcal{C} = \{(c_1, c_2) \mid \dot{c}_1 + c_2 = 0\}$. Also every controller represented by $kc_1 + c_2 = 0$, with $k \neq 1$, regularly implements $\mathcal{K}$. We would like to find a parametrization of *all* 1×2 polynomial matrices $C(\xi) = (C_1(\xi) \quad C_2(\xi))$ such that $\mathcal{C} = \ker(C_1 \quad C_2)$ regularly implements $\mathcal{K}$.

We now recall the definition of a stabilizing controller (see [2, section 3]).

Definition 6. *Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{q+k}$. The controller $\mathcal{C} \in \mathfrak{L}^k$ is said to stabilize $\mathcal{P}_{\text{full}}$ through c if the manifest controlled behavior $(\mathcal{K}_{\text{full}}(\mathcal{C}))_w$ is stable and the interconnection of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$ is regular. The controller $\mathcal{C}$ is then called a* stabilizing controller.

The following result was shown in [2, Theorem 6].

Proposition 7. *Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{q+k}$. There exists a stabilizing controller $\mathcal{C}$ if and only if $(\mathcal{P}_{\text{full}})_w$ is stabilizable and in $\mathcal{P}_{\text{full}}$ w is detectable from c.*

We now formulate the second main problem that is solved in this paper. Let $\mathcal{P}_{\text{full}} = \ker(R_1 \quad R_2)$ be a minimal representation of the plant. Then the problem is as follows: *give a parametrization, in terms of the polynomial matrices R_1 and R_2, of* all *polynomial matrices C such that the controller* $\ker(C)$ *is a stabilizing controller.*

Example 8. Consider the full plant behavior $\mathcal{P}_{\text{full}}$ represented by

$$\begin{aligned} w_1 + \dot{w}_2 + \dot{c}_1 + c_2 &= 0, \\ w_2 + c_1 + c_2 &= 0, \\ \dot{c}_1 + c_1 + \dot{c}_2 + c_2 &= 0. \end{aligned}$$

A stabilizing controller is given by $\mathcal{C} = \{(c_1, c_2) \mid \dot{c}_2 + 2c_1 + c_2 = 0\}$. Indeed, by eliminating c from the full controlled behavior $\mathcal{K}_{\text{full}}(\mathcal{C})$ (as described in section 2) we find that $(\mathcal{K}_{\text{full}}(\mathcal{C}))_w = \ker(R)$, with

$$R(\xi) = \begin{pmatrix} 0 & \xi + 1 \\ -1 & 2 \end{pmatrix},$$

which is Hurwitz. Yet another class of stabilizing controllers is represented by $C(\xi) = (\xi(\xi+1) + k, \xi + 1 + k)$, $k \in \mathbb{R}$. We want to find a parametrization of *all* 1×2 polynomial matrices $C(\xi)$ such that $\ker(C)$ is a stabilizing controller.

For the special case of full interconnection (see section 4), the problem of parametrizing all stabilizing controllers was considered earlier in [4]. Of course, in the context of feedback stabilization this parametrization problem dates back to the famous result of Youla [12].

4. Implementability and controller parametrization: The full interconnection case. In this section we treat the full interconnection case. First, we briefly review some facts on implementability and present a new condition for regular implementability. Then we establish a parametrization of all controllers that regularly implements a given behavior. Finally, we parametrize all stabilizing controllers.

Let $\mathcal{P} \in \mathfrak{L}^q$ be a plant behavior. A controller for $\mathcal{P}$ is a system behavior $\mathcal{C} \in \mathfrak{L}^q$. The *full interconnection* of $\mathcal{P}$ and $\mathcal{C}$ is defined as the system with behavior $\mathcal{P} \cap \mathcal{C}$. This *controlled behavior* is again an element of $\mathfrak{L}^q$. The full interconnection is called *regular* if $\mathtt{p}(\mathcal{P} \cap \mathcal{C}) = \mathtt{p}(\mathcal{P}) + \mathtt{p}(\mathcal{C})$.

Let $\mathcal{K} \in \mathfrak{L}^q$ be a given behavior, to be interpreted as a "desired" behavior. If $\mathcal{K}$ can be achieved as controlled behavior, i.e., if there exists $\mathcal{C} \in \mathfrak{L}^q$ such that $\mathcal{K} = \mathcal{P} \cap \mathcal{C}$, then we call $\mathcal{K}$ *implementable by full interconnection* (w.r.t. $\mathcal{P}$). If $\mathcal{K}$ can be achieved by regular interconnection, i.e., if there exists $\mathcal{C}$ such that $\mathcal{K} = \mathcal{P} \cap \mathcal{C}$ and $\mathtt{p}(\mathcal{P}\cap\mathcal{C}) = \mathtt{p}(\mathcal{P}) + \mathtt{p}(\mathcal{C})$, then we call $\mathcal{K}$ *regularly implementable by full interconnection*. Obviously, a given $\mathcal{K} \in \mathfrak{L}^q$ is implementable by full interconnection w.r.t. $\mathcal{P}$ if and only if $\mathcal{K} \subseteq \mathcal{P}$. Indeed, if $\mathcal{K} \subseteq \mathcal{P}$, then with "controller" $\mathcal{C} = \mathcal{K}$ we have $\mathcal{K} = \mathcal{P} \cap \mathcal{C}$. Thus, if $\mathcal{P} = \ker(R)$ and $\mathcal{K} = \ker(K)$ are minimal representations, then $\mathcal{K}$ is implementable w.r.t. $\mathcal{P}$ if and only if there exists a polynomial matrix F such that $R = FK$. The property of *regular* implementability turns out to be equivalent with the existence of such a polynomial matrix F with, in addition, $F(\lambda)$ full row rank for all λ, as stated in the following theorem.

THEOREM 9. *Let $\mathcal{P}, \mathcal{K} \in \mathfrak{L}^q$. Let $\mathcal{P} = \ker(R)$ and $\mathcal{K} = \ker(K)$ be minimal representations. Then the following statements are equivalent:*

1. *$\mathcal{K}$ is regularly implementable w.r.t. $\mathcal{P}$ by full interconnection.*
2. *There exists a polynomial matrix F with $F(\lambda)$ full row rank for all $\lambda \in \mathbb{C}$, such that $R = FK$.*
3. *$\mathcal{K} + \mathcal{P}_{\text{cont}} = \mathcal{P}$.*

Here, $\mathcal{P}_{\text{cont}}$ denotes the controllable part of $\mathcal{P}$.

Proof. The equivalence of statements 1 and 3 was proven in [2, Lemma 7]. We will only prove the equivalence of 1 and 2 here.

$(1 \Rightarrow 2)$ Let C be such that $\binom{R}{C}w = 0$ is a minimal representation of $\mathcal{K}$. Then there exists a unimodular U such that $\mathrm{col}(R, C) = UK$. This implies $R = FK$, with F consisting of the upper rows of U.

$(2 \Rightarrow 1)$ Assume $R = FK$. Let V be such that $\mathrm{col}(F, V)$ is unimodular. Define $C = VK$. Then $\binom{R}{C}w = 0$ is a minimal representation of $\mathcal{K}$, and thus $\mathcal{K}$ is regularly implemented by the controller $\mathcal{C} = \ker(C)$. $\square$

From the above, for a given regularly implementable $\mathcal{K}$ it is easy to obtain a controller that regularly implements it. Indeed, if $R = FK$ with $F(\lambda)$ full row rank for all λ, let V be such that $\mathrm{col}(F, V)$ is unimodular. Then clearly the controller $\ker(VK)$ does the job. Note that this construction requires the representation $\mathcal{K} = \ker(K)$ to be minimal. In the subsequent development, in particular in section 5, we will need an expression in terms of K for a regularly implementing controller in the general case that K is *not* full row rank. This issue is dealt with in the following lemma.

LEMMA 10. *Let $\mathcal{P}, \mathcal{K} \in \mathfrak{L}^q$. Let K be such that $\mathcal{K} = \ker(K)$ (not necessarily minimal), and let R be such that $\mathcal{P} = \ker(R)$ is a minimal representation. Construct a polynomial matrix W as follows:*

1. *Let M be an MRA of R such that $M(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$;*
2. *Let Q be a full row rank MLA of KM;*
3. *Let W be a polynomial matrix such that $\mathrm{col}(Q, W)$ is unimodular.*

Then $\mathcal{K}$ is regularly implementable by full interconnection w.r.t. $\mathcal{P}$ if and only if

$$\begin{pmatrix} R \\ WK \end{pmatrix} w = 0 \tag{1}$$

is a minimal representation of $\mathcal{K}$. A controller that regularly implements $\mathcal{K}$ is then represented by the minimal representation $WKw = 0$.

