Renormalization and non-rigidity
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The study of time evolution of the systems under consideration plays an important role in many natural sciences. Experiments and simulations in these fields often show very complicated, chaotic behavior. However, a rigorous understanding of chaotic dynamical systems are far from complete.

There are not many real world systems which can be modeled by one dimensional dynamical systems. That is, systems described by iteration of a map of the interval. Nevertheless, during the last forty years an extensive and rather complete theory has been developed to explain their dynamics. The surprising fact is that many of the one dimensional phenomena are observed in nature. Although one dimensional systems are very simple models, they contain mechanisms which are relevant for real world systems. The natural strategy is to explore, how far we can extend the one dimensional theory and get a better understanding of higher dimensional systems.

The central theme of the one-dimensional theory is the geometric rigidity of the attractors. The main technique is renormalization. Renormalization is a method to study the microscopic geometric properties of attractors. It was introduced into dynamics in the late seventies by P. Coullet and C.P. Tresser (Coullet and Tresser 1978) and independently M.J. Feigenbaum (Feigenbaum 1978). Initially the goal was to study the dynamics at the accumulation of period doubling. Systems which are at the accumulation of period doubling have very specific combinatorial behavior. This behavior occurs when a system is at transition to chaos, when it is at the boundary of chaos in the space of systems.

The attractors of the maps at transition to chaos have a special property. They are Cantor sets and on arbitrarily small scale the attractor can be identified with a rescaled version of the attractor of another one-dimensional map. This allows to introduce an operator on the set of one-dimensional maps at transition which assigns to a map, the map which describes its attractor at the smaller scale. This operator acts as a microscope. For maps at the transition we can describe the dynamics at arbitrarily small scale. That is, we can apply the renormalization operator infinitely many times to study the dynamics. It was conjectured in (Coullet and Tresser 1978) and (Feigenbaum 1978), that the maps at transition form exactly the stable manifold of a unique fixed point $f_*$ of the renormalization operator. This conjecture explains why the fine scale structure of the attractor is independent of the original map being considered. The microscopic geometry of an attractor at
transition to chaos is \textit{universal}. The fine scale geometrical structure can not be deformed. The attractors are \textit{rigid}.

During the following thirty years the renormalization idea was extended and applied to general types of combinatorics of one-dimensional maps. Our understanding of one-dimensional dynamics is a consequence of the maturity of one-dimensional renormalization.

A general theory for smooth dynamics is still completely out of reach. There are two natural directions in which one can extend the theory using the results from one-dimensional smooth dynamics. The first one is one-dimensional dynamics with low smoothness and the second is dynamics of Hénon maps.

Models of real world systems are usually very high dimensional or even infinite dimensional as in fluid dynamics. However, there is a phenomenon of dimension reduction, the essence of the dynamics happens on low dimensional attractors. On some cases these attractors can be described, even in terms of one-dimensional systems. This is the reason why one-dimensional dynamical systems are more than just toy models. The theory for one-dimensional systems is well developed in the case when the systems are smooth. Unfortunately, the one-dimensional systems which arise from applications are usually not smooth. In dissipative systems the states are groups in so-called stable manifolds, different states in such a stable manifold have the same future. The packing of the stable manifold usually does not occur in a smooth way. For example, the Lorenz flow is a flow on three dimensional space approximating a convection problem in fluid dynamics. The stable manifolds are two dimensional surfaces packed in a non smooth foliation. This flow can be understood by a map on the interval whose smoothness is usually below $C^2$.

The first part of the thesis discusses renormalization of one-dimensional maps with low smoothness. The first group of results deals with maps which are $C^2$. All maps under consideration will be maps with a \textit{quadratic} tip. These maps are unimodal, they have a single maximum at their critical point, it is denoted by $c$ and this maximum is well approximated by a quadratic polynomial. The collection of unimodal maps with quadratic tip and a certain smoothness is denoted by $U^r$.

