Chapter 6

Brane Solutions and Generating Geodesic Flows

The main goal behind this chapter is to illustrate the power of the sigma-model technique in constructing solutions in a purely algebraic way.

We show first that via torus reduction over the worldvolume of a brane we obtain a link between timelike $p$-branes, e.g. $Dp$-branes, and instantons ($D(-1)$-instanton), and similarly between $Sp$-branes and $S(-1)$-branes\footnote{S-branes have been briefly introduced in chapter 1. We shall define them at greater length in section 6.1.}. The wordvolume reduction of an $Sp$-brane over a Euclidean torus (spacelike reductions) leads to a nonlinear $\sigma$-model coupled to gravity, describing the dynamics of an $S(-1)$-brane. We will restrict to $\sigma$-models whose target space (moduli space) metrics $G_{ij}$ belong to Riemannian maximally non-compact symmetric spaces $G/H$ with $H$ the maximal compact subgroup of $G$. By the same token, worldvolume reduction of a $p$-brane over a Lorentzian torus results in a nonlinear $\sigma$-model based on pseudo-Riemannian symmetric spaces $G/H^*$ with $H^*$ is a non-compact version of $H$.

Next we shall show that the branes are described by a geodesic motion on moduli spaces. Our approach is based on the construction of the \textit{the minimal generating geodesic solution}: a geodesic with the minimal number of free parameters such that all the geodesics are generated by isometry transformations of the moduli space. We will mainly focus on the Kaluza-Klein theory that follows from the reduction of pure Einstein gravity, where $G$ is $\text{GL}(p+1, \mathbb{R})$ in $D > 3$ and $\text{SL}(p+2, \mathbb{R})$ in $D = 3$. In the case of $S$-branes this approach allows the construction of new vacuum time-dependent solutions (fluxless $Sp$-brane solutions). In the case of timelike branes we obtain some stationary solutions of pure gravity which still have to be identified.

This chapter is mainly based on work done in [E], [F] and [G].
6.1 Solutions in (Super)gravity

Many (super)gravity solutions of equations of motion have the structure of a $p$-brane. These solutions have played an essential role in strengthening our belief in dualities in the non-perturbative limit.

6.1.1 $p$-brane Solutions

We have seen in chapter 2 that the spectrum of string theory contains higher rank gauge fields. Therefore, a further generalization of strings is possible, namely $p$-branes, $(p + 1)$-dimensional extended objects in $d$-dimensional spacetime of supergravity theory. The $p$-brane electrically couples to a $(p + 1)$-rank gauge field $A_{(p+1)}$, or magnetically to a $(d - p - 3)$-form gauge field $A_{(d-p-3)}$. Another characteristic of brane solutions is that the geometry has a flat $(p + 1)$-dimensional worldvolume. Generically, we discriminate between two kinds of $p$-brane solutions:

Timelike $p$-branes

They are related to D-branes and M-branes. A D$p$-brane is a static supersymmetric object appearing in string theory and has three descriptions. The first description is the one of chapter 2, namely the D$p$-brane which can be viewed as $(9 - p)$ spacelike boundary conditions for the open string in its perturbative limit. In the supergravity picture D$p$-branes are supersymmetric solitonic solutions extended in $p$ spacelike dimensions and one timelike dimension. We say that the D$p$-brane has a $(p + 1)$-dimensional worldvolume containing time and a $(9 - p)$-dimensional transverse space. They can roughly be seen as higher-dimensional extensions of Reissner-Nordström black hole of four-dimensional Einstein-Maxwell theory. The third picture is less evident and considers stable D$p$-branes as tachyonic kink solutions on the worldvolume of unstable D$(p+1)$-branes [137].