Proof. Factor $R = DR_1$, with D square and nonsingular and $R_1(\lambda)$ full row rank for all λ. Then $\mathcal{P}_{\text{cont}} = \ker(R_1)$ (see [5, Theorem 5.2.14]). Let M^+ be a polynomial left inverse of M and R_1^+ a polynomial right inverse of R_1. Define $S := M^+(I - R_1^+ R_1)$. By direct verification we then have $\binom{R_1}{S}(R_1^+ \;\; M) = I_q$. It follows that $R_1^+ R_1 + MS = I_q$. We claim that there exists a polynomial matrix T such that $TR = QK$. In order to prove this, we show that $\ker(R) \subseteq \ker(QK)$. Indeed, let w be such that $Rw = 0$. Since $\mathcal{P} = \mathcal{K} + \mathcal{P}_{\text{cont}}$, there exist $w_1 \in \mathcal{K}$ and $w_2 \in \mathcal{P}_{\text{cont}}$ such that $w = w_1 + w_2$. Hence, since $w_2 \in \ker(R_1)$ and $QKM = 0$, $QKw = QKw_2 = QK(R_1^+ R_1 + MS)w_2 = 0$. Next, note that

$$\begin{pmatrix} I_p & 0 \\ -T & Q \\ 0 & W \end{pmatrix} \begin{pmatrix} R \\ K \end{pmatrix} = \begin{pmatrix} R \\ 0 \\ WK \end{pmatrix}.$$

The leftmost matrix in this equation is unimodular. Thus we have that $(w \in \mathcal{K})$ if and only if ($Kw = 0$ and $Rw = 0$), which in turn is equivalent to ($WKw = 0$ and $Rw = 0$). This proves that (1) is indeed a kernel representation of $\mathcal{K}$.

Finally, we show that the representation (1) is minimal. Indeed,

$$\begin{pmatrix} R \\ WK \end{pmatrix} \begin{pmatrix} R_1^+ & M \end{pmatrix} = \begin{pmatrix} D & 0 \\ WKR_1^+ & WKM \end{pmatrix}.$$

It is easily seen that, by construction, WKM has full row rank. Since D is nonsingular, we conclude that $\text{col}(R, WK)$ must have full row rank as well. ∎

We now establish, for a given plant $\mathcal{P} \in \mathfrak{L}^q$ and a given regularly implementable behavior $\mathcal{K} \in \mathfrak{L}^q$, a parametrization of all controllers $\mathcal{C} \in \mathfrak{L}^q$ that regularly implement $\mathcal{K}$ by full interconnection. This problem was considered before in [4] for the case where the plant behavior $\mathcal{P}$ is controllable and the given subbehavior $\mathcal{K}$ is autonomous. Here, we will establish a parametrization for arbitrary $\mathcal{P}$ and arbitrary (regularly implementable) $\mathcal{K}$.

THEOREM 11. *Let $\mathcal{P} \in \mathfrak{L}^q$, with minimal representation $\mathcal{P} = \ker(R)$. Let $\mathcal{K} \in \mathfrak{L}^q$ be regularly implementable by full interconnection, and let $\mathcal{K} = \ker(K)$. Construct a polynomial matrix W as in Lemma 10. Then for any $\mathcal{C} \in \mathfrak{L}^q$, $\mathcal{C} = \ker(C)$, the following statements are equivalent:*

1. *$\mathcal{C} = \ker(C)$ is a minimal representation, and $\mathcal{C}$ regularly implements $\mathcal{K}$.*
2. *There exist a polynomial matrix F and a unimodular polynomial matrix U such that $C = FR + UWK$.*

Proof. ($2 \Rightarrow 1$) First note that since $\mathcal{K}$ is regularly implementable, by Lemma 10 the polynomial matrix $\text{col}(R, WK)$ has full row rank. Since

$$\begin{pmatrix} I_p & 0 \\ F & U \end{pmatrix} \begin{pmatrix} R \\ WK \end{pmatrix} = \begin{pmatrix} R \\ FR + UWK \end{pmatrix}, \tag{2}$$

this implies that also $C = FR + UWK$ has full row rank, so $Cw = 0$ is a minimal representation of $\mathcal{C}$. It also follows from (2) that $\mathcal{C}$ implements $\mathcal{K}$. Clearly, the interconnection of $\mathcal{P}$ and $\mathcal{C}$ is regular.

($1 \Rightarrow 2$) Assume that C has full row rank and that $\mathcal{C}$ regularly implements $\mathcal{K}$. Then both

$$\begin{pmatrix} R \\ C \end{pmatrix} w = 0 \quad \text{and} \quad \begin{pmatrix} R \\ WK \end{pmatrix} w = 0$$

are minimal representations of $\mathcal{K}$. Consequently, there exists a unimodular polynomial matrix $V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}$ such that $V\,\mathrm{col}(R, WK) = \mathrm{col}(R, C)$. This implies $R = V_{11}R + V_{12}WK$. Since $\mathrm{col}(R, WK)$ has full row rank, this yields $V_{11} = I_p$ and $V_{12} = 0$. It follows that V_{22} is unimodular. We also have $C = V_{21}R + V_{22}WK$. This completes the proof of the theorem. □

Note that since $\mathrm{col}(R, WK)$ has full row rank, the linear map $(F, U) \mapsto FR + UWK$ is one-one, so different parameters (F, U) yield different controllers C.

To conclude this section, we parametrize all controllers that stabilize a given plant behavior by full interconnection. The problem of stabilization by full interconnection is formulated as follows. Let $\mathcal{P} \in \mathfrak{L}^q$ be a given plant behavior. Find a controller behavior $\mathcal{C} \in \mathfrak{L}^q$ such that the controlled behavior $\mathcal{K} = \mathcal{P} \cap \mathcal{C}$ is autonomous and stable and the interconnection is regular. It was proved in [10] that such a stabilizing controller $\mathcal{C}$ exists if and only if $\mathcal{P}$ is stabilizable.

Let $\mathcal{P} = \ker(R)$ be a minimal representation. Assume that $\mathcal{P}$ is stabilizable; equivalently, $R(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}^+ = \{\lambda \in \mathbb{C} \mid \mathrm{Re}(\lambda) \geq 0\}$. The following theorem yields a parametrization of all stabilizing controllers.

THEOREM 12. *Let $\mathcal{P} \in \mathfrak{L}^q$ be stabilizable. Let $\mathcal{P} = \ker(R)$ be a minimal representation, and let R_1 be such that $\ker(R_1)$ is a minimal representation of the controllable part $\mathcal{P}_{\mathrm{cont}}$ of $\mathcal{P}$. Let C_0 be such that $\mathrm{col}(R_1, C_0)$ is unimodular. Then for any $\mathcal{C} \in \mathfrak{L}^q$ with $\mathcal{C} = \ker(C)$ the following statements are equivalent:*

1. *$\mathcal{P} \cap \mathcal{C}$ is autonomous and stable, the interconnection is regular, and the representation $\mathcal{C} = \ker(C)$ is minimal.*
2. *There exist a polynomial matrix F and a Hurwitz polynomial matrix D such that $C = FR + DC_0$.*

Proof. By combining [5, Theorems 5.2.14 and 5.2.30], note that $R = D_1R_1$, with D_1 Hurwitz. $(2 \Rightarrow 1)$ Assume that $C = FR + DC_0$. We have

$$\begin{pmatrix} R \\ C \end{pmatrix} = \begin{pmatrix} D_1 & 0 \\ FD_1 & D \end{pmatrix} \begin{pmatrix} R_1 \\ C_0 \end{pmatrix}.$$

This implies that $\mathrm{col}(R, C)$ has full row rank, so the interconnection of $\mathcal{P}$ and $\mathcal{C}$ is regular. Also, for some nonzero constant c, $\det \mathrm{col}(R, C) = c\det(D_1)\det(D)$, so the interconnection is autonomous and stable.

$(1 \Rightarrow 2)$ Since $\mathcal{P} \cap \mathcal{C}$ is stable and the interconnection is regular, $\mathrm{col}(R, C)$ is Hurwitz. Also, $\mathrm{col}(R_1, C_0)$ is unimodular, so there exist polynomial matrices F_{11}, F_{12}, F_{21}, and F_{22} such that

$$\begin{pmatrix} R \\ C \end{pmatrix} = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} \begin{pmatrix} R_1 \\ C_0 \end{pmatrix}.$$

This implies that $F_{11} = D_1$ and $F_{12} = 0$. The left-hand side of the above equation is Hurwitz, so F_{22} must be Hurwitz. The proof is completed by taking $F = F_{21}$ and $D = F_{22}$. □

The above result generalizes the result from [4] for controllable $\mathcal{P}$. If, in the above, we assume that $\mathcal{P}$ is controllable, then we can take $R = R_1$, and we recover the parametrization obtained in [4].

