Chapter 4

Hénon Renormalization

In this chapter we study the renormalization of Hénon-like maps. It was known from the work (de Carvalho et al. 2005), that there exists a short curve in the Hénon-family $F_{a,b} : (x, y) \mapsto (a - x^2 - by, x)$ which consisting of infinitely renormalizable Hénon-maps of period doubling type. In this work we study numerically, the extension of this curve in the parameter space up to the conservative map. In particular, we describe the combinatorial changes which occur along this curve. These changes are called, “top-down breaking process” of Hénon renormalization. The second part of our study is to describe, how the one-dimensional Cantor set deforms into the Cantor set of the infinitely renormalizable conservative map. To explain this, we compute the distribution of angles of the invariant line fields along the Cantor set. It is known that for highly dissipative maps, the geometry of the Cantor set is different from the corresponding unimodal Cantor set. Finally, we show how this geometry becomes more complicated for maps close to the conservative map.

4.1 Introduction

The Renormalization theory for the Hénon family was initiated in the work of Collet, Eckmann and Koch (Collet et al. 1980). It was shown that the one-dimensional renormalization fixed point $f_*$ is also a hyperbolic fixed point for nearby dissipative two-dimensional maps. Later, a subsequent article by Gambaudo, van Strien and Tresser (Gambaudo et al. 1989) demonstrated that, similar to the one-dimensional situation, the infinitely renormalizable two-dimensional maps which are close to $f_*$ have an attracting Cantor set $\mathcal{O}$ on which the map acts as an adding machine. However, the geometry of the Cantor sets and global topological properties of these maps are very interesting to study. Recently, de Carvalho, Lyubich and Martens, (de Carvalho et al. 2005), discovered that for these maps universality features can coexist with unbounded geometry. This happens due to the lack of rigidity, which makes it quite different from the familiar one-dimensional theory. In this work, we study numerically the approximation of the stable manifold of the renormalization operator in parameter space and explain numerical computations related to the geometry of the
invariant Cantor sets of these maps. The precise statement of the results are formulated below.

**Structure of the problem and Numerical results:** This study is organized in the following way.

In § 4.3, we explain the construction of the locus of period $2^n$ points such that the trace $Tr$ of the first derivative of the $2^n$th iterate of the Hénon map satisfies $Tr = 0$. This locus consists of all $(x, y, a, b)$ such that $(x, y)$ is an attracting period $2^n$ point for the Hénon map with parameters $(a, b)$. This is a smooth surface which projects to the $(a, b)$ parameter plane by a local diffeomorphism. We conjecture that as $n \to \infty$, this locus of period $2^n$ points will converge to the space of infinitely renormalizable maps. Furthermore, we show that graphically this locus of parameters, $\Gamma_{2^n} = \{(a(b), b) \mid 0 \leq b \leq 1\}$, will be a smooth curve in the $(a, b)$ parameter plane.

In § 4.5, we describe the possible extension of the renormalization theory globally in the parameter space up to the boundary, where the map become conservative. To describe this, we use the topological definition of renormalizability, which was introduced in (de Carvalho et al. 2005). In particular, we describe the “top-down breaking process” of Hénon renormalization on the curve $\Gamma_{2^n}$. To explain, we compute numerically the heteroclinic tangencies for the fixed points and for the periodic points up to the period $2^{n-1}$, and describe their asymptotic behavior as $n \to \infty$. Finally, we conjecture that these heteroclinic tangencies satisfy the following relation,

$$\lim_{n \to \infty} \frac{b_n^2}{b_{n-1}} = 1.$$

In the second part of this work we focus on the geometry of the Cantor set of infinitely renormalizable Hénon-like maps. It was shown in (de Carvalho et al. 2005), for highly dissipative maps the corresponding Cantor set is not contained in a smooth curve. It is interesting to study the geometry of the maps close to $b = 1$. We notice that, for high $b$ values, the corresponding Cantor set has complicated geometry (compare to the situation of the degenerate map, where the Cantor set lies on a smooth curve). This means, the geometry of the Cantor set turns out to be, more away from the degenerate case. To describe this, we compute the distribution of angles of invariant line fields for various values of $b$ on the curve $\Gamma_{2^n}$ and compare these distributions with the distribution of line fields for the degenerate map. These results are presented with more details in section 4.6.

Finally, in last section 4.7, we construct the renormalization of Hénon boxes around the point $l_p$, the extreme right most point in the orbit, and compute the average angles versus $b$ value, in each of zooming levels around the point $l_p$. These pictures are illustrated in Figure 4.43. It has been proved in (Lyubich and Martens 2008), that the average Jacobian $b$ is topologically invariant. This gives us, if we take any other Hénon family and compute
their average angles by constructing the renormalization boxes around the point $l_p$, then we see, same kind of graph (piece-wise affine nature) as Figure 4.43. From this, we conjecture that there exist universal angles on the Cantor set around the point $l_p$.

## 4.2 Notation

Let $\Omega_h, \Omega_v \subset \mathbb{C}$ be neighborhoods of $[-1, 1] \subset \mathbb{R}$ and $\Omega = \Omega_h \times \Omega_v$. Let $B = [-1, 1] \times [-1, 1]$ and $\tau > 0$. Consider the class $\mathcal{H}_\Omega(\epsilon)$, consists of maps $F : B \to B$ of the following form,

$$F(x, y) = (f_a(x) - \epsilon(x, y), x),$$

where $f_a : [-1, 1] \to [-1, 1]$ is a unimodal map which admits a holomorphic extension to $\Omega_h$ and $\epsilon : B \to \mathbb{R}$ admits a holomorphic extension to $\Omega$ and finally $|\epsilon| \leq \tau$. The critical point $c$ of $f$ is non-degenerate, if $Df(c) < 0$. A map $F \in \mathcal{H}_\Omega(\epsilon)$ is said to be Hénon-like map, if $F$ maps vertical lines to horizontal lines.