Such branes couple electrically to the gauge fields according to

$$
\int d^{p+1} \sigma A_{\mu_1 \cdots \mu_{p+1}} \partial_{\alpha_1} X^{\mu_1} \cdots \partial_{\alpha_{p+1}} X^{\mu_{p+1}} \epsilon^{\alpha_1 \cdots \alpha_{p+1}},
$$

(6.1.1)

similarly to the point particle ($p = 0$), which couples to a 1-form gauge field, and the NS-NS 2-form $B_{\mu\nu}$ which couples to a string worldsheet. Thus the electric charge of such an object can be determined through a generalization of Gauss’ law:

$$
Q_e \sim \int_{S^{d-p-2}} \ast F_{(p+2)},
$$

(6.1.2)

where $\ast F_{(p+2)}$ represents the Hodge dual of $A_{(p+1)}$ field strength, and the $S^{d-p-2}$ is a sphere surrounding the $p$-brane. This charge is conserved due to the equations of
motion of the gauge field. Moreover there is the dual magnetic \((d-p-2)\)-brane which couples to \(A_{(d-p-3)}\) dual to \(A_{(p+1)}\). Its topological magnetic charge is found to be

\[
Q_m \sim \int_{\mathbb{S}^{p+2}} F_{(p+2)},
\]

which is conserved due to the Bianchi identity. Here we perform the integration over the transversal directions of \(p\)-brane. The charges satisfy the following Dirac’s quantization condition for electric and magnetic monopoles

\[
Q_e \cdot Q_m = 2\pi n, \quad n \in \mathbb{Z}.
\]

The consistent bosonic truncation of the supergravity action that one needs to find the solitonic \(p\)-brane solutions reads

\[
S = \frac{1}{2\kappa^2} \int d^d x \sqrt{|g|} \left( R - \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - \frac{1}{2(p+2)!} e^{a \phi} F^2_{(p+2)} \right),
\]

i.e., brane solutions are supported by the metric, possible the dilaton and a \((p+1)\)-form gauge potential.

Naively, one can think about deriving the equations of motion which follow from 6.1.5, and then solving them for a solitonic \(p\)-brane solution. But this is highly non-trivial. Instead one can write down a convenient Ansatz for such a solution which is given by

\[
ds^2 = e^{2A(r)} \eta_{\mu\nu} dx^\mu dx^\nu + e^{2B(r)} dr^2 + e^{2C(r)} d\Sigma_k^2,
\]

where \(A, B\) and \(C\) are arbitrary functions, \(\eta\) is the usual Minkowski metric in \(p+1\) dimensions \(\eta = \text{diag}(-, +, \cdots, +)\) and \(d\Sigma_k^2\) is the metric of a maximally symmetric space with unit curvature \(k = -1, 0, 1\) (see appendix A.4). We refer to such \(p\)-branes as timelike branes, stationary solutions (time-independent solutions). If we assume that the above Ansatz is consistent with the ISO\((p, 1) \times SO(d-p-1)\) symmetry of spacetime, with the ISO along the worldvolume directions, the Ansatz 6.1.6 becomes

\[
ds^2 = e^{2A(r)} dx^\mu dx^\nu \eta_{\mu\nu} + e^{2B(r)} dy^a dy^{b \delta_{ab}}, \quad \phi = \phi(r),
\]

\[
\mu, \nu = 0, \cdots, p \quad a, b = p+1, \cdots, d-1,
\]

with \(r \equiv \sqrt{y^{a} y^{b} \delta_{ab}}\) the isotropic radial distance in the transverse space. There are two possible solutions of the equations of motion, resulting in an electric or magnetic \(p\)-brane [138, 139]. The metric of those solutions are given by

\[
ds^2 = h^{-\frac{2(p-2)}{d}} dx^2_{(p+1)} + h^{-\frac{2(p+1)}{d-2}} dy^2_{(d-p-1)},
\]

where the harmonic function \(h\) satisfies \(\nabla^2 h = 0\), and the parameter \(\Delta\) is given by

\[
\Delta = a^2 + \frac{2(p+1)(d-p-3)}{d-2}.
\]
For \( d - p - 1 > 2 \), \( h \) can be written as \( h(r) = 1 + \left( \frac{r_0}{r} \right)^{d-p-3} \), where \( r_0 \) is an integration constant related to the charge in the magnetic case. For the electric-brane solution the scalar and gauge fields read

\[
e^\phi = h^{2a}, \quad F_{\alpha \mu_1 \cdots \mu_{p+1}} = \frac{2}{\sqrt{\Delta}} \epsilon_{\mu_1 \cdots \mu_{p+1}} \partial_\alpha (h^{-1}). \tag{6.1.10}
\]