The main results lead to the fact that renormalization on the space of $C^2$ unimodal maps is not hyperbolic and the convergence to the analytic fixed point can be arbitrarily slow.

\textbf{Theorem 1.0.1} Let $d_n > 0$ be any sequence with $d_n \to 0$. There exists an infinitely renormalizable $C^2$ unimodal map $f \in U^2$ such that

$$\text{dist}_0 (R^n f, f^*_n) \geq d_n.$$ 

The distance is measured in the $C^0$ topology.
Corollary 1.0.2 The analytic unimodal map $f^\omega_*$ is not a hyperbolic fixed point in the space of $C^2$ maps.

We introduce a new type of differentiability of a unimodal map, called $C^{2+|\cdot|}$, which is the minimal needed to be able to apply the classical proofs of a priori bounds for the invariant Cantor sets of infinitely renormalizable maps, see for example (Martens 1994), (Martens et al. 1988), (de Melo and van Strien 1993). This type of differentiability will allow us to represent any $C^{2+|\cdot|}$ unimodal map as

$$f = \phi \circ q,$$

where $q$ is a quadratic polynomial and $\phi$ has still enough differentiability to control cross-ratio distortion.

**Theorem 1.0.3** If $f$ is an infinitely renormalizable $C^{2+|\cdot|}$ unimodal map then

$$\lim_{n \to \infty} \text{dist}_0 (R^n f, f^\omega_*) = 0.$$  

A construction similar to the one provided for $C^2$ unimodal maps leads to the following result:

**Theorem 1.0.4** Let $d_n > 0$ be any sequence with $\sum_{n \geq 1} d_n < \infty$. There exists an infinitely renormalizable $C^{2+|\cdot|}$ unimodal map $f$ such that

$$\text{dist}_0 (R^n f, f^\omega_*) \geq d_n.$$  

The analytic unimodal map $f^\omega_*$ is not a hyperbolic fixed point in the space of $C^{2+|\cdot|}$ maps.

Our second set of theorems deals with renormalization of $C^{1+\text{Lip}}$ unimodal maps with a quadratic tip.

**Theorem 1.0.5** There exists an infinitely renormalizable $C^{1+\text{Lip}}$ unimodal map $f$ which is not $C^2$ but

$$Rf = f.$$  

The topological entropy of a system defined on a non-compact space is defined to be the Supremum of the topological entropies contained in compact invariant subsets. As a consequence of a Theorem of Davie (Davie 1999), we get that renormalization on $U^{2+\alpha}$ has entropy zero, for any $\alpha > 0$.

**Theorem 1.0.6** The renormalization operator acting on the space of $C^{1+\text{Lip}}$ unimodal maps has infinite entropy.
The last theorem illustrates a specific aspect of the chaotic behavior of the renormalization operator on the space of $C^{1+Lip}$ unimodal maps.

**Theorem 1.0.7** There exists an infinitely renormalizable $C^{1+Lip}$ unimodal map $f$ such that $\{c_n\}_{n \geq 0}$ is dense in a Cantor set. Here $c_n$ is the critical point of $R^n f$.

The second possibility is to use the successful one-dimensional renormalization theory to study two-dimensional dynamics. In the case of dissipative dynamics we should start with the Hénon family. The maps in this family act on a two-dimensional domain and are given by

$$F_{a,b}(x, y) = (f_a(x) - by, x),$$

where $b \geq 0$, is the Jacobian and $f_a(x)$ is a unimodal map. This family arises when one creates chaos from a homoclinic bifurcation in a dissipative system. Strongly dissipative Hénon maps, $b << 1$, are perturbations of one dimensional dynamics and one-dimensional renormalization theory is a powerful starting point for the development of a theory. The Hénon family has many realistic applications because of its relevance in the creation of chaos.