Remark 13. In the special case that the plant $\mathcal{P}$ to be stabilized is given together with an input/output partition $w = (y, u)$, our parametrization result of Theorem 12 specializes to the well-known Youla parametrization of all stabilizing controllers. For simplicity, assume that $\mathcal{P}$ is controllable. Assume that, in $\mathcal{P}$, G is the transfer matrix

from u to y. Let $P^{-1}Q$ be a left coprime factorization of G. Then $\mathcal{P} = \ker(P \quad -Q)$. Choose polynomial matrices X and Y such that

$$\begin{pmatrix} P & -Q \\ X & Y \end{pmatrix}$$

is unimodular. According to Theorem 12, a parametrization of all stabilizing controllers $\ker(Q_c \quad P_c)$ is given by $(Q_c \quad P_c) = F(P \quad -Q) + D(X \quad Y)$, where F is arbitrary polynomial and D is Hurwitz. In transfer matrix form this yields $C := -P_c^{-1}Q_c = -(DY - FQ)^{-1}(DX + FP) = -(Y - D^{-1}FQ)^{-1}(X + D^{-1}FP)$. Finally, denote $D^{-1}F$ by T, and let T vary over all proper stable rational matrices to obtain the original Youla parametrization $C = -(Y - TQ)^{-1}(X + TP)$ (see [12]).

5. All controllers that regularly implement a given behavior: The observable case. We now turn to the partial interconnection case. Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{q+k}$, with system variable (w, c), where w takes its values in $\mathbb{R}^q$ and c in $\mathbb{R}^k$. In this section and the next we study the problem of parametrizing, for a given regularly implementable $\mathcal{K} \in \mathfrak{L}^q$, all controllers $\mathcal{C} \in \mathfrak{L}^k$ that regularly implement $\mathcal{K}$ through c w.r.t. $\mathcal{P}_{\text{full}}$. We first assume that in the full plant behavior $\mathcal{P}_{\text{full}}$, c is observable from w. Starting from this assumption, in the present section we establish a parametrization. Then, in the next section we will lift the observability assumption and describe a parametrization for the general case.

Implementability by *partial* interconnection was already defined in section 3. Necessary and sufficient conditions for a given $\mathcal{K} \in \mathfrak{L}^q$ to be (regularly) implementable by partial interconnection can be given in terms of the manifest plant behavior and hidden behavior associated with the full plant behavior $\mathcal{P}_{\text{full}}$, which are defined as follows.

DEFINITION 14. *The* manifest plant behavior *is the behavior* $(\mathcal{P}_{\text{full}})_w \in \mathfrak{L}^q$ *obtained from* $\mathcal{P}_{\text{full}}$ *by eliminating* c*. The* hidden behavior $\mathcal{N}$ *consists of those* w *trajectories that appear in* $\mathcal{P}_{\text{full}}$ *with* c *equal to zero, i.e.,* $\mathcal{N} = \{w \mid (w, 0) \in \mathcal{P}_{\text{full}}\}$.

Conditions for implementability and regular implementability were obtained in [11, Theorem 1] and [2, Theorem 4], respectively, as follows.

PROPOSITION 15.

1. $\mathcal{K} \in \mathfrak{L}^q$ *is implementable by partial interconnection through* c *w.r.t.* $\mathcal{P}_{\text{full}}$ *if and only if* $\mathcal{N} \subseteq \mathcal{K} \subseteq (\mathcal{P}_{\text{full}})_w$.
2. $\mathcal{K} \in \mathfrak{L}^q$ *is regularly implementable by partial interconnection through* c *w.r.t.* $\mathcal{P}_{\text{full}}$ *if and only if* $\mathcal{N} \subseteq \mathcal{K} \subseteq (\mathcal{P}_{\text{full}})_w$ *and* $\mathcal{K}$ *is regularly implementable w.r.t.* $(\mathcal{P}_{\text{full}})_w$ *by full interconnection.*

For a given $\mathcal{K} \in \mathfrak{L}^q$, an important role will be played by the subbehavior $\mathcal{L}_{\text{full}}(\mathcal{K})$ of $\mathcal{P}_{\text{full}}$ defined as the interconnection of $\mathcal{P}_{\text{full}}$ and $\mathcal{K}$ through w:

$$\mathcal{L}_{\text{full}}(\mathcal{K}) := \{(w, c) \in \mathcal{P}_{\text{full}} \mid w \in \mathcal{K}\}. \tag{3}$$

The proof of the following lemma is straightforward and is left to the reader.

LEMMA 16. *If* $\mathcal{K} \subseteq (\mathcal{P}_{\text{full}})_w$*, then* $(\mathcal{L}_{\text{full}}(\mathcal{K}))_w = \mathcal{K}$.

Now let $\mathcal{K} \in \mathfrak{L}^q$ be implementable through c w.r.t. $\mathcal{P}_{\text{full}}$. We first consider the problem of finding one controller $\mathcal{C} \in \mathfrak{L}^k$ that implements $\mathcal{K}$. We will derive a representation of one such controller in terms of representations of $\mathcal{P}_{\text{full}}$ and $\mathcal{K}$. Let $\mathcal{P}_{\text{full}} = \ker(R_1 \quad R_2)$ and $\mathcal{K} = \ker(K)$. Then clearly the behavior $\mathcal{L}_{\text{full}}(\mathcal{K})$ defined by (3) is represented by

$$\begin{pmatrix} R_1 & R_2 \\ K & 0 \end{pmatrix} \begin{pmatrix} w \\ c \end{pmatrix} = 0.$$

Note that the hidden behavior $\mathcal{N}$ is equal to $\ker(R_1)$. Since $\mathcal{N} \subseteq \mathcal{K}$ there exists a polynomial matrix F such that $K = FR_1$. Now define a controller behavior $\mathcal{C}^* \in \mathfrak{L}^k$ by

$$\mathcal{C}^* := \ker(FR_2). \tag{4}$$

This controller indeed implements $\mathcal{K}$, as shown in the following lemma.

LEMMA 17. $\mathcal{K} = (\mathcal{K}_{\text{full}}(\mathcal{C}^*))_w$.

Proof. We have

$$\begin{pmatrix} R_1 & R_2 \\ 0 & FR_2 \end{pmatrix} = \begin{pmatrix} I & 0 \\ F & -I \end{pmatrix} \begin{pmatrix} R_1 & R_2 \\ K & 0 \end{pmatrix}.$$

Hence $\mathcal{L}_{\text{full}}(\mathcal{K})$ is equal to the full controlled behavior $\mathcal{K}_{\text{full}}(\mathcal{C}^*)$. The conclusion then follows from Lemma 16. $\square$

The following lemma states that if c is observable from w and if a given subbehavior of the manifest plant behavior is obtained by elimination of c from a subbehavior of $\mathcal{P}_{\text{full}}$, then this subbehavior of $\mathcal{P}_{\text{full}}$ is unique, as shown in the following lemma.

LEMMA 18. *Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{q+k}$ with system variable (w, c). Assume that c is observable from w. Let $\mathcal{K}^1_{\text{full}}, \mathcal{K}^2_{\text{full}} \in \mathfrak{L}^{q+k}$ be subbehaviors of $\mathcal{P}_{\text{full}}$. Then we have $(\mathcal{K}^1_{\text{full}})_w = (\mathcal{K}^2_{\text{full}})_w$ if and only if $\mathcal{K}^1_{\text{full}} = \mathcal{K}^2_{\text{full}}$.*

Proof. Assume $(w, c) \in \mathcal{K}^1_{\text{full}}$. Then $w \in (\mathcal{K}^1_{\text{full}})_w = (\mathcal{K}^2_{\text{full}})_w$, so there exists c' such that $(w, c') \in \mathcal{K}^2_{\text{full}}$. Thus, $(0, c - c') = (w, c) - (w, c') \in \mathcal{P}_{\text{full}}$, so $c = c'$. It follows that $(w, c) \in \mathcal{K}^2_{\text{full}}$. $\square$

LEMMA 19. *Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{q+k}$ with system variable (w, c). Let $\mathcal{K} \in \mathfrak{L}^q$ be implementable through c w.r.t. $\mathcal{P}_{\text{full}}$. Let $\mathcal{C}$ be a controller such that $\mathcal{K} = (\mathcal{K}_{\text{full}}(\mathcal{C}))_w$. Then we have the following:*