Similarly, the magnetic solution has

\[
e^\phi = h^{-2a}, \quad F_{a_1 \cdots a_{p+2}} = -\frac{2}{\sqrt{\Delta}} \epsilon_{a_1 \cdots a_{p+2}} \partial_r (h). \tag{6.1.11}
\]

The simplest example in type II theories is the electric 1-brane, coupling to the NS-NS 2-form \( B_{\mu \nu} \), called the fundamental string \( F_1 \). This solution can be obtained from the solution above by setting \( p = 1 \), \( a = -1 \), and \( d = 10 \). Its magnetic dual is called the Neveu-Schwarz 5-brane \( NS5 \). As seen in chapter 2, type II also contain RR-gauge fields allowing for a separate class of solutions, D-branes solutions (see chapter 2).

The eleven dimensional supergravity theory only involves one 3-form gauge field, and no dilaton. The only sources one can associate to a 3-form are 2-brane or 5-brane solutions, so we take \( \Delta = 4 \). The resulting solutions are called the electric M2-brane [140] and magnetic M5-brane [141]. Note that the compactification of the M2-brane along its spatial directions yields exactly the \( F_1 \) solution of type IIA supergravity. The \( NS5 \) solution can be obtained by compactifying the M5-brane along a transverse direction.

A special case of \( p \)-brane is the so-called domain wall, a \((d-2)\)-brane with one transverse direction, separating space into two regions.

It has been shown that all those brane solutions preserve half of the supersymmetries of the supergravity theories. This implies that such solutions have to satisfy some first-order differential equations which arise from demanding that the supersymmetry variation of the fermion vanishes. These first order-equations are now referred to as Bogomol’nyi, Prasad and Sommerfeld or BPS equations [142, 143]. In [144] Witten and Olive gave the condition to preserve supersymmetry of solitons in supersymmetric theories. The term BPS equation is now generically used for equations of motion that are inferred by rewriting the action as a sum of squares. Supersymmetric solutions, in general, belong to this class\(^2\). In the literature these supersymmetric branes are also called extremal. The word extremal originates from the fact the branes are subject to a relation between the mass and the charge of \( D_p \)-branes. In other words, when the mass equals the charge a brane is called extremal [146], otherwise the brane is called non-extremal.

\(^2\)Stationary non-extremal and time-dependent solutions (discussed later) cannot preserve supersymmetry in ordinary supergravity theories. However, it has been argued in [E] and [145] that these solutions often can be found from first-order equations called fake-or pseudo-BPS equations.
Spacelike $p$-branes

There is another kind of $p$-brane solutions where the time direction belongs to the transversal space and hence they have Euclidean worldvolume. This means that one chooses a Dirichlet condition for the time-direction. Such $p$-branes are called *spacelike* branes or Sp-$p$-branes for short, and they are explicitly time-dependent.

Similarly to Dp-brane, a Sp-$p$-brane solution is carried by a metric, a dilaton and a $(p+1)$-form gauge potential. The metric, which describes a time-dependent geometry, is schematically given by

$$\begin{align*}
\text{d} s^2 &= e^{2A(t)} \delta_{\mu\nu} \text{d}x^\mu \text{d}x^\nu - e^{2B(t)} \text{d}t^2 + e^{2C(t)} \text{d}\Sigma^2,
\end{align*}\quad (6.1.12)$$

where $\delta$ is the usual flat Euclidean metric in $p+1$ dimensions $\delta_{\mu\nu} = \text{diag}(+,+,\cdots,+)$.

The transverse space consists of time and $(d-p-2)$ dimensions. Ansatz 6.1.12 has ISO$(p+1)$ worldvolume symmetry, and Lorentzian SO$(d-p-2,1)$ transversal symmetry in the $k = -1$ case and can be asymptotically flat (in contrast to $k = +1$ solutions). Those solutions are the spacelike branes introduced by Gutperle and Strominger [147], who conjectured that such branes correspond to specific time-dependent processes in string theory. Nonetheless, in this chapter we will also define S-brane in the generalized sense, i.e. for all the other possible slicings.