According to the topological construction, a Hénon map is said to be renormalizable if there exists a domain $D \subset B$ such that $F^2 : D \to D$. The construction of the domain $D$ is inspired by renormalization of unimodal maps. In particular it is a topological construction. The precise analytical definition of renormalization can be found in (de Carvalho et al. 2005). If the renormalizable Hénon map is given by $F(x, y) = (f(x) - \epsilon(x, y))$ then the domain, $D \subset B$, is essentially a vertical strip which is bounded by two curves of the form

$$f(x) - \epsilon(x, y) = \text{Const.}$$

These curves are graphs over the $y$–axis with a slope of order $\tau > 0$. The domain $D$ satisfies similar combinatorial properties as the domain of renormalization of a unimodal map. Namely,

$$F(D) \cap D = \emptyset,$$

and

$$F^2(D) \subset D.$$ 

However, the restriction $F^2|_D$ is not a Hénon-like map as it does not map vertical lines into horizontal lines. In (de Carvalho et al. 2005), a non-linear change of variables was used to define the renormalization of $F$. This is given by

$$RF = \phi^{-1} \circ (F^2|_U) \circ \phi,$$

where $U$ is a certain neighborhood of the “critical value” $v = (f(0), 0)$ and $\phi$ is an explicit non-linear change of variables. The set of $n$–times renormalizable maps is denoted by $\mathcal{H}_\Omega^n(b) \subset \mathcal{H}_\Omega(b)$. The set of infinitely renormalizable maps is denoted by

$$W_\Omega(b) = \bigcap_{n \geq 1} \mathcal{H}_\Omega^n(b)$$
4. Hénon Renormalization

It was shown that the degenerate map $F_*(x, y) := (f_*(x), x)$, where $f_*$ is the fixed point of the one-dimensional renormalization operator, is a hyperbolic fixed point for $R$ with a one-dimensional unstable manifold (consisting of one-dimensional maps) and that the renormalizations $R^n F$ of infinitely renormalizable maps converge at a super-exponential rate towards the space of unimodal maps (de Carvalho et al. 2005). For any infinitely renormalizable map $F$, there exists a hierarchical family of boxes $B^n_\sigma$, with $2^n$ on each level and organized by the inclusion in the dyadic tree, such that

$$O = O_F = \bigcap_{n \geq 1} \bigcup_{\sigma} B^n_\sigma$$

is the Cantor set on which $F$ acts as an adding machine. Furthermore, the diameters of the boxes $B^n_\sigma$ shrink at least exponentially with rate $O(\lambda^{-n})$, where $\lambda = \frac{1}{\sigma} = 2.6...$ and $\sigma$ is the universal scaling factor of one-dimensional renormalization fixed point. This means that the Hausdorff dimension of the Cantor set is less than one. This makes it possible to control the distortion of the renormalizations. Ultimately, this leads to the following asymptotic formula,

$$R^n F(x, y) = (f_n(x) - b^{2^n} a(x) y (1 + O(\rho^n)), x),$$

where $f_n \to f_*$ exponentially fast and

$$b = b_F = \exp \int_O \log \text{Jac} F \, d\mu,$$

is the average Jacobian of $F$. Here $\mu$ is the unique invariant measure on $O$ and the Jacobian is the absolute value of the determinant of the derivative $\rho \in (0, 1)$ and $a(x)$ is a universal function. This is a new universality feature of two-dimensional dynamics: $f_*$ controls the zeroth order shape of the renormalization and $a(x)$ gives the first order control. Also in (de Carvalho et al. 2005), they had noticed striking differences between the one- and two-dimensional situations. Namely, the Cantor set $O$ is not rigid. That means that if $F$ and $G$ are two infinitely renormalizable maps with $b_F < b_G$, then a conjugacy $h : O_F \to O_G$, does not admit a smooth extension to $\mathbb{R}^2$. Thus, in dimension two, universality and rigidity phenomena do not necessarily coexist. This non-rigidity phenomenon is also observed in one-dimensional unimodal maps. There the influence of the smoothness of the maps has been considered played a vital role for the non-rigidity, see (Chandramouli et al. 2008), for more details.
4.3 Hénon cycles

4.3.1 Construction of the period $2^n$ points

Consider the Hénon family

$$F_{a,b} (x, y) \mapsto (f_a(x) - b y, x)$$

where $0 \leq b \leq 1$, $a > 0$ and $f_a(x)$ is a unimodal map. For these maps the Jacobian $b_{F_{a,b}} = b$, is constant. In the case of the degenerate map ($b = 0$), there is an unique $a^*$ for which the map $F_{a^*,0}$ is infinitely renormalizable. This is the accumulation point of period doubling bifurcations. Here, our numerical computations show that there is a curve,

$$b \mapsto (a(b), b) \text{ for } b \in [0, 1],$$

which is attached to the point $(a^*, 0)$ in the parameter plane, consisting of infinitely renormalizable Hénon-like maps. To show this, we constructed the “attracting period $2^n$ locus”, consisting of all $(x, y, a, b)$ such that the trace $Tr$ of the first derivative of the $2^n$th iterate of the Hénon map satisfies $Tr = 0$. This means, start with the sequence of one dimensional quadratic maps $f_{a_n}$, which have the critical orbit of period $2^n$ and converge to the Feigenbaum map. For each of these maps, we extend it to a curve in the Hénon parameter plane which has the most attracting period $2^n$ orbit. We explain this construction in the following.

**Algorithm:** Consider the Hénon map

$$F_{a,b} (x, y) = (a - x^2 - b y, x)$$

(4.1)

where $0 \leq b \leq 1$. For $b = 0$, we can easily compute the sequence of parameters

$$\{a_0^{2^1}, a_0^{2^2}, a_0^{2^3}, \cdots, a_0^{2^n} \cdots \},$$

for the quadratic map $f_a(x) = a - x^2$, as strongly contracting periodic points. We obtain this sequence $\{a_0^{2^n}\}$, by solving the following polynomial

$$f_a^{2^n}(0) = 0,$$

for each $n = 1, 2, \cdots, 15$.