1. $\mathcal{K}_{\text{full}}(\mathcal{C}) \subseteq \mathcal{L}_{\text{full}}(\mathcal{K})$;
2. *if c is observable from w, then $\mathcal{K}_{\text{full}}(\mathcal{C}) = \mathcal{L}_{\text{full}}(\mathcal{K})$.*

Proof. (1) If $(w, c) \in \mathcal{K}_{\text{full}}(\mathcal{C})$, then $w \in (\mathcal{K}_{\text{full}}(\mathcal{C}))_w = \mathcal{K}$. Also, $(w, c) \in \mathcal{P}_{\text{full}}$. It follows that $(w, c) \in \mathcal{L}_{\text{full}}(\mathcal{K})$. (2) By Lemma 16 $(\mathcal{K}_{\text{full}}(\mathcal{C}))_w = \mathcal{K} = (\mathcal{L}_{\text{full}}(\mathcal{K}))_w$. If c is observable from w, then this implies $\mathcal{K}_{\text{full}}(\mathcal{C}) = \mathcal{L}_{\text{full}}(\mathcal{K})$. $\square$

For the special case that, in $\mathcal{P}_{\text{full}}$, c is observable from w, the following theorem reduces the problem of parametrizing all controllers that regularly implement $\mathcal{K}$ via interconnection through c w.r.t. $\mathcal{P}_{\text{full}}$ to that of parametrizing all controllers that regularly implement $(\mathcal{L}_{\text{full}}(\mathcal{K}))_c$ via *full interconnection* w.r.t. $(\mathcal{P}_{\text{full}})_c$ (see also [6, Prop. 1]).

THEOREM 20. *Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{q+k}$ with system variable (w, c). Assume that c is observable from w. Let $\mathcal{K} \in \mathfrak{L}^q$ be regularly implementable through c. Let $\mathcal{L}_{\text{full}}(\mathcal{K})$ be the interconnection of $\mathcal{P}_{\text{full}}$ and $\mathcal{K}$ through w. Let $\mathcal{C} \in \mathfrak{L}^k$. Then the following two statements are equivalent:*

1. *$\mathcal{C}$ regularly implements $\mathcal{K}$ by interconnection through c.*
2. *$\mathcal{C}$ regularly implements $(\mathcal{L}_{\text{full}}(\mathcal{K}))_c$ via full interconnection w.r.t. $(\mathcal{P}_{\text{full}})_c$.*

Proof. Let $\mathcal{P}_{\text{full}} = \ker(R_1 \;\; R_2)$ be a minimal representation. Let V be unimodular such that $VR_1 = \text{col}(R_{11}, 0)$ with R_{11} full row rank. Partition $VR_2 = \text{col}(R_{12}, R_{22})$. Then $R_{22}c = 0$ is a minimal representation of $(\mathcal{P}_{\text{full}})_c$. $\mathcal{K}$ is implementable, so (using Lemma 19), there exists a polynomial matrix, say C^* (with C^* any polynomial matrix such that $\mathcal{C}^* = \ker(C^*)$ with $\mathcal{C}^*$ given by (4)), such that

$$\begin{pmatrix} R_{11} & R_{12} \\ 0 & R_{22} \\ 0 & C^* \end{pmatrix} \begin{pmatrix} w \\ c \end{pmatrix} = 0$$

is a kernel representation of $\mathcal{L}_{\text{full}}(\mathcal{K})$. Hence a kernel representation of $(\mathcal{L}_{\text{full}}(\mathcal{K}))_c$ is given by

$$\begin{pmatrix} R_{22} \\ C^* \end{pmatrix} c = 0. \tag{5}$$

(1 $\Rightarrow$ 2) Assume that $\mathcal{C}$ regularly implements $\mathcal{K}$; i.e., $(\mathcal{K}_{\text{full}}(\mathcal{C}))_w = \mathcal{K}$ and the interconnection is regular. Let $\mathcal{C} = \ker(C)$ be a minimal representation of $\mathcal{C}$. Then the polynomial matrix

$$\begin{pmatrix} R_{11} & R_{12} \\ 0 & R_{22} \\ 0 & C \end{pmatrix} \tag{6}$$

has full row rank and (by Lemma 19) represents $\mathcal{L}_{\text{full}}(\mathcal{K})$. It is then immediate that $(\mathcal{P}_{\text{full}})_c \cap \mathcal{C} = (\mathcal{L}_{\text{full}}(\mathcal{K}))_c$. Obviously, $\text{col}(R_{22}, C)$ has full row rank, so the full interconnection of $(\mathcal{P}_{\text{full}})_c$ and $\mathcal{C}$ is regular.

(2 $\Rightarrow$ 1) Conversely, assume that $\mathcal{C}$ regularly implements $(\mathcal{L}_{\text{full}}(\mathcal{K}))_c$ w.r.t. $(\mathcal{P}_{\text{full}})_c$ by full interconnection, and that $\mathcal{C} = \ker(C)$ is a minimal representation. Then $\binom{R_{22}}{C}c = 0$ is a minimal kernel representation of $(\mathcal{L}_{\text{full}}(\mathcal{K}))_c$. Since (5) is also a kernel representation, it is easily seen that $\mathcal{K}_{\text{full}}(\mathcal{C}) = \mathcal{L}_{\text{full}}(\mathcal{K})$, which, by Lemma 16, implies $(\mathcal{K}_{\text{full}}(\mathcal{C}))_w = (\mathcal{L}_{\text{full}}(\mathcal{K}))_w = \mathcal{K}$. In addition, (6) has full row rank so the interconnection is regular. $\square$

Finally, we arrive at the main result of this section. We establish, for the case that in $\mathcal{P}_{\text{full}}$ c is observable from w, a parametrization of all controllers that regularly implement a given $\mathcal{K}$ w.r.t. $\mathcal{P}_{\text{full}}$. The idea is to compute representations of $(\mathcal{P}_{\text{full}})_c$ and $(\mathcal{L}_{\text{full}}(\mathcal{K}))_c$ and to parametrize all controllers that regularly implement $(\mathcal{L}_{\text{full}}(\mathcal{K}))_c$ by full interconnection w.r.t. $(\mathcal{P}_{\text{full}})_c$ using Theorem 11. Then, by Theorem 20, this yields a parametrization of all controllers that regularly implemement $\mathcal{K}$ through c w.r.t. $\mathcal{P}_{\text{full}}$.

THEOREM 21. *Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{q+k}$ with system variable (w, c) and c observable from w. Let $\mathcal{P}_{\text{full}} = \ker(R_1 \;\; R_2)$ be a minimal representation. Let $\mathcal{K} \in \mathfrak{L}^q$ be regularly implementable through c w.r.t. $\mathcal{P}_{\text{full}}$. Let $\mathcal{K} = \ker(K)$ be a minimal representation. Construct polynomial matrices V_1, V_2, F_1, and W as follows:*

1. *Let V_2 be a full row rank MLA of R_1.*
2. *Choose V_1 such that $\text{col}(V_1, V_2)$ is unimodular.*
3. *Let M be an MRA of V_2R_2 with $M(\lambda)$ full column rank for all λ.*
4. *Let F_1 be such that $K = F_1V_1R_1$.*
5. *Let Q be a full row rank MLA of $F_1V_1R_2M$.*
6. *Choose W such that $\text{col}(Q, W)$ is unimodular.*

Then for any $\mathcal{C} \in \mathfrak{L}^k$ with $\mathcal{C} = \ker(C)$, the following statements are equivalent:

1. *$\mathcal{C} = \ker(C)$ is a minimal representation and $\mathcal{C}$ regularly implements $\mathcal{K}$ through c w.r.t. $\mathcal{P}_{\text{full}}$.*
2. *There exist a polynomial matrix G and a unimodular U such that*

$$C = (UWF_1V_1 + GV_2)R_2.$$

Proof. V_2 is a full row rank MLA of R_1, so the unimodular matrix $V = \text{col}(V_1, V_2)$ satisfies $VR_1 = \text{col}(V_1R_1, 0)$ with V_1R_1 full row rank. Also $VR_2 = \text{col}(V_1R_2, V_2R_2)$. Obviously, $(\mathcal{P}_{\text{full}})_c = \ker(V_2R_2)$ is a minimal representation, and the hidden behavior $\mathcal{N}$ is represented by $\ker(V_1R_1)$. Let F_1 be such that $K = F_1V_1R_1$ (such F_1 exists since