Due to the time-dependence the S-brane solutions belonging to type II supergravities are not supersymmetric. Consequently, the solutions are more complicated to write down. Hence we prefer to focus on S$(-1)$-branes of type IIB supergravity. This brane can be viewed as the time-dependent twin of the Euclidean D$(-1)$-instanton. The action of the S$(-1)$-brane follows from the truncation of type IIB supergravity $(d = 10)$ to its scalar sector

$$S = \int \text{d}^d \sqrt{|g|} x \left[ \mathcal{R} - \frac{1}{2} (\partial \phi)^2 - \frac{1}{2} e^{2\phi} (\partial \chi)^2 \right],$$

(6.1.13)

where $\phi$ is the dilaton and we denote the axion with $\chi$ instead of $C_{(0)}$ defined in chapter 2. As we will see in the ensuing sections, the axion and the dilaton form a SL(2, $\mathbb{R}$)/SO(2) nonlinear $\sigma$-model, and also the equations of motion derived from 6.1.13 tells us that the scalar fields trace out a geodesic on the target space.

For $p = -1$ all space is transverse, so the part involving $A$ is not present in the Ansatz 6.1.12. We choose the gauge where $e^{2C} = t^2$ and the Ansatz becomes

$$\begin{align*}
ds^2 &= -f(t) \text{d}t^2 + t^2 \left[ \frac{1}{1-kr^2} \text{d}r^2 + r^2 \text{d}\Omega^2_{d-2} \right],
\end{align*}\quad (6.1.14)$$

For $k = 0$ we have flat space, for $k = +1$ a sphere and finally for $k = -1$ a hyperboloid. This follows from the fact that when $t$ goes to infinity the metric describes a flat

\[\text{In this chapter a Sp-$p$-brane has a } p+1\text{-dimensional Euclidean worldvolume just like a Dp-brane has a } (p+1)\text{-dimensional Lorentzian worldvolume.}\]
Minkowski spacetime only for $k = -1$ (if $f \to 1$). Only for $k = -1$ there is an expected string theory interpretation. The two scalars depend only on $t$.

The metric solution of the equations of motion following from 6.1.13 behaves as

$$ds^2 = -\frac{dt^2}{||v||^2(2(d-1)(d-2) - k)} + t^2 \left( \frac{1}{1-kv^2} dv^2 + v^2 d\Omega_{d-2}^2 \right),$$  \hspace{1cm} (6.1.15)

where $||v||^2$ is a strictly positive number which turns out to be the affine velocity\footnote{The geodesic motion and the affine parameter are defined in appendix A.3.2.} labelling the geodesics traced out by the scalar fields on the targetspace (scalar manifold). The scalar fields solution are given by

$$\phi(t) = \log \left[ C_1 \cosh(||v|h(t) + C_2) \right], \quad \chi(t) = \pm \frac{1}{C_1} \tanh \left[ ||v||h(t) + C_2 \right] + C_3.$$  \hspace{1cm} (6.1.16)

The $C_i$'s are constants of integrations and the harmonic function $h$ is

$$h(t) = \frac{1}{\sqrt{a(2-d)}} \ln \sqrt{at^{2-d} + \sqrt{at^{2(d-2)-k}}} + c,$$  \hspace{1cm} (6.1.17)

with

$$a = \frac{||v||^2}{2(d-1)(d-2)}. \hspace{1cm} (6.1.18)$$

The two scalar fields and the geodesic are related via the following relation

$$||v||^2 = (\partial_h \phi)^2 + e^{2\phi}(\partial_h \chi)^2.$$  \hspace{1cm} (6.1.19)

More about this later.

For the three different values of $k$ we have

- For $k = -1$ one has the S($-1$)-brane of type IIB supergravity [148].
- For $k = 0$ the brane describes a so-called power-law universe in the FLRW-coordinates.
- For $k = +1$ the solution is not really an S($-1$)-brane. Actually it describes a transition from a Big-Bang to a Big Crunch for a closed universe. We recommend [149] for a nice explanation.