The next step is to increment $b$ as $b_i$, where $b_i = b_{i-1} + \delta$, with $\delta = 10^{-10}$ and we compute the sequence of parameters $\{a_i^{2^n}\}$, corresponding to the sequence of strongly contracting periodic points. This means, for each $b_i$ we need to find a vector $v_i^{2^n} = (x_i^{2^n}, y_i^{2^n}, a_i^{2^n})$
in such a way that \( (x_i^{2^n}, y_i^{2^n}) \) is a periodic point of period \( 2^n \) at the parameter \( (a_i^{2^n}, b_i) \), and the trace of the first derivative of \( 2^n \)th map is equal to 0. This leads to the following equations.

\[
F_{(a,b)}^{2^n} \left( \begin{array}{c} x \\ y \end{array} \right) - \left( \begin{array}{c} x \\ y \end{array} \right) = 0 \tag{4.2}
\]

\[
Tr D \left( F_{(a,b)}^{2^n} \left( \begin{array}{c} x \\ y \end{array} \right) \right) = 0 \tag{4.3}
\]

Let \( X^0 = x, Y^0 = y \) and \( F_{a,b}^i (x, y) = (X^i, Y^i) \). Note that all of the

\[
(X^{k+1}, Y^{k+1}) = (a - (X^k)^2 - b Y^k, X^k)
\]

for \( k \geq 0 \), can be expressed explicitly as functions of \( x, y, \) and \( a \). Use subscripts to indicate the partial derivatives,

\[
X^k_x = \frac{\partial X^k}{\partial x}, \quad X^k_y = \frac{\partial X^k}{\partial y}, \quad X^k_a = \frac{\partial X^k}{\partial a},
\]

and the second derivatives as

\[
X^k_{xx}, X^k_{xy}, X^k_{xa}, X^k_{yx}, X^k_{yy}, X^k_{ya}.
\]

Rewrite the Equations (4.2), (4.3) as,

\[
\phi_1 \equiv X^{2^n} - X^0 = 0 \tag{4.4}
\]

\[
\phi_2 \equiv Y^{2^n} - Y^0 = 0 \tag{4.5}
\]

\[
\phi_3 \equiv X_x^{2^n} + Y_y^{2^n} = 0 \tag{4.6}
\]

We employ the Newton algorithm to solve the above equations. Let \( u_i^{2^n}(t) = (x_i^{2^n}, y_i^{2^n}, a_i^{2^n}) \) be the initial point such that \( (x_i^{2^n}, y_i^{2^n}) \) is a periodic point of period \( 2^n \) with parameter \( a_i^{2^n} \). Then the updated vector \( u_i^{2^n}(t + 1) \) is given by

\[
u_i^{2^n}(t + 1) = u_i^{2^n}(t) - (D\phi)^{-1} \cdot \phi \left( u_i^{2^n}(t) \right) \tag{4.7}\]

where

\[
\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}
\]

and

\[
D\phi = \begin{pmatrix} \phi_{1x} & \phi_{1y} & \phi_{1a} \\ \phi_{2x} & \phi_{2y} & \phi_{2a} \\ \phi_{3x} & \phi_{3y} & \phi_{3a} \end{pmatrix}
\]
4.3. Hénon cycles

Computation of $D\phi$ will involve not only the first partial derivative but also the second derivatives of $(X^{2^n}, Y^{2^n})$. We calculate these derivatives recursively. Thus, we have

$$
\begin{align*}
\phi_{1x} &= -2X^{2^n}x - bY^{2^n}x - 1 \\
\phi_{1y} &= -2X^{2^n}y - bY^{2^n}y \\
\phi_{1a} &= -2X^{2^n}a - bY^{2^n}a + 1 \\
\phi_{2x} &= X^{2^n}x; \quad \phi_{2y} = X^{2^n}y; \quad \phi_{2a} = X^{2^n}a \\
\phi_{3x} &= -2(X^{2^n})^2 - 2X^{2^n}x - bY^{2^n}x + X^{2^n}y \\
\phi_{3y} &= -2X^{2^n}y - 2X^{2^n}x - bY^{2^n}y + X^{2^n}y \\
\phi_{3a} &= -2X^{2^n}a - 2X^{2^n}x - bY^{2^n}a + X^{2^n}a
\end{align*}
$$

Once we have these derivatives, it is straightforward to obtain the updated vector $u^{2^n}_i(t+1)$, using the Equation (4.7). We continue this process until the error term $e^{2^n}_i = u^{2^n}_i(t+1) - u^{2^n}_i(t) \leq 10^{-13}$, then the algorithm will stop. Let $v^{2^n}_i = u^{2^n}_{i+1}$ be the final updated vector obtained from the Newton process. It will act as initial vector for the next increment of $b_{i+1}$. Suppose that, we start in the attracting basin of the period $2^n$ orbit, then one can easily find the orbit, simply by repeated iteration. Then slowly change $b$ from $b_i$ to $b_i + \delta b$, and compute the corresponding parameter $a^{2^n}_i$, by repeating the above Newton algorithm, so as to plot the corresponding “most-attracting” curve in the $(a,b)-$parameter plane. We call this curve parameter curve, with period $2^n$ and it is denoted by $\Gamma_{2^n}$. These curves are illustrated in Figure 4.1, for $n = 1, \ldots, 15$.

For $b$ close to 0, it was shown that these curves $\Gamma_{2^n}$, as $n \to \infty$ will converge to a fixed curve $\Gamma_{2\infty}$, which consist of infinitely renormalizable maps (de Carvalho et al. 2005). Figure 4.2, illustrates the fact that these smooth curves, $\Gamma_{2^n}$, will not intersect each other, for $n \geq 1$. It is difficult to see that these curves $\Gamma_{2^n}$, for $n \geq 7$, are separated from each other in the $(a,b)$ parameter plane. To emphasize this fact, we calculated the ratios of successive period doubling, strongly contracting points of these one-dimensional quadratic maps. We observed that these ratios will converge to the Feigenbaum constant $\delta$ as $n \to \infty$, for all $a^{2^n}_i$ corresponding to each $b_i$, where $0 \leq b_i \leq 1$. That is,

$$
\frac{(a^{2^{n-1}}_i - a^{2^n}_i)}{(a^{2^n}_i - a^{2^{n+1}}_i)} \to 4.699201609102996...
$$

It is interesting to study the geometry as well as the topological properties of these maps on this curve $\Gamma_{2\infty}$ and also the bifurcation pattern that occurs. We discuss these issues in the next section.
4. Hénon Renormalization

**Figure 4.1:** most attracting curve $\Gamma_{2^n}$ in the $(a, b)$–parameter plane

**Figure 4.2:** the parameter curves $\Gamma_{2^n}$ for $n < 7$
4.3.2 Construction of period \( k \) points with Fibonacci combinatorics

In this section we construct the parameter curves of period\(- k \), with Fibonacci combinatorics. Consider the Hénon map

\[
F_{a,b}(x, y) = (f_a(x) - by, x)
\]

with \( b \geq 0 \) and \( f_a(x) \) a unimodal map. Consider the following kneading sequence

\[
1, 10, 1001, 1001110, 10011011001, 10011101001100110, \\
100110110011001110011011001, \cdots.
\]

For each of the above kneading sequence there a quadratic map \( f_a(x) = a - x^2 \) with parameter \( a \). Here, we compute the sequence of parameters \( a_k^b \) for the quadratic map \( f_a(x) = a - x^2 \), such that the trace of the first derivative of \( k^{th} \) iterate of Hénon map is zero and the other condition is, the corresponding periodic orbit has to satisfy the Fibonacci combinatorics.