$\mathcal{N} \subseteq \mathcal{K}$). Then the controller $\ker(F_1V_1R_2)$ implements $\mathcal{K}$, so by Lemma 19 $\mathcal{L}_{\text{full}}(\mathcal{K})$ is represented by

$$\begin{pmatrix} V_1R_1 & V_1R_2 \\ 0 & V_2R_2 \\ 0 & F_1V_1R_2 \end{pmatrix} \begin{pmatrix} w \\ c \end{pmatrix} = 0;$$

thus $(\mathcal{L}_{\text{full}}(\mathcal{K}))_c = \ker\left(\begin{smallmatrix} V_2R_2 \\ F_1V_1R_2 \end{smallmatrix}\right)$. Now, M is an MRA of V_2R_2 so

$$\begin{pmatrix} V_2R_2 \\ F_1V_1R_2 \end{pmatrix} M = \begin{pmatrix} 0 \\ F_1V_1R_2M \end{pmatrix}.$$

Since Q is a full row rank MLA of $F_1V_1R_2M$, clearly $\left(\begin{smallmatrix} I & 0 \\ 0 & Q \end{smallmatrix}\right)$ is a full row rank MLA of $\left(\begin{smallmatrix} V_2R_2 \\ F_1V_1R_2 \end{smallmatrix} M\right)$. If we choose W such that $\text{col}(Q, W)$ is unimodular, then the matrix

$$\begin{pmatrix} I & 0 \\ 0 & Q \\ 0 & W \end{pmatrix}$$

is unimodular. By applying Theorem 11, a parametrization of all controllers that regularly implement $(\mathcal{L}_{\text{full}}(\mathcal{K}))_c$ by full interconnection w.r.t. $(\mathcal{P}_{\text{full}})_c$ is then given by $C = GV_2R_2 + U(0 \quad W)\left(\begin{smallmatrix} V_2R_2 \\ F_1V_1R_2 \end{smallmatrix}\right) = (GV_2 + UWF_1V_1)R_2$, where G ranges over all polynomial matrices and U ranges over all unimodular polynomial matrices, of course of suitable dimensions. Finally, by Theorem 20, the same parametrization holds for all controllers $\ker(C)$ that regularly implement $\mathcal{K}$ through c w.r.t. $\mathcal{P}_{\text{full}}$. □

6. Parametrization of all regular controllers: The nonobservable case. We now treat the nonobservable case. Consider the system $\mathcal{P}_{\text{full}}$ represented by $R_1w + R_2c = 0$. We no longer assume that c is observable from w, but show that the general case can be reduced to the observable case. This reduction requires two steps. First, we reduce the general case to the case that R_2 has full column rank, and next reduce the latter to the case that $R_2(\lambda)$ has full column rank for all λ, i.e., the observable case.

1. *Reduction to the case that R_2 has full column rank.* Let V be a unimodular matrix such that $R_2 = \begin{pmatrix} \tilde{R}_2 & 0 \end{pmatrix} V$, with $\tilde{R}_2$ full column rank k'. Define $\mathcal{P}'_{\text{full}} \in \mathfrak{L}^{q+k'}$ as the system (with control variable c') represented by $R_1w + \tilde{R}_2c' = 0$.
2. *Reduction to the observable case.* Assume now that in $\mathcal{P}_{\text{full}}$ the matrix R_2 has full column rank. Let L be a square, nonsingular polynomial matrix such that $R_2 = \tilde{R}_2L$, with $\tilde{R}_2(\lambda)$ full column rank for all $\lambda \in \mathbb{C}$. Using the Smith form of R_2 it is easily seen that this is always possible. Define $\mathcal{P}'_{\text{full}}$ as the system (with control variable c') represented by $R_1w + \tilde{R}_2c' = 0$. In the system $\mathcal{P}'_{\text{full}}$, c' is observable from w.

As it will turn out, in both reduction steps, $\mathcal{K} \in \mathfrak{L}^q$ is regularly implementable through c w.r.t. $\mathcal{P}_{\text{full}}$ if and only if it is regularly implementable through c' w.r.t. $\mathcal{P}'_{\text{full}}$. Also, every controller that regularly implements $\mathcal{K}$ w.r.t. $\mathcal{P}'_{\text{full}}$ will turn out to lead to a *set* of controllers that implement $\mathcal{K}$ w.r.t. $\mathcal{P}_{\text{full}}$. In the following two subsections, we will treat the two reduction steps separately.

6.1. Reduction to the case that R_2 has full column rank. In this subsection the parametrization problem for the original plant $\mathcal{P}_{\text{full}}$ is reduced to the parametrization problem for a plant $\mathcal{P}'_{\text{full}}$ in which the R_2-matrix has full column rank. In the following, let V be a unimodular matrix such that $R_2 = (\tilde{R}_2 \quad 0)V$, with $\tilde{R}_2$ full column rank $k' = \text{rank}(R_2)$. Define $\mathcal{P}'_{\text{full}}$ as the system represented by $R_1 w + \tilde{R}_2 c' = 0$.

THEOREM 22. *Let $\mathcal{K} \in \mathfrak{L}^q$. Then $\mathcal{K}$ is regularly implementable through c w.r.t. $\mathcal{P}_{\text{full}}$ if and only if $\mathcal{K}$ is regularly implementable through c' w.r.t. $\mathcal{P}'_{\text{full}}$. Let $\mathcal{C} \in \mathfrak{L}^k$, with $\mathcal{C} = \ker(C)$ a minimal representation. Then the following two statements are equivalent:*

1. *The controller $\mathcal{C}$ regularly implements $\mathcal{K}$ through c w.r.t. $\mathcal{P}_{\text{full}}$.*
2. *There exist a polynomial matrix C_{11}, polynomial matrices C_{12} and C_{21} of full row rank, and a unimodular matrix U such that*

$$C = U \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & 0 \end{pmatrix} V \tag{7}$$

and such that the controller $\mathcal{C}_{21} = \ker(C_{21})$ regularly implements $\mathcal{K}$ through c' w.r.t. $\mathcal{P}'_{\text{full}}$.

Proof. We first prove the equivalence of statements 1 and 2. Partition $CV^{-1} = (C_1 \quad C_2)$ with the number of columns of C_1 equal to $k' = \text{rank}(R_2)$. Choose a unimodular matrix U such that $U^{-1}C_2 = \text{col}(C_{12}, 0)$ with C_{12} full row rank. Partition $U^{-1}C_1 = \text{col}(C_{11}, C_{21})$. Then we have

$$\mathcal{K}_{\text{full}}(\mathcal{C}) = \{(w, V^{-1}c') \mid (w, c') \in \ker(M)\}, \text{ where } M = \begin{pmatrix} R_1 & \tilde{R}_2 & 0 \\ 0 & C_{11} & C_{12} \\ 0 & C_{21} & 0 \end{pmatrix}.$$

Define $\mathcal{K}'_{\text{full}}(\mathcal{C}_{21}) := \{(w, c_1) \mid R_1 w + \tilde{R}_2 c_1 = 0 \text{ and } C_{21}c_1 = 0\}$, the full controlled behavior of $\mathcal{P}'_{\text{full}}$ using the controller $\mathcal{C}_{21} := \ker(C_{21})$. Using the full row rank of C_{12} we then have $(\mathcal{K}_{\text{full}}(\mathcal{C}))_w = (\ker(M))_w = (\mathcal{K}'_{\text{full}}(\mathcal{C}_{21}))_w$. From this we conclude that $\mathcal{C}$ implements $\mathcal{K}$ through c w.r.t. $\mathcal{P}_{\text{full}}$ if and only if $\mathcal{C}_{21}$ implements $\mathcal{K}$ through c' w.r.t. $\mathcal{P}'_{\text{full}}$. Furthermore, again by full row rank of C_{12}, regularity of either of the interconnections implies the same for the other one.

Finally, the statement that $\mathcal{K}$ is regularly implementable through c w.r.t. $\mathcal{P}_{\text{full}}$ if and only if $\mathcal{K}$ is regularly implementable through c' w.r.t. $\mathcal{P}'_{\text{full}}$ follows immediately from the equivalence of statements 1 and 2. ∎

6.2. Reduction to the observable case. In the previous subsection it was shown that our parametrization problem can be reduced to a problem for a plant behavior with R_2-matrix of full row rank. In the present subsection we reduce the full column rank case to the observable case. Let $\mathcal{P}_{\text{full}} = \ker(R_1 \quad R_2)$ be a minimal representation, with R_2 full column rank. Let L be square and nonsingular such that $R_2 = \tilde{R}_2 L$, with $\tilde{R}_2(\lambda)$ full column rank for all λ. Let $\mathcal{P}'_{\text{full}}$ be the (observable) system represented by $R_1 w + \tilde{R}_2 c' = 0$.