In fact, time-dependent backgrounds (solutions), e.g. S-branes, are badly understood in string theory because string theory is not yet formulated in such a background. One of the reasons is, as previously mentioned, that (most) time-dependent backgrounds are not supersymmetric. Similarly to the timelike $p$-brane solutions, e.g. D-branes, there have been some conjectures about the open string picture of S-branes...
as timelike tachyonic kinks. D-branes play a central role in the holographic description of string theory such as the AdS/CFT correspondence [5]. Similarly the S-branes are conjectured to play a similar role as D-branes in the context of holography. However, in this case the holographic duality would be a dS/CFT correspondence [150]. In order to understand this correspondence better one has to improve the understanding of S-branes. Quite recently there was a proposal to understand cosmological solutions as Wick-rotations of supersymmetric domain-walls (DW) [151]. This suggests a relation between DW/QFT correspondence [152] (this is a non-conformal extension of the AdS/CFT correspondence) and a hypothetical COSM/QFT correspondence. Central in that discussion is the concept of pseudo-supersymmetry [151]. It is interesting to formulate pseudo supersymmetry in 10- or 11-dimensional supergravity and interpret S-branes as pseudo-BPS objects. Such a formulation of S-branes would improve the understanding of the dS/CFT correspondence. For more about S-brane solutions we refer the reader to [148, 149, 153–157].

6.2 From \(p\)-brane to (-1)-brane

Since the worldvolume of \(p\)-brane solutions (timelike and spacelike) is translation invariant \((x^a \to x^a + \epsilon^a)\), all these solutions have the property that the worldvolume directions correspond to Killing directions. In order to fulfill this property, the matter fields that carry the solutions must also be translation invariant. This implies that one can effectively dimensionally reduce the solution over the worldvolume. Thus a \(p\)-brane can be mapped to a \((-1)\)-brane solution in \(D = d - p - 1\) dimensions whose equations of motion can be derived from the action:

\[
S = \int d^D x \sqrt{|g|} \left[ R - \frac{1}{2} G_{ij}(\Phi) \partial_\mu \Phi^i \partial^\mu \Phi^j \right],
\]

where \(G_{ij}\) is the metric defined in eq.5.2.1 on the moduli space that appears after dimensional reduction over a torus. For timelike branes, time is included in the reduction and the corresponding moduli spaces are pseudo-Riemannian \(G/H^*\), in contrast to moduli spaces (Riemannian \(G/H\)) appear after spacelike reduction. If we compare 6.1.12 and 6.1.6 with 5.3.13, we realize that one has to set the Kaluza-Klein vectors to zero. In addition, the worldvolume of the theory is identified with \(\mathcal{M}_{mn}\). In general, the metric of the torus \(\mathcal{M}_{mn}\) breaks the worldvolume symmetries (ISO\((p, 1)\) and ISO\((p + 1)\)), since we will obtain extra terms multiplying the \(dx dx\)-terms on the worldvolume. If we reduce to \(D = 3\) one can dualize all Kaluza-Klein vectors to scalars (see section 5.3.3). These Kaluza-Klein vectors will lead to off-diagonal terms that mix worldvolume directions with transversal directions (product of \(x\)-direction and angular direction).

The metric Ansatz for the \((-1)\)-brane is

\[
ds^2_D = \epsilon f^2(\rho) d\rho^2 + g^2(\rho) g_D^{ab} dx^a dx^b, \quad \Phi^i = \Phi^i(\rho),
\]

(6.2.2)
where the function $f$ corresponds to gauge freedom of reparameterizing the coordinate $\rho$.

Now we have two cases

- $\epsilon$ is -1, then the radial coordinate $\rho$ corresponds to time, i.e. $\rho = t$, and $g_{ab}$ is a metric on a Euclidean maximally symmetric space (see A.4), the three possible FLRW-geometries.

- For $\epsilon = 1$, 6.2.2 describes an instanton geometry with $\rho = r$ the direction of the tunnelling process. Timelike $D(-1)$-branes are solutions of Euclidean supergravities.