We start with this known sequence of Fibonacci periodic points of period \( k \) and slowly change the \( b \) value and we compute the corresponding sequence of period \( k \) points, such that the periodic orbit will follow the above kneading sequence. This is a similar construction as that described in section § 4.3.1, but here the condition we imposed is that the periodic orbit should satisfy the Fibonacci combinatorics. We illustrate these curves in Figure 4.3, Figure 4.4, and Figure 4.5. We notice that the parameter curves corresponding to the periods 3, 8, 21, 55, \( \cdots \), will move in the backward direction, whereas the other periods 2, 5, 13, 34, \( \cdots \) will move in the forward direction. We call these curves, good parameter curves. Furthermore, we conjecture that the sequence of these good parameter curves will converge super exponentially to a particular curve, called, Fibonacci parameter curve and is denoted by \( \Gamma_{Fib} \). The maps in this curve are defined to be the Fibonacci Hénon maps.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fibonacci_curve_2.png}
\includegraphics[width=0.5\textwidth]{fibonacci_curve_3.png}
\caption{Fibonacci parameter curves of periods 2 and 3}
\end{figure}
The sequence of good parameter curves are shown in Figure 4.6. Here, it is difficult to see that the Fibonacci parameter curve of period 5, 13 and 34 are separated. So, we plotted these curves in a smaller scale and shown in Figure 4.7, that these curves are actually separated.
4.4 Flow of periodic orbits

We describe how the periodic orbits move along the curve $\Gamma_{2^n}$ as we vary the parameter from $b = 0$ to $b = 1$. For a Hénon map $F_{a,b}$ with parameters on this curve $\Gamma_{2^n}$, we compute the attracting periodic orbit of length $2^n$, and project this orbit onto the $x$–axis, plotting these points against the corresponding $b$ values. We call this, flow of periodic orbits. We illustrated this flow for different periods in Figure 4.8 and Figure 4.9.

It is known that, for $b$ close to zero, crossings in the periodic flow will happen. This is shown in Figure 4.10. This is because of the occurrence of Hénon renormalization boxes on top lying of each other. This will lead to the destruction of the geometry of the Cantor set and so produces non-rigidity. This was explained more in (de Carvalho et al. 2005).

We notice that, for higher values of $b$, the same phenomenon will occurs, with even more crossings happening everywhere in the periodic orbit. This means that the corresponding renormalization boxes will overlap, in many places in the orbit. This appears to destroy the geometry of the corresponding Cantor set and produce non-rigidity.
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Figure 4.8: projection of periodic orbit of periods $2^5$ and $2^6$

Figure 4.9: projection of periodic orbit $2^7$ and $2^8$

Figure 4.10: crossing of periodic flow of period $2^6$
We repeated the same experiment for the Hénon maps with Fibonacci combinatorics. The projected flow of these periodic orbits are shown in Figure 4.11. These flows of periodic orbits along on the curve $\Gamma_{Fib}$ indicates that there will still be a Cantor set.

At this point, we do not have a renormalization theory for Hénon maps with Fibonacci combinatorics (maps on $\Gamma_{Fib}$). Further research is needed to develop a renormalization theory for Fibonacci Hénon maps. This experiment motivates the conjecture that such a theory can be developed.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4.11.png}
\caption{Top: projection of periodic orbit 13; Bottom: for the periodic orbit 21 with Fibonacci combinatorics}
\end{figure}
4.5 Break-up process of Hénon renormalization

In an attempt to describe a global renormalization theory, we focused the heteroclinic web and we used the topological definition of renormalizability, as was introduced in (de Carvalho et al. 2005), and considered extending this globally in the parameter domain. In this section we discuss the breaking procedure of renormalizability, along the curve $\Gamma_{2n}$.

**Definition 6** A Hénon-like map is said to be 2-renormalizable if it has two saddle fixed points. One is a regular saddle $\beta_0$, with positive eigenvalues and the other is a flip saddle $\beta_1$ with negative eigenvalues, such that the unstable manifold $W^u(\beta_0)$ intersects the stable manifold $W^s(\beta_1)$ in a single orbit.

It is illustrated in Figure 4.12. If $F$ is 2-renormalizable then there exists a disc $D$ which is bounded by the local unstable manifold of the point $\beta_0$ and local stable manifold of the point $\beta_1$, such that $F^2|_D$ is invariant.

![Figure 4.12: A renormalizable Hénon-like map](image)

When the unstable manifold $W^u(\beta_0)$, touches or crosses the stable manifold $W^s(\beta_1)$, then it it not 2-renormalizable. In this case, there is no disk of period 2, hence no period 2 cycle exists. This is illustrated in Figure 4.13 and Figure 4.14.

**Definition 7** First bifurcation moment: The unstable manifold $W^u(\beta_0)$ touches the stable manifold $W^s(\beta_1)$ at a point $p_0$. This is the point where the first bifurcation happens. It is illustrated in Figure 4.13.
4.5. Break-up process of Hénon renormalization

In the previous section 4.3.1 we explained the construction of the periodic curves of period $2^n$. Let $\Gamma_{2^n}$ be the parameter curve for a fixed $n > 1$, as shown in Figure 4.15. For each point on this curve we have the parameters $(a_i, b_i)$ such that the Hénon map has a strongly attracting periodic point $(x_i, y_i)$ of period $2^n$. We slowly change the parameter $b$ along this curve and compute the first bifurcation moment. This will happen at some point $b_1$ on $\Gamma_{2^n}$, such that, at this point, the corresponding Hénon map has heteroclinic tangency. This is shown in the Figure 4.16.