THEOREM 23. *Let $\mathcal{K} \in \mathfrak{L}^q$. Then $\mathcal{K}$ is regularly implementable through c w.r.t. $\mathcal{P}_{\text{full}}$ if and only if $\mathcal{K}$ is regularly implementable through c' w.r.t. $\mathcal{P}'_{\text{full}}$. Let $\mathcal{C} \in \mathfrak{L}^k$ with $\mathcal{C} = \ker(C)$ a minimal representation. Then the following two statements are equivalent:*

1. *The controller $\mathcal{C}$ regularly implements $\mathcal{K}$ through c w.r.t. $\mathcal{P}_{\text{full}}$.*

2. *The controller $\mathcal{C}'$ represented in latent variable representation (with latent variable ℓ) by*

$$\begin{pmatrix} I \\ 0 \end{pmatrix} c' = \begin{pmatrix} L \\ C \end{pmatrix} \ell \tag{8}$$

regularly implements $\mathcal{K}$ through c' w.r.t. $\mathcal{P}'_{\text{full}}$.

Proof. We first prove the equivalence of statements 1 and 2. The manifest controlled behavior resulting from the interconnection of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$ is equal to

$$(\mathcal{K}_{\text{full}}(\mathcal{C}))_w = \{w \mid \text{ there exists } c \text{ such that } R_1 w + \tilde{R}_2 L c = 0,\, Cc = 0\},$$

which, since L is nonsingular, equals

$$\{w \mid \text{ there exists } c', c \text{ such that } R_1 w + \tilde{R}_2 c' = 0,\, c' = Lc,\, Cc = 0\}.$$

The latter is equal to $(\mathcal{K}'_{\text{full}}(\mathcal{C}'))_w$, the manifest controlled behavior resulting from the interconnection of $\mathcal{P}'_{\text{full}}$ and $\mathcal{C}'$. Thus, $\mathcal{C}$ implements $\mathcal{K}$ w.r.t. $\mathcal{P}_{\text{full}}$ if and only if $\mathcal{C}'$ implements $\mathcal{K}$ w.r.t. $\mathcal{P}'_{\text{full}}$. Next, we prove that the interconnection of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$ is regular if and only if the interconnection of $\mathcal{P}'_{\text{full}}$ and $\mathcal{C}'$ is regular. Note that $\mathcal{K}'_{\text{full}}(\mathcal{C}')$ has latent variable representation

$$\begin{pmatrix} R_1 & \tilde{R}_2 \\ 0 & I \\ 0 & 0 \end{pmatrix} \begin{pmatrix} w \\ c' \end{pmatrix} = \begin{pmatrix} 0 \\ L \\ C \end{pmatrix} \ell.$$

Hence the output cardinality of $\mathcal{K}'_{\text{full}}(\mathcal{C}')$ equals

$$\mathtt{p}(\mathcal{K}'_{\text{full}}(\mathcal{C}')) = \operatorname{rank} \begin{pmatrix} R_1 & \tilde{R}_2 & 0 \\ 0 & I & L \\ 0 & 0 & C \end{pmatrix} - \operatorname{rank} \begin{pmatrix} 0 \\ L \\ C \end{pmatrix}$$

(see [2, Lemma 8]). Using elementary row and column operations and the fact that L is nonsingular, this can be shown to be equal to rank $\left(\begin{smallmatrix} R_1 & R_2 \\ 0 & C \end{smallmatrix}\right)$, which equals $\mathtt{p}(\mathcal{K}_{\text{full}}(\mathcal{C}))$. Also, $\mathtt{p}(\mathcal{C}') = \operatorname{rank} \left(\begin{smallmatrix} I & L \\ 0 & C \end{smallmatrix}\right) - \operatorname{rank} \left(\begin{smallmatrix} L \\ C \end{smallmatrix}\right) = \operatorname{rank}(C) = \mathtt{p}(\mathcal{C})$. Finally, $\mathtt{p}(\mathcal{P}_{\text{full}}) = \operatorname{rank}(R_1 \;\; R_2) = \operatorname{rank}(R_1 \;\; \tilde{R}_2) = \mathtt{p}(\mathcal{P}'_{\text{full}})$. This proves our claim.

Again, the first statement of the theorem follows immediately from the equivalence of statements 1 and 2. ∎

According to this theorem, the controller $\mathcal{C} = \ker(C)$ works for $\mathcal{P}_{\text{full}}$ if and only if the controller $L\ker(C)$ (with control variable c') works for the observable system $\mathcal{P}'_{\text{full}}$. What we are looking for here is a parametrization of *all* such polynomial matrices C. The next theorem reduces the parametrization of these C's to the parametrization of all polynomial matrices C' such that $\ker(C')$ regularly implements $\mathcal{K}$ through c' w.r.t. $\mathcal{P}'_{\text{full}}$, which was already established in Theorem 21.

THEOREM 24. *Let $\mathcal{P}_{\text{full}} = \ker(R_1 \;\; R_2)$ be a minimal representation, with R_2 full column rank. Let L be square and nonsingular such that $R_2 = \tilde{R}_2 L$, with $\tilde{R}_2(\lambda)$ full column rank for all λ. Let $\mathcal{P}'_{\text{full}}$ be the (observable) system represented by $R_1 w + \tilde{R}_2 c' = 0$. Let $\mathcal{C} \in \mathfrak{L}^k$ with minimal representation $\mathcal{C} = \ker(C)$. Then for every $\mathcal{K} \in \mathfrak{L}^q$ that is regularly implementable through c w.r.t. $\mathcal{P}_{\text{full}}$ the following two statements are equivalent:*

1. *The controller $\mathcal{C}$ regularly implements $\mathcal{K}$ through c w.r.t. $\mathcal{P}_{\text{full}}$.*

2. *There exists a square, nonsingular polynomial matrix X and a full row rank polynomial matrix C' such that $C = X^{-1}C'L$, where $(C'(\lambda) \;\; X(\lambda))$ has full row rank for all $\lambda \in \mathbb{C}$ and the controller $\mathcal{C}' = \ker(C')$ regularly implements $\mathcal{K}$ through c' w.r.t. $\mathcal{P}'_{\text{full}}$.*

Proof. ($1 \Rightarrow 2$) Let $\mathcal{C} = \ker(C)$ regularly implement $\mathcal{K}$ through c w.r.t. $\mathcal{P}_{\text{full}}$. By Theorem 23, the controller $L\ker(C)$ regularly implements $\mathcal{K}$ through c' w.r.t. $\mathcal{P}'_{\text{full}}$. Let

$$\begin{pmatrix} A & B \\ Y & X \end{pmatrix} \begin{pmatrix} L \\ C \end{pmatrix} = \begin{pmatrix} D \\ 0 \end{pmatrix}, \tag{9}$$

where the leftmost matrix is unimodular and D has full row rank. Obviously, this can always be done. We claim that $L\ker(C) = \ker(Y)$. Indeed, ($c' \in L\ker(C)$) $\Leftrightarrow$ (there exists ℓ such that $c' = L\ell$ and $C\ell = 0$) $\Leftrightarrow$ (there exists ℓ such that $\left(\begin{smallmatrix} A & B \\ Y & X \end{smallmatrix}\right)\left(\begin{smallmatrix} c' \\ 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} D \\ 0 \end{smallmatrix}\right)\ell$) $\Leftrightarrow$ ($Yc' = 0$). The last equivalence follows from the fact that D has full row rank, so it induces a surjective differential operator. Next, we prove that Y has full row rank. Let p be a polynomial row vector such that $pY = 0$. Since $YL + XC = 0$ and C has full row rank, we obtain $pX = 0$. Since $(Y \;\; X)$ has full row rank, this yields $p = 0$. In the same way it can be proved that X has full row rank.

Now define $C' := -Y$. Then the controller $\mathcal{C}' = \ker(C')$ regularly implements $\mathcal{K}$ through c' w.r.t. $\mathcal{P}'_{\text{full}}$. Of course, coming from a unimodular matrix, $(C'(\lambda) \;\; X(\lambda))$ has full row rank for all $\lambda \in \mathbb{C}$. We show that X is nonsingular. Since it has full row rank, it suffices to show that it is square. This follows immediately from (9):

$$\text{rowdim}(X) = \text{rowdim}(C) + \text{rowdim}(L) - \text{rank}(\text{col}(L, C)).$$

Since L is nonsingular, $\text{rank}(\text{col}(L, C)) = \text{rank}(L)$, so $\text{rowdim}(X) = \text{rowdim}(C)$. Of course, also $\text{coldim}(X) = \text{rowdim}(C)$, so X is square. Finally, we have $C = X^{-1}C'L$.