If we reparameterize the coordinate $\rho$ to $h(\rho)$ via

$$dh(\rho) = g^{1-D}f d\rho,$$

then the equations of motion for the scalars are derived from the one dimensional action

$$S = \int G_{ij}\partial h \Phi^i \partial h \Phi^j dh.$$  \hfill (6.2.4)

This action demonstrates that the solutions describe geodesic motion on the moduli space with $h(\rho)$ as an affine parameter (see appendix A.3.2). From equation 6.2.3 we read off that $h(r)$ is a harmonic function on the $(-1)$-brane geometry. In terms of the affine parameter the velocity $\|v\|$ is constant such that

$$\|v\|^2 = G_{ij}\partial h \Phi^i \partial h \Phi^j.$$ \hfill (6.2.5)

The Ricci tensor following from the metric 6.2.2 is given by

$$R_{\rho\rho} = (D - 1)|-\frac{\ddot{g}}{g} + \frac{\dot{g}\dot{f}}{gf}|,$$ \hfill (6.2.6)

$$R_{ab} = -\epsilon \left(\frac{\ddot{g}g - \frac{\dot{g}^2 f^2}{f^2} + (D - 2)\frac{\dot{g}^2}{f^2}}{f^2}\right)g^{D-1}_{ab} + R^{D-1}_{ab},$$ \hfill (6.2.7)

where a dot denotes differentiation with respect to $\rho$. The Einstein equation for $(-1)$-branes is expressed as

$$R_{\rho\rho} = \frac{1}{2}G_{ij}\partial h \Phi^i \partial h \Phi^j = \frac{1}{2}\|v\|^2(\partial h(\rho))^2, \quad R_{ab} = 0.$$ \hfill (6.2.8)

The combination of the Einstein equation together with 6.2.5 leads to the following first-order equation

$$\dot{g}^2 = \frac{\|v\|^2}{2(D - 2)(D - 1)}f^2 g^{D-2D} + \epsilon k f^2.$$ \hfill (6.2.9)
This equation gives rise to a solution only when the right-hand side remains positive. Note that there is no equation of motion for $f$ due to the fact that it corresponds to the reparametrization freedom that one has of $\rho$. Interestingly enough, the scalar field solution has no influence on solving the metric.

To summarize, one has two kinds of worldvolume reduction ($WR$) of branes:

- **Worldvolume reduction of spacelike branes:** the resulting moduli space is Riemannian $G/H$ with a compact isotropy group. Therefore the metric $G_{ij}$ is positive definite, and then $||v||^2 > 0$. Thus the scalar fields trace out spacelike geodesics on the moduli space $G/H$. We have seen also that $Sp$-branes reduced over their worldvolumes lead to a system containing gravity and scalars fields only. Generically, a solution which is carried by a metric and scalar fields alone has a simpler mathematical structure than those solutions that are carried by non-trivial $p$-form potentials. When we have solved the lower-dimensional (scalar) equations of motion we can lift up the solution to the original fields. This way one might obtain a solution carried by a non-trivial $p$-form potential as well. We will see later in this chapter that via a worldvolume uplifting ($WU$) the $S(-1)$-brane of a pure Kaluza-Klein theory becomes a fluxless $Sp$-brane. We refer to [158, 159] for a description of spacelike branes, in maximal supergravity, in terms of a geodesic motion. We thus have the following map

$$Sp\text{-brane} \xrightarrow{WR} S(-1)\text{-brane} \xrightarrow{WU} Sp\text{-brane}. \quad (6.2.10)$$

- **Worldvolume reduction of timelike branes:** for this case the reduction gives rise to a pseudo-Riemannian moduli space $G/H^*$ with a non compact isotropy group. Hence the metric $G_{ij}$ has an indefinite signature and as a result $||v||^2$ can be zero, positive or negative. Therefore the geodesic curves traced out by the scalar fields on $G/H^*$ are labeled according to the sign of $||v||^2$, i.e. null–like, spacelike and timelike. As an example of a geodesic motion on the moduli space, we consider the supersymmetric IIB instanton [160]. That solution corresponds to the lightlike geodesics on $SL(2, \mathbb{R})/SO(1, 1)$ (the Euclidean axion-dilaton system) whereas the non-supersymmetric IIB instantons correspond to spacelike and timelike geodesics [161] on $SL(2, \mathbb{R})/SO(1, 1)$. This way we obtain

$$p\text{-brane} \xrightarrow{WR} (-1)\text{-brane} \xrightarrow{WU} p\text{-brane}. \quad (6.2.11)$$