Let $\Gamma_{2^n}^1$ be a piece of the curve on $\Gamma_{2^n}$ such that it is a graph over $[0, b_1]$. We call it as the first window, denoted by $\Gamma_{2^n}^1$ on $\Gamma_{2^n}^1$. In this window, for any map $F_{a,b}$ with $(a, b) \in \Gamma_{2^n}^1$, then $F$ is infinitely $2-$renormalizable. In particular, it has a Cantor attractor $\mathcal{O}_F$ and a
The orbit of \( D_n \) is denoted by \( \mathcal{C}_n \), where

\[
\mathcal{C}_n = \{ D_n, F(D_n), F^2(D_n), \ldots, F^{2^n-1}(D_n) \}.
\]

This is called a cycle. Therefore, the Cantor set \( \mathcal{O}_F \) is

\[
\mathcal{O}_F = \bigcap_{n \geq 1} \bigcup_{i=0}^{2^n-1} F^i(D_n)
\]

Note that in this first window \( \Gamma_{2^n} \), all maps are infinitely 2-renormalizable with the Cantor set \( \mathcal{O}_F \) satisfying

\[
C_1 \supset C_2 \supset C_3 \supset \cdots \supset \mathcal{O}_F.
\]
This means that all cycles will survive in this window $\Gamma_{2n}^1$.

Let $\Gamma_{2n}^2$ be a piece of curve on $\Gamma_{2n}$ such that it is a graph over $[b_1, b_2]$, we call it the second window on $\Gamma_{2n}$. Here, $b_2$ is the point where the second heteroclinic tangency occurs. This means that, the unstable manifold $W^u(\beta_1)$ touches the stable manifold $W^s(\beta_2)$ in a single orbit, where $\beta_1$ is a saddle fixed point and $\beta_2$ is a period$-2$ point. This is shown in Figure 4.17. Notice that, if we flip the second picture in Figure 4.17, it looks like the first Figure 4.16.

**Figure 4.16:** first heteroclinic tangency at $b_1$; $\beta_0, \beta_1$ are fixed points; $W^u(\beta_0)$ is the unstable manifold of $\beta_0$ and $W^s(\beta_1)$ is the stable manifold of $\beta_1$.

**Figure 4.17:** second heteroclinic tangency at $b_2$; $\beta_1$ is a fixed point and $\beta_2$ is period$-2$ point; $W^u(\beta_1)$ is the unstable manifold of $\beta_1$ and $W^s(\beta_2)$ is the stable manifold of $\beta_2$. 
Let \( F \) be any map in \( \Gamma_{2^n}^2 \), then \( F \) is not \( 2 \)-renormalizable but it is \( 4 \)-renormalizable. This means that, there exists an invariant disk \( D_4 \) and a non-affine rescaling \( \phi \) such that

\[
R_4 F = \phi^{-1} F^4 |_{D_4} \phi.
\]

Furthermore, \( R_4 F \) is infinitely \( 2 \)-renormalizable.

Observe that, in this window \( \Gamma_{2^n}^2 \), there is no cycle of period 2 and therefore, the invariance of the disk \( D_1 \) disappears because of the heteroclinic tangency of \( W^u(\beta_0) \) and \( W^s(\beta_1) \). This we call the breaking of the cycle. However, all the other cycles will survive. Therefore,

\[
C_2 \supset C_3 \supset \cdots \supset \mathcal{O}_F.
\]

In particular, the Cantor set

\[
\mathcal{O}_F = \bigcap_{n \geq 2} \bigcup_{i=0}^{2^n-1} F^i(D_n)
\]

will survive.

Similarly, there exist \( b_3 \) on \( \Gamma_{2^n}^3 \), such that the third window \( \Gamma_{2^n}^3 \), is the graph over \([b_2, b_3]\). For any map \( F \in \Gamma_{2^n}^3 \), \( F \) is not \( 2 \)-renormalizable and not \( 4 \)-renormalizable, but it is \( 8 \)-renormalizable. Therefore, there exists a non-affine rescaling \( \phi \)

\[
R_8 F = \phi^{-1} F^8 |_{D_8} \phi.
\]

such that \( R_8 F \) is infinitely \( 2 \)-renormalizable. At this point \( b_3 \), the unstable manifold \( W^u(\beta_2) \) intersects the stable manifold \( W^s(\beta_3) \) in a single orbit, where \( \beta_3 \) is periodic point of period \( 2^3 \). This is illustrated in Figure 4.18.

Similarly, as before, observe that there is no period 2 and period 4 cycle. This is because of the heteroclinic tangency at \( b_3 \). But the other cycles

\[
C_3 \supset C_4 \supset \cdots \supset \mathcal{O}_F
\]

will survive with the corresponding Cantor set

\[
\mathcal{O}_F = \bigcap_{n \geq 3} \bigcup_{i=0}^{2^n-1} F^i(D_n).
\]

**Definition 8** A Hénon like map is said to be \( 2^n \)-renormalizable if there exists \( \beta_n \), a saddle of period \( 2^{n-1} \), and there exists \( \beta_{n-1} \), a saddle of period \( 2^{n-2} \), such that the following holds:

- The unstable manifold \( W^u(\beta_{n-1}) \) intersects the stable manifold \( W^s(\beta_n) \) in a single orbit.
4.5. Break-up process of Hénon renormalization

Figure 4.18: third heteroclinic tangency at $b_3$; $\beta_2$ is period 2 point and $\beta_3$ is period 4 point; $W^u(\beta_2)$ is unstable manifold of $\beta_2$ and $W^s(\beta_3)$ is the stable manifold of $\beta_3$.

- A piece of local stable manifold of $\beta_n$ and a piece of local unstable manifold of $\beta_{n-1}$ bound a topological disk $D_n$, which is invariant under $F^{2^n}$.

- $\text{int}(F^i(D_n))$ are piecewise disjoint, for $i = 0, 1, \cdots, 2^{n-1}$.

Using the above definition, we continue the process of computing the heteroclinic tangencies $\{b_k\}$, such that for each $b_k$ there exists a window $\Gamma^k_{2^n}$, is a piece of curves on $\Gamma_{2^n}$ and moreover it is a graph over $[b_{k-1}, b_k]$. Notice that, in each of these windows, the cycles $C_n$, for $n = 1, \cdots, k$ will break. This process of breaking the cycles corresponding to the heteroclinic tangency positions is called the top-down breaking process of Hénon renormalization.