($2 \Rightarrow 1$) We will prove that $L\ker(C) = \ker(C')$. The implication will then follow from Theorem 23. Clearly, $-C'L + XC = 0$. Let A and B be polynomial matrices such that $\left(\begin{smallmatrix} A & B \\ -C' & X \end{smallmatrix}\right)$ is unimodular. Let D be defined by

$$\begin{pmatrix} A & B \\ -C' & X \end{pmatrix} \begin{pmatrix} L \\ C \end{pmatrix} = \begin{pmatrix} D \\ 0 \end{pmatrix}.$$

By the same argument as was used in the first part of this proof, it suffices to prove that D has full row rank. Indeed, using the fact that A is square, we have that $\text{rank}(D) = \text{rank}(\text{col}(D, 0)) = \text{rank}(\text{col}(L, C)) = \text{rank}(L) = \text{rowdim}(L) = \text{coldim}(A) = \text{rowdim}(A) = \text{rowdim}(D)$. $\square$

Thus, for any given full row rank C' that works for the observable system $\mathcal{P}'_{\text{full}}$, a *set* of polynomial matrices C that work for $\mathcal{P}_{\text{full}}$ is obtained by dividing $C'L$ by those nonsingular polynomial matrices X that have the properties that $(C'(\lambda) \;\; X(\lambda))$ has full row rank for all λ, and the quotient $X^{-1}C'L$ is a polynomial matrix again.

7. All stabilizing controllers: The partial interconnection case. In this section we return to the stabilization problem. Whereas in Theorem 12 we gave a parametrization in the full interconnection case, we now solve the problem of parametrizing, for a given plant $\mathcal{P}_{\text{full}} = \ker(R_1 \;\; R_2)$, all stabilizing controllers (see section 3, Definition 6) for the case of partial interconnection. This is done along the same lines as the parametrization of all regularly implementing controllers: we first establish a parametrization under the condition that in $\mathcal{P}_{\text{full}}$ c is observable from

w. Then we lift the assumption and treat the general case. As already mentioned in section 3, necessary and sufficient conditions for the existence of a stabilizing controller for $\mathcal{P}_{\text{full}}$ are that $(\mathcal{P}_{\text{full}})_w$ is stabilizable and that w is detectable from c (see Proposition 7). For the observable case the following lemma is instrumental.

LEMMA 25. *Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{q+k}$ with system variable (w, c). Assume that c is observable from w. Assume that $(\mathcal{P}_{\text{full}})_w$ is stabilizable and that w is detectable from c. Let $\mathcal{C} \in \mathfrak{L}^k$. Then the following two statements are equivalent:*

1. *$\mathcal{C}$ stabilizes $\mathcal{P}_{\text{full}}$ through c.*
2. *$\mathcal{C}$ stabilizes $(\mathcal{P}_{\text{full}})_c$ by full interconnection.*

Proof. $(1 \Rightarrow 2)$ $\mathcal{K} := (\mathcal{K}_{\text{full}}(\mathcal{C}))_w$ is stable and the interconnection is regular. Let $\mathcal{L}_{\text{full}}(\mathcal{K})$ be the interconnection of $\mathcal{P}_{\text{full}}$ and $\mathcal{K}$ through w. Then, by Lemma 19, $\mathcal{K}_{\text{full}}(\mathcal{C}) = \mathcal{L}_{\text{full}}(\mathcal{K})$. According to Theorem 20, $\mathcal{C}$ regularly implements $(\mathcal{L}_{\text{full}}(\mathcal{K}))_c$ by full interconnection with $(\mathcal{P}_{\text{full}})_c$. We claim that $(\mathcal{L}_{\text{full}}(\mathcal{K}))_c$ is stable. Indeed, let $c \in (\mathcal{L}_{\text{full}}(\mathcal{K}))_c$. There exists w such that $(w, c) \in \mathcal{L}_{\text{full}}(\mathcal{K}) \subseteq \mathcal{P}_{\text{full}}$. Let R_2^+ be a polynomial left-inverse of R_2. Then we have $c = -R_2^+ R_1 w$. Hence $c(t) \to 0$ $(t \to \infty)$ (note that the components of w are products of polynomials and stable exponentials).

$(2 \Rightarrow 1)$ Let $\mathcal{P}_{\text{full}} = \ker\begin{pmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{pmatrix}$ be a minimal representation, with R_{11} full row rank (see also the proof of Theorem 20). Then $(\mathcal{P}_{\text{full}})_c = \ker(R_{22})$ is a minimal representation. Represent $\mathcal{C} = \ker(C)$ minimally. Then $\operatorname{col}(R_{22}, C)$ is Hurwitz. Using this, together with the fact that R_{11} has full row rank, it is immediate that the interconnection of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$ is regular. We now prove that $(\mathcal{K}_{\text{full}}(\mathcal{C}))_w$ is stable. Let $w \in (\mathcal{K}_{\text{full}}(\mathcal{C}))_w$. There exists c such that $(w, c) \in \mathcal{K}_{\text{full}}(\mathcal{C})$ so $R_{11} w + R_{12} c = 0$ and $\begin{pmatrix} R_{22} \\ C \end{pmatrix} c = 0$. Thus, the components of c are products of polynomials and stable exponentials. Since w is detectable from c, $R_{11}(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}^+$. This implies that $w(t) \to 0$ $(t \to \infty)$. ◻

The following theorem then gives a parametrization of all stabilizing controllers for the observable case.

COROLLARY 26. *Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{q+k}$ satisfy the assumptions of Lemma 25. Let $\mathcal{P}_{\text{full}} = \ker(R_1 \;\; R_2)$ be a minimal representation. Construct polynomial matrices V_2, S, and C_0 as follows:*

1. *Let V_2 be a full row rank MLA of R_1.*
2. *Factorize $V_2 R_2 = TS$ with T square, nonsingular and $S(\lambda)$ full row rank for all $\lambda \in \mathbb{C}$.*
3. *Let C_0 be such that $\operatorname{col}(S, C_0)$ is unimodular.*

Then for any $\mathcal{C} \in \mathfrak{L}^k$ with $\mathcal{C} = \ker(C)$ the following statements are equivalent:

1. *$\mathcal{C}$ stabilizes $\mathcal{P}_{\text{full}}$ through c and the representation $\mathcal{C} = \ker(C)$ is minimal.*
2. *There exist a polynomial matrix F and a Hurwitz polynomial matrix D such that $C = FS + DC_0$.*

Proof. This is an immediate Corollary of Theorem 12 and Lemma 25. ◻

Thus we have obtained a parametrization of all stabilizing controllers for the observable case. In order to arrive at a parametrization for the general case, we can perform the same two reduction steps as in section 6. We will describe both steps separately now; the proofs are left to the reader.

The first step concerns the reduction of a general $\mathcal{P}_{\text{full}}$ to a full plant behavior $\mathcal{P}'_{\text{full}}$ with R_2-matrix full column rank. Let V be a unimodular matrix such that $R_2 = (\tilde{R}_2 \;\; 0)V$, with $\tilde{R}_2$ full column rank. Let $\mathcal{P}'_{\text{full}}$ be represented by $R_1 w + \tilde{R}_2 c' = 0$.

COROLLARY 27. *$(\mathcal{P}_{\text{full}})_w$ is stabilizable if and only if $(\mathcal{P}'_{\text{full}})_w$ is stabilizable, and in $\mathcal{P}_{\text{full}}$, w is detectable from c if and only in $\mathcal{P}'_{\text{full}}$, w is detectable from c'. Furthermore,*

if $\mathcal{C} \in \mathfrak{L}^k$ with minimal representation $\mathcal{C} = \ker(C)$, then the following two statements are equivalent:

1. *The controller $\mathcal{C}$ stabilizes $\mathcal{P}_{\text{full}}$ through c.*
2. *There exist a polynomial matrix C_{11}, polynomial matrices C_{12} and C_{21} of full row rank, and a unimodular matrix U such that*

$$C = U \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & 0 \end{pmatrix} V, \tag{10}$$

and such that the controller $\mathcal{C}_{21} = \ker(C_{21})$ stabilizes $\mathcal{P}'_{\text{full}}$ through c'.

The next step concerns the reduction of a full plant behavior $\mathcal{P}_{\text{full}}$ with full column rank R_2-matrix to a behavior $\mathcal{P}'_{\text{full}}$ in which the control variable c' is observable from w. Let L be square, nonsingular, such that $R_2 = L\tilde{R}_2$, with $\tilde{R}_2(\lambda)$ full column rank for all λ. Let $\mathcal{P}'_{\text{full}}$ be represented by $R_1 w + \tilde{R}_2 c' = 0$.