Rigorous understanding of Hénon map is fragmented. There are three well understood phenomena. The first one is the Newhouse phenomenon (Newhouse 1974). There are smooth maps (also in the Hénon family) which have periodic attractors of arbitrarily high period. This behavior is quite different form the chaotic maps constructed by M. Benedicks and L. Carleson (Benedicks and Carleson 1991). They proved that for a set of parameters with positive measure the corresponding Hénon map has a non-trivial attractor with an ergodic invariant measure, describing the statistical long term behavior of typical orbits. This fundamental work from the late eighties was recently refined by L.S. Young and Q.D. Wang to apply higher dimensions, Hénon-like maps, (Wang and Young 2005).

The third part of our knowledge of Hénon maps deals with maps in a neighborhood of the accumulation of period doubling. This is an area in parameter space where chaos is created. The first study of this area was done by P. Collet, J-P. Eckmann and H. Koch, (Collet et al. 1980). They used analytical tools to extend the one-dimensional renormalization operator to a space of strongly dissipative Hénon-like maps and proved the hyperbolicity of the operator. A. de Carvalho, M.Lyubich, and M. Martens constructed a renormalization operator on the space of strongly dissipative Hénon-like maps using geometric ingredients, (de Carvalho et al. 2005). The specific construction and the hyperbolicity of this renormalization operator allowed to study the geometry of Cantor attractors of Hénon maps at the accumulation of period doubling. It opened a source of surprising phenomena. The results obtained discuss the geometric (non)-rigidity of the Cantor attractors of maps at the accumulation of period doubling, the topology of such maps as well as the bifurcation pattern in a neighborhood of the accumulation of period doubling. The main theme is that
the theory for two-dimensional dissipative dynamics is far from a straightforward general-
ization of the one-dimensional theory, even for maps which are the simplest combinatorial
type, period doubling. However, renormalization is again a very powerful tool which is
able to describe the dynamics of Hénon maps.

The second part of the thesis discusses renormalization for Hénon maps. It is a nu-
merical study. The present renormalization theory deals with strongly dissipative Hénon
maps. These maps form a short curve in parameter space of a generic Hénon family. An
important question is whether the observed phenomena of (non-)rigidity and universality
can be extended to maps which are (not strongly) dissipative and even up to the conserva-
tive maps. Briefly speaking, can we extend the curve of infinitely renormalizable strongly
dissipative Hénon maps up to the conservative maps? The first numerical study shows
that, indeed, the curve extend that far. More importantly, the study describes the combi-
natorial changes which occur along this curve. These changes are denoted by “top down
breaking of the boxes”.

Most of the results for Hénon maps discuss strongly dissipative maps, $b \ll 1$. We do
not yet have the tools to study maps which are not strongly dissipative, maps which are
not small perturbations of one-dimensional maps. The numerical description of “top down
breaking of boxes” indicates how one can proceed to rigorously extend the curve up to the
conservative maps.

One-dimensional dynamics is controlled by the critical points of these systems. In-
finitely renormalizable Hénon maps also have a topologically defined critical points which
plays a crucial role. At the present moment we are at the starting point of developing a
renormalization theory for Hénon maps with more general combinatorial types. Part of
the problem is to describe the combinatorial type of Hénon map.

History inspires us to consider Hénon-like maps of Fibonacci type. Unfortunately, the
situation is far more complex than the period doubling case for Hénon maps. There are
infinitely many critical points. However, a numerical study presented in this thesis shows
that there is a curve in the Hénon family whose maps have an invariant Cantor set of
Fibonacci type. This is strong support for the possibility of constructing a renormalization
operator for Hénon maps of Fibonacci type.

Infinitely renormalizable Hénon maps of period doubling type have a Cantor attractor.
This Cantor set has geometrical aspects which are exactly the same as the counter part
in the Cantor attractors of infinitely renormalizable one-dimensional systems. This phe-
nomenon is called universality. Contrary to the one-dimensional situation, these Hénon
Cantor sets are not rigid. There are parts of the Hénon Cantor set where the geometry on
asymptotically small scale is different from the one-dimensional situation. By changing the
Jacobian $b$ one can change the asymptotic geometry of the Cantor set. The non-rigidity
was up to recently an unexpected phenomenon. Strongly dissipative two-dimensional sys-
tems are geometrically different from the one-dimensional world. Although, two and one-
dimensional systems do have some universal geometrical aspects.