The breaking of these cycles will happen as we continue the process of constructing the pieces of windows on this curve $\Gamma_{2^n}$, as $n \to \infty$, in such a way that

$$
\Gamma_{2^n} = \bigcup_{k=1}^{n} \Gamma^k_{2^n} \cup \{a^{*}_{b=1}\},
$$

where $\Gamma^k_{2^n}$ is the graph over $[b_{k-1}, b_k]$ and $a^{*}_{b=1}$ is the parameter, with strongly contracting periodic orbit of period $2^n$, for the corresponding Hénon map with $b = 1$.

This means that, if any map $F$ is in $\Gamma^k_{2^n}$, then $F$ is $2^k$-renormalizable. In particular, the Cantor set

$$
\mathcal{O}_F = \bigcap_{n \geq k} \bigcup_{i=0}^{2^n-1} F^i(D_n)
$$
will survive.

We present these computations up to the 8th heteroclinic tangency position on the curve $\Gamma_{2n}$ and illustrated in Figures 4.19; 4.20; 4.21; 4.22 and 4.23. In these pictures the unstable manifold $W^u(\beta_{n-1})$ is plotted by constructing the manifold around the periodic point $\beta_{n-1}$ of period $2^{n-2}$, by taking 25000 points on each side with in radius of $10^{-9}$ on the line segment in the direction of unstable eigen-vector and extend this manifold by iterating the Hénon system up to 30 times. To get the stable manifold $W^s(\beta_n)$ of the periodic point $\beta_n$ of period $2^{n-1}$, we computed the unstable manifold of the inverse map by taking the same measurements as above, but the number of times the manifold extended was reduced to only two, as the stable manifold grows a lot faster than the unstable manifold.

![Figure 4.19: Fourth heteroclinic tangency at $b_4$; $\beta_4$ is period 8 point and $\beta_5$ is period 16 point; $W^u(\beta_4)$ is the unstable manifold of $\beta_4$ and $W^s(\beta_5)$ is the stable manifold of $\beta_5$.](image)

Note that the degenerate map $F_{a^*, 0}$ on $\Gamma_{2\infty}$ has the collection of disks

$$D_1 \supset D_2 \supset D_4 \supset D_8 \cdots \supset D_n \supset \cdots$$

with $F^{2^n}(D_n) \subset D_n$ and the cycle $C_n = Orb(D_n)$. For small perturbation of the parameter $(a^*, 0)$ to $(a, 0)$, (with $b = 0$), the map has a period $-m$ collection of disks such that

$$D_1 \supset D_2 \supset \cdots \supset D_m.$$ 

However there is no domain of period $2^k$, $k \geq m + 1$. This means that the higher boxes will break first at deep levels for deformations of a degenerate map. This gives us the following observation.
4.5. Break-up process of Hénon renormalization

Figure 4.20: fifth heteroclinic tangency at $b_5$; $\beta_5$ is period $2^4$ point and $\beta_4$ is period $2^3$ point; $W^u(\beta_5)$ is unstable manifold of $\beta_5$ and $W^s(\beta_4)$ is the stable manifold of $\beta_4$.

Figure 4.21: sixth heteroclinic tangency at $b_6$; $\beta_6$ is period $2^5$ point and $\beta_5$ is period $2^4$ point; $W^u(\beta_6)$ is the unstable manifold of $\beta_6$ and $W^s(\beta_5)$ is the stable manifold of $\beta_5$.

Observation: This bifurcation process on $\Gamma_{2^\infty}$ is exactly opposite to the bifurcation process in the case of the degenerate map, where the cycles of higher period breaks first on deep levels.

We are interested in computing the heteroclinic tangencies for fixed points as well as for the periodic points on the curve $\Gamma_{2^\infty}$ (using Definition 8). These numerical values are presented in Table 4.1.
4. Hénon Renormalization

Figure 4.22: seventh heteroclinic tangency at $b_7$; $\beta_7$ is period $2^6$ point and $\beta_6$ is period $2^5$ point; $W^u(\beta_7)$ is unstable manifold of $\beta_7$ and $W^s(\beta_6)$ is the stable manifold of $\beta_6$.

Figure 4.23: eighth heteroclinic tangency at $b_8$; $\beta_8$ is period $2^7$ point and $\beta_7$ is period $2^6$ point; $W^u(\beta_8)$ is the unstable manifold of $\beta_8$ and $W^s(\beta_7)$ is the stable manifold of $\beta_7$.

On the curve $\Gamma_{2^n}$, $n \leq 9$, we noticed the top-down breaking procedure of the cycles. This happens at specific bifurcation moments $b_i(n) \in \Gamma_{2^n}$, these are illustrated in the Table 4.1, it indicates a convergence

$$b_i(n) \in \Gamma_{2^n} \rightarrow b_i \in \Gamma_{2^\infty}.$$ 

The breaking of the boxes from the top-down process seems to be the combinatorial explanation for why the stable manifold of renormalization can be extended up to the conservative map.
### 4.5. Break-up process of Hénon renormalization

**Table 4.1:** \( b_i \) is the heteroclinic tangency position on the curves \( \Gamma_{2^n} \)

<table>
<thead>
<tr>
<th>( b_i )</th>
<th>period 2(^5)</th>
<th>period 2(^6)</th>
<th>period 2(^7)</th>
<th>period 2(^8)</th>
<th>period 2(^9)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b_1 )</td>
<td>0.0311405</td>
<td>0.03099086</td>
<td>0.03095879</td>
<td>0.03095192</td>
<td>0.03095045</td>
</tr>
<tr>
<td>( b_2 )</td>
<td>0.1715389</td>
<td>0.16961800</td>
<td>0.16920269</td>
<td>0.16911365</td>
<td>0.16909453</td>
</tr>
<tr>
<td>( b_3 )</td>
<td>0.4255566</td>
<td>0.41529040</td>
<td>0.41295748</td>
<td>0.41245413</td>
<td>0.41234636</td>
</tr>
<tr>
<td>( b_4 )</td>
<td>0.6814848</td>
<td>0.65226240</td>
<td>0.64433498</td>
<td>0.64249999</td>
<td>0.64213923</td>
</tr>
<tr>
<td>( b_5 )</td>
<td>0.8669798</td>
<td>0.82551980</td>
<td>0.80765004</td>
<td>0.80270981</td>
<td>0.80158498</td>
</tr>
<tr>
<td>( b_6 )</td>
<td>-</td>
<td>0.93085480</td>
<td>0.90799999</td>
<td>0.89849968</td>
<td>0.88802243</td>
</tr>
<tr>
<td>( b_7 )</td>
<td>-</td>
<td>-</td>
<td>0.96499998</td>
<td>0.95309998</td>
<td>0.95549798</td>
</tr>
<tr>
<td>( b_8 )</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0.98224989</td>
<td>0.97748565</td>
</tr>
<tr>
<td>( b_9 )</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