COROLLARY 28. *$(\mathcal{P}_{\text{full}})_w$ is stabilizable if and only if $(\mathcal{P}'_{\text{full}})_w$ is stabilizable, and in $\mathcal{P}_{\text{full}}$, w is detectable from c if and only if in $\mathcal{P}'_{\text{full}}$, w is detectable from c'. Furthermore, for $\mathcal{C} \in \mathfrak{L}^k$ with minimal representation $\mathcal{C} = \ker(C)$, the following two statements are equivalent:*

1. *The controller $\mathcal{C}$ stabilizes $\mathcal{P}_{\text{full}}$ through c.*
2. *There exist a square, nonsingular polynomial matrix X and a full row rank polynomial matrix C' such that $C = X^{-1}C'L$, where $(C'(\lambda) \quad X(\lambda))$ has full row rank for all $\lambda \in \mathbb{C}$ and the controller $\mathcal{C}' = \ker(C')$ stabilizes $\mathcal{P}'_{\text{full}}$ through c'.*

8. Worked-out examples. In order to illustrate the theory developed in this paper, we now present some worked-out examples. The examples are those that were already presented in the problem formulation in section 3.

Example 29. Let $\mathcal{P}_{\text{full}}$ with manifest variable $w = (w_1, w_2)$ and control variable $c = (c_1, c_2)$ be represented by

$$\begin{aligned} w_1 + \dot{w}_2 + \dot{c}_1 + c_2 &= 0, \\ c_1 + c_2 &= 0. \end{aligned}$$

Clearly, $(\mathcal{P}_{\text{full}})_w = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^2)$. For $\mathcal{K}$ take the behavior represented by $w_1 + \dot{w}_2 = 0$. $\mathcal{K}$ is regularly implementable through (c_1, c_2) w.r.t. $\mathcal{P}_{\text{full}}$. We have

$$R_1(\xi) = \begin{pmatrix} 1 & \xi \\ 0 & 0 \end{pmatrix} \text{ and } R_2(\xi) = \begin{pmatrix} \xi & 1 \\ 1 & 1 \end{pmatrix}.$$

R_2 has full column rank. Factorize $R_2 = \tilde{R}_2 L$ with $\tilde{R}_2 = I_2$, the 2×2 identity matrix, and $L = R_2$. The resulting system $\mathcal{P}'_{\text{full}}$ represented by $R_2 w + \tilde{R}_2 c' = 0$ is observable. We first parametrize all controllers that regularly implement $\mathcal{K}$ w.r.t. $\mathcal{P}'_{\text{full}}$. For this, we perform the steps described in Theorem 21: $V_2 = (0, 1)$, $V_1 = (1, 0)$, $V_2\tilde{R}_2 = (0, 1)$, so $M = \text{col}(1, 0)$. Next, $V_1 R_1 = K = (1, \xi)$, so $F_1 = 1$. We have $F_1 V_1 \tilde{R}_2 M = 1$. Thus Q, as full row rank MLA of 1, is void. We take $W = 1$. A parametrization of all full row rank controller representations C' that regularly implement $\mathcal{K}$ w.r.t. $\mathcal{P}'_{\text{full}}$ is given by $C'(\xi) = (u, g(\xi))$, with $0 \neq u \in \mathbb{R}$ and g an arbitrary polynomial with real coefficients.

Next we parametrize all controllers C that regularly implement $\mathcal{K}$ w.r.t. the original full plant behavior $\mathcal{P}_{\text{full}}$. According to Theorem 24, for any choice of $u \neq 0$ and polynomial g, we should find all nonzero polynomials $x(\xi)$ that divide $C'L =$

$(u\xi + g(\xi), u + g(\xi))$ such that $(x(\lambda), u, g(\lambda)) \neq 0$ for all $\lambda \in \mathbb{C}$. Since $u \neq 0$, this constraint is automatically satisfied. Thus we need only compute all common factors $x(\xi)$ of the polynomials $u\xi + g(\xi)$ and $u + g(\xi)$. If $x(\xi)$ is such a common factor, then it must also divide the difference $u(\xi - 1)$. Hence there are two possibilities:

1. $g(1) \neq -u$. In this case $u\xi + g(\xi)$ and $u + g(\xi)$ are coprime. The only common factor is then $x(\xi) = 1$.
2. $g(1) = -u$. In this case $x(\xi) = \xi - 1$ is the only common factor.

Thus we find that a parametrization of all controllers that regularly implement $\mathcal{K}$ for $\mathcal{P}_{\text{full}}$ is given by $C(\xi) = (u\xi + g(\xi), u + g(\xi))$, $u \neq 0$ and g arbitrary polynomial, or $C(\xi) = (\frac{u\xi+g(\xi)}{\xi-1}, \frac{u+g(\xi)}{\xi-1})$, $u \neq 0$ and g arbitrary polynomial such that $g(1) = -u$. Since $g(1) = -u$ if and only if there exists a polynomial h such that $g(\xi) = -u + h(\xi)(\xi - 1)$, the latter is equivalent to $C(\xi) = (u + h(\xi), h(\xi))$, $u \neq 0$ and h arbitrary polynomial.

Example 30. Consider the full plant behavior $\mathcal{P}_{\text{full}}$ represented by

$$\begin{aligned} w_1 + \dot{w}_2 + \dot{c}_1 + c_2 &= 0, \\ w_2 + c_1 + c_2 &= 0, \\ \dot{c}_1 + c_1 + \dot{c}_2 + c_2 &= 0. \end{aligned}$$

We will parametrize all controllers $C(\frac{\mathrm{d}}{\mathrm{d}t})c = 0$ that stabilize $\mathcal{P}_{\text{full}}$ through c. We have

$$R_1 = \begin{pmatrix} 1 & \xi \\ 0 & 1 \\ 0 & 0 \end{pmatrix},\ R_2 = \begin{pmatrix} \xi & 1 \\ 1 & 1 \\ \xi+1 & \xi+1 \end{pmatrix},\ \tilde{R}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & \xi+1 \end{pmatrix},\ L = \begin{pmatrix} \xi & 1 \\ 1 & 1 \end{pmatrix}.$$

In $\mathcal{P}'_{\text{full}}$, represented by $R_1 w + \tilde{R}_2 c' = 0$, c' is observable from w. We first parametrize all controllers $C'(\frac{\mathrm{d}}{\mathrm{d}t})c' = 0$ that stabilize $\mathcal{P}'_{\text{full}}$. Performing the steps of Corollary 26, we obtain $V_2 = (0, 0, 1)$, $V_2\tilde{R}_2 = TS$ with $T(\xi) = \xi + 1$ and $S = (0, 1)$. Choose $C_0 = (1, 0)$. The required parametrization is then $C'(\xi) = (d(\xi), f(\xi))$ with d an arbitrary Hurwitz polynomial and f an arbitrary polynomial. We compute $C'(\xi)L(\xi) = (\xi d(\xi) + f(\xi), d(\xi) + f(\xi))$. A parametrization for the original plant $\mathcal{P}_{\text{full}}$ is obtained by computing, for any choice of d and f, all nonzero common factors $x(\xi)$ of the polynomials $\xi d(\xi) + f(\xi)$ and $d(\xi) + f(\xi)$ with the property that $(x(\lambda), d(\lambda), f(\lambda)) \neq 0$ for all λ. Let d and f be given, with d Hurwitz. If $x(\xi)$ is a common factor, then it is also a common factor of $(\xi - 1)d(\xi)$. Hence the following possibilities occur:

1. $x(\xi) = c$, constant, unequal to zero. These $x(\xi)$'s satisfy the requirements.
2. $x(\xi) = c(\xi - 1)$, with $c \neq 0$, equivalently, $d(1) + f(1) = 0$. Since d is Hurwitz, $d(1) \neq 0$, so we have $(x(1), d(1), f(1)) \neq 0$, and the rank condition holds.
3. $x(\xi)$ is not constant and divides $d(\xi)$. In this case there is λ such that $x(\lambda) = 0$ and $d(\lambda) = 0$. However, then also $f(\lambda) = 0$, violating the rank condition.

By applying Corollary 28, we conclude that a parametrization of all stabilizing controllers for $\mathcal{P}_{\text{full}}$ is given by $C(\xi) = (\xi d(\xi) + f(\xi), d(\xi) + f(\xi))$, with d Hurwitz polynomial, f arbitrary polynomial, or $C(\xi) = \frac{1}{\xi-1}(\xi d(\xi) + f(\xi), d(\xi) + f(\xi))$, with d Hurwitz polynomial and f polynomial such that $d(1) + f(1) = 0$.

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