The numerically constructed curve of infinitely renormalizable dissipative Hénon maps
ends at conservative map. This conservative map has an invariant Cantor set. The geometry
of this Cantor set is not at all similar to the Cantor attractor of the dissipative maps.

Our third numerical study on Hénon maps discusses how the one-dimensional Cantor set
deforms into the Cantor set of the conservative map. To describe this deformation we
studied the invariant line field which is carried on the Cantor set. This line field has zero
characteristic exponent. One could think about this line field as if it was aligned along the
Cantor set. However, one should be careful. It has been shown that this line field is not
continuous for truly two-dimensional Hénon maps (de Carvalho et al. 2005). The Cantor
set does not lie on a smooth curve.

We study numerically, the distribution of the angles of the lines in the line field with re-
spect to a fixed direction. Initially, for strongly dissipative maps, the angles are distributed
in a Cantor set. This is not surprising. However, if we consider infinitely renormalizable
maps on the curve closer towards the end with the conservative map, the distributions are
assigning weight to all angles. These distribution of angles in extreme cases, $b = 0$ and
$b = 0.95$, are illustrated in Figure 1.1.

![Figure 1.1: Left: the distribution of angles for $b = 0.0$; Right: $b = 0.95$.](image)

Observe, these Cantor set are always having Hausdorff dimension smaller than one. It
is not filling more and more the space. The appearance of more and more angles is a result
from the more and more complex geometry of the Cantor set. It gets more and more away
from being on a smooth curve.

This refined understanding might play a crucial role in further studies of Hénon maps.
Simple questions like the existence of wandering domains is closely related to the geometry
of the line field. The non-existence of wandering domains is still open.

The short term goals of this thesis is to contribute to our understanding at the accumu-
lation of period doubling and get a complete understanding of this type of dynamics.
The second short term goal is to develop a renormalization theory which can be applied to more general types of combinatorics, beyond period doubling and study the corresponding dynamics. This will provide fundamental pieces of the larger Hénon puzzle.

The long term goal is to understand two-dimensional dynamics. The conjecture which describes the behavior of smooth dynamics in general was formulated by J. Palis, [P]. It is the central theme of smooth dynamics. The essence of the conjecture is as follows. Almost every map in a generic family has finitely many attractors: almost every orbit accumulates at one of them. Furthermore, each attractor carries an invariant measure which describes the statistical behavior of a typical orbit in its basin. Systems with zero entropy can be understood in purely topological terms. Namely, the Morse-Smale systems are dense among zero entropy systems.

The conjecture has a long history. In particular, it took several decades to observe that, as well topological as measure theoretical ingredients are necessary to understand smooth dynamics. The first context in which the Palis Conjecture was proved is unimodal dynamics on the interval. The main techniques used to prove the Palis Conjecture in one dimension are centered around renormalization. Indeed, the fine geometrical properties of unimodal maps are closely related to the phenomena described in the conjecture.

The Palis Conjecture is the long term goal of smooth dynamics. We are still far from such a general understanding. However, it as been proved in one-dimension.

The natural next step is to go to two-dimensional dynamics, the Hénon family. The results by M. Benedicks and L. Carleson are the first fundamental steps towards the Palis Conjecture for Hénon maps. The renormalization work done at the accumulation of period doubling was used to show that the Morse-Smale maps are dense in the set of strongly dissipative Hénon maps with entropy zero, (Lyubich and Martens 2008). Although, even this result on density of Morse-Smale maps is more involved than the one-dimensional counterpart, renormalization technique are able to deal with the situation.

As in one-dimension, renormalization should become an intrinsic part of a comprehensive picture of two-dimensional dynamics.
Renormalization of $C^{1+\text{Lip}}$ and $C^2$ unimodal maps