**Conjecture** The points \( b_n \), of the heteroclinic tangencies on \( \Gamma_{2^n} \) satisfy the following relation

\[
\lim_{n \to \infty} \frac{b_n^2}{b_{n-1}} = 1.
\]

**Remark 4.5.1** The above conjecture has been verified numerically, for another family of Hénon-map

\[
F_{a, b} : (x, y) \mapsto \left( a - x^4 - \frac{1}{2} x^2 - b y, x \right).
\]
4.6 Line fields on the Cantor set

In this section we describe the distribution of angles of the invariant line field of the Cantor set. It is known that, for the degenerate map, the Cantor attractor $\mathcal{O}_F$ lies along a smooth curve. Let $F \in \Gamma_{2^\infty}$, be an infinite $2$–renormalizable non-degenerate Hénon map. From the work (de Carvalho et al. 2005), it possesses the Cantor attractor $\mathcal{O} = \mathcal{O}_F$ on which it acts as adding machine. Furthermore, they showed that $F$ does not have a continuous invariant line field on $\mathcal{O}$. This leads to interesting consequences that the attractor $\mathcal{O}$ does not lie on a smooth curve, which is contrary to one dimensional situation. The question raised here is, for high $b$ values (increasingly $b$ close to 1), does this Cantor set move further away from the degenerate case? To answer this question, we construct the line fields on the Cantor set and analyze the distribution of angles of the line fields on the Cantor set.

Let $F$ be a Hénon map with fixed parameters $(a, b)$ on the curve $\Gamma_{2^\infty}$, so it is infinitely $2$–renormalizable. Then, it has a sequence of invariant disks

$$D_1 \supset D_2 \supset D_3, \supset \cdots D_n \cdots$$

where $F^{2^n}(D_n) \subset D_n$. Let $\beta_n = (x_0^{2^n}, y_0^{2^n}) \in D_n$ be the periodic point of period $2^n$. One can easily find the complete orbit of $\beta_n$, simply by repeated iteration. This orbit is denoted by

$$\text{Orb}_{2^n}(\beta_n) = \bigcup_{k \geq 0} F_{a,b}^k (x_0^{2^n}, y_0^{2^n}) .$$

Using the algorithm which is described in section 4.3, one can compute the periodic point $\alpha_n = (x_0^{2^n-1}, y_0^{2^n-1}) \in D_{n-1}$ of period $2^n-1$, which is an immediate neighbor of $\beta_n$ in the combinatorial sense.

We now approximate the line field around the point $\beta_n$, by constructing a line segment $l(\beta_n, \alpha_n)$, passing through the two periodic points $\beta_n$ and $\alpha_n$. Let $\theta$ be the angle between the line $l(\beta_n, \alpha_n)$ made with the vertical axes (which is asymptotically equivalent to the local stable manifold of $W^s(\beta_n)$). We measure this angle by

$$\sin \theta = \frac{(x_0^{2^n} - x_0^{2^n-1})}{\text{dist}(\beta_n, \alpha_n)}$$

where $\text{dist}(\beta_n, \alpha_n)$ stands for the distance between the two periodic points $\beta_n$ and $\alpha_n$. The next is to find the image of this pair $(\beta_n, \alpha_n)$ under the Hénon map $F_{a,b}$ and approximate the line field around the point $F_{a,b}(\beta_n)$, compute the corresponding vertical angle. Repeat this process of approximation of line fields at each point in the orbit $\text{Orb}_{2^n}(\beta_n)$, with their corresponding line segments and make a list of these angles $\theta_i$ for $i = 1, \ldots, 2^n$. We plot the histogram for the list of these angles, considering a $2^9$ subintervals on $[-1,1]$ and the number of angles present in each sub interval on vertical axes. We call this,
distribution of angles. We compute these distributions for various parameters \((a, b)\) on the curve \(\Gamma_{2n}, n = 11\), starting with \(b = 0\) and varying up to the maps close to the conservative case. These distributions are presented in the following: Figure 4.24, Figure 4.25, Figure 4.26, Figure 4.27 and Figure 4.28.

Notice that from these distributions, as the parameter \(b\) changes, the distribution becomes increasingly fat (we don’t want to give a precise definition of fat). In the case of degenerate map, these angles are distributed in a Cantor set. As \(b\) increases, the other angles are generated slowly. Finally, for the maps close to the conservative map the distribution is weighted with all angles. This means that more angles are generated compared to the situation of degenerate map. This illustrates the complexity of the geometry of the
corresponding Cantor set, indicates that it is does not lie on a smooth curve any more. This type of Cantor set is called a *Twisted Cantor set*. It is illustrated in Figure 4.29.

![Figure 4.26](image_url)

**Figure 4.26:** Left: distribution of the angles of line fields for the Hénon map $F_{a,b}$ with parameters $(a, b) = (2.182768010790645, 0.4)$; $\beta_n = (1.578302824145569, 0.113227149452719)$; $\alpha_n = (1.578297165342688, 0.113278026146582)$; Right: $(a, b) = (2.439153110310706, 0.5)$; $\beta_n = (1.640300814146864, 0.131617526429658)$; $\alpha_n = (1.640294843823455, 0.13166397820682)$.

![Figure 4.27](image_url)

**Figure 4.27:** Left: distribution of the angles of line fields for the Hénon map $F_{a,b}$ with parameters $(a, b) = (2.721829067829454, 0.6)$; $\beta_n = (1.70927675739446, 0.14770692997098)$; $\alpha_n = (1.70927086709486, 0.147747102534229)$; Right: $(a, b) = (3.031843160671423, 0.7)$; $\beta_n = (1.785002442125404, 0.16187072855787)$; $\alpha_n = (1.785004856836326, 0.161855764134139)$.

The same phenomenon is also described by plotting these angles, taking time on horizontal axes and corresponding angles on vertical axes. In these pictures observe that the dispersion of angles are slowly started, see Figure 4.30 and become more when the maps move away from the degenerate case, see Figure 4.31 and Figure 4.32. Finally, the comparison of the list of angles for the degenerate map and the map with high $b$ value are presented in the Figure 4.33.
4.6. Line fields on the Cantor set

Figure 4.28: Left: distribution of angles of the line fields for the Hénon map $F_{a,b}$ with parameters $(a, b) = (3.370236565105158, 0.8); \beta_n = (1.867105947046825, 0.174689500267944); \alpha_n = (1.867107962878618, 0.174677917712148); Right: $(a, b) = (3.933243998542534, 0.95); \beta_n = (2.001337170806964, 0.191500529607978); \alpha_n = (2.001338052860656, 0.191495919395331).

Figure 4.29: Top: distribution of angles for the degenerate map; Below: for the map $b = 0.95$. 
Figure 4.30: time versus angle for $b = 0.0$ to $b = 0.3$

Figure 4.31: time versus angle for $b = 0.4$ to $b = 0.7$
4.6. Line fields on the Cantor set

Figure 4.32: *time versus angle for $b = 0.8$ and $b = 0.95$*

Figure 4.33: *comparison of angles; Top: $b = 0.0$; Below: $b = 0.95$*
We plotted the lines, which are the approximation of line field along the Cantor set, for different values of $b$ on the curve $\Gamma_{2^n}$. Notice that, in the case of degenerate map the lines has only few directions. But as we consider the maps close to the conservative map, the lines has all other possible directions. The appearance of more and more directions is a result of complexity of the geometry of the Cantor set. These line fields are illustrated in Figure 4.34, 4.35, 4.36, 4.37 and Figure 4.38.

**Figure 4.34:** Left: line fields for $b = 0.0$; Right: $b = 0.1$

**Figure 4.35:** Left: line fields for $b = 0.2$; Right: $b = 0.3$
4.6. Line fields on the Cantor set

**Figure 4.36**: Left: line fields for $b = 0.4$; Right: $b = 0.5$

**Figure 4.37**: Left: line fields for $b = 0.6$; Right: $b = 0.7$

**Figure 4.38**: Left: line fields for $b = 0.8$; Right: $b = 0.95$
4.7 Distributional Universality

It was discovered, in the work (de Carvalho et al. 2005), that the Cantor set $O$ does not have bounded geometry and so it is not quasiconformally equivalent to the standard Cantor set of one-dimensional unimodal map. Moreover, the Cantor set $O$ cannot be embedded into a smooth curve. These properties are different from their one-dimensional counterparts. They come from a *tilting and bending phenomenon*: near the “tip” of a Hénon-like map the renormalization boxes are not rectangles but rather slightly tilted and bent like parallelograms. This tilt significantly affects the $b$--scale geometry of $O$. For highly dissipative maps, the Jacobian $b$ is replaced by $b^{2^n}$ under the $n$th renormalization, the geometry gets affected at arbitrarily small scales.

We calculate the amount of “tilting” for the Hénon renormalization boxes, zooming in to the deep levels around the point $\beta_n$, which is an approximation of the “tip”. From Definition 8, the existence of the invariant disk $D_n$ is called the Hénon renormalization box.

The line fields constructed in the previous section, are aligned in the direction of these Hénon renormalization boxes. For each of these boxes, we compute the distribution of angles. Now at this point, we separate each distribution into two different distributions, one with the angles pointing in the upward direction and other one with the angles pointing downward direction. The first one we call the *distribution with upward angles* and the second one the *distribution with downward angles*. In each of these distributions we compute the average of the angles. This average angle gives us, the amount of tilting of the corresponding boxes. We illustrate this “tilting phenomenon” by plotting the $b$ value on horizontal axes and the corresponding average angle of the distribution on vertical axes. It is shown in the Figure 4.39 and Figure 4.40. Here, $n.\,\text{lev}$ indicates that the zoom level of the boxes around the point $\beta_n$. Notice that, from these pictures, as the $b$ value increases the average angle is also increased. This can be observed only after the $4\text{th}$ zoom level of the boxes. This emphasizes the fact that for high $b$ value the “tilt” will happen more.

Similar phenomenon is also observed if we construct the renormalization boxes around the point $l_p$. It is illustrated in Figure 4.41. Here, the zooming of the boxes considered around the point $l_p$, which is the right most periodic point of the projected orbit on $x$--axes. Figure 4.41, is magnified for the period $2^{13}$ and illustrated in Figure 4.43 and Figure 4.44.
Figure 4.39: $b$ versus average for downward angles; “n lev” indicates the nth zoom level around the point $\beta_n$.
0. Hénon Renormalization

Figure 4.40: \( b \) versus average for downward angles; “n lev” indicates the \( n \)th zoom level around the point \( \beta_n \).
Figure 4.41: $b$ versus average for downward angles; “n lev” indicates the nth zoom level around the point $l_p$. 

4.7. Distributional Universality
Figure 4.42: $b$ versus average for downward angles; “n lev” indicates the nth zoom level around the point $l_p$; of periodic orbit $2^{13}$
4.7. Distributional Universality

Figure 4.43: $b$ versus average for downward angles; periodic orbit of period $2^{13}$; $11.\ lev$ indicates the zoom level at the point $l_p$

Figure 4.44: $b$ versus average for upward angle; periodic orbit of period $2^{13}$; $11.\ lev$ indicates the zoom level at the point $l_p$

It has been proved that in the work (Lyubich and Martens 2008), the average Jacobian $b$ is topologically invariant. From the above Figure 4.43, and Figure 4.44, one can conclude that, if we take any other Hénon family and compute the average angles by constructing the distributions in the corresponding renormalization boxes around the point $l_p$, which is the right most periodic point in the orbit then we get a similar piece-wise affine nature as above. This means that, these angles are universal, related to the parameter dependence. We call this phenomenon *Distributional universality*.

This refined understanding might play a crucial role in further studies of Hénon maps. Simple questions like the existence of wandering domains is closely related to the geometry of the line field. The non-existence of wandering domains is still open.