In the case of the reduction of timelike branes, the correspondence between geodesics and branes is probably best known in terms of four-dimensional black holes (0-branes) and the three-dimensional instantons [130, 162–164].

### 6.2.1 $(-1)$-brane Geometries

Here we want to look for metric solutions belonging to the action 6.2.1 which only depend on the coordinate $\rho$. 

S(-1)-brane Geometries

We first consider the spacelike (-1)-branes ($\rho = t$). For this case the target space is Riemannian and all geodesics have strictly positive affine velocity squared $||v||^2 > 0$. The solution to the Einstein equations 6.2.9 gives the following $D$-dimensional metric

$$ds_D^2 = -\frac{dt^2}{at^{-2(D-2)} - k} + t^2 d\Sigma_k^2, \quad a = \frac{||v||^2}{2(D-1)(D-2)},$$

(6.2.12)

while the scalar fields trace out geodesics curves with the harmonic function $h(t)$ as affine parameter. The harmonic function $h$ is given by

$$h(t) = \frac{1}{\sqrt{a(2-D)}} \ln |\sqrt{at^{2-D}} + \sqrt{at^{2-(2-D)} - k}| + b.$$  

(6.2.13)

We take $b=0$ in what follows since $b$ just corresponds to a shift in the affine parameter $h$. For $k = 1$ the metric 6.2.12 has a coordinate singularity.

Timelike (-1)-brane Geometries: D(-1)-instanton

As mentioned above the timelike (-1) brane ($\rho = r$) is an instanton. Its geometry entirely depends on the character of the geodesic curve (spacelike, nulllike or timelike). Some of these solutions have appeared in the literature before [130, 149, 161, 165, 166].

- $||v||^2 > 0$
  
  For this class of instantons we will be using the gauge $f = g$. In the table below we present the conformal factor $f$ that determines the metric and the radial harmonic function $\rho$. Note that for all three values of $k$ the solutions have singularities.

- $||v||^2 = 0$
  
  We take the Euclidean “FLRW gauge” for which $f = 1$. It is clear from (6.2.9) that for $k = -1$ we do not find a solution and that for $k = 0$ we find flat space in Cartesian coordinates ($g = 1$) and for $k = +1$ we find flat space in spherical coordinates ($g = r$). This makes sense since a lightlike geodesic motion comes with zero “energy-momentum”\(^5\). The harmonic function is

$$k = 0 \quad h(r) = cr + b,$$

$$k = 1 \quad h(r) = \frac{c}{r^{D-2}} + b.$$  

(6.2.14)

\(^5\)The fact that the $k = -1$ solution does not exists reflects that there does not exist a hyperbolic slicing of the Euclidean plane.
Table 6.2.1: The Euclidean geometries with $||v^2|| > 0$ in the gauge $f = g$. The real number $b$ is an integration constant.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$f(r)$</th>
<th>$h(r)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-1$</td>
<td>$f(r) = \left( \frac{</td>
<td></td>
</tr>
<tr>
<td>$0$</td>
<td>$f(r) = \left( \frac{(D-2)</td>
<td></td>
</tr>
<tr>
<td>$+1$</td>
<td>$f(r) = \left( \frac{</td>
<td></td>
</tr>
</tbody>
</table>

where $c$ is a constant. In Euclidean IIB supergravity the axion-dilaton parameterize SL(2, $\mathbb{R}$)/SO(1, 1) and for $||v||^2 = 0$ and $k = 1$ we have the standard half-supersymmetric D-instanton [160].

- $||v||^2 < 0$
  For $k = 0$ and $k = -1$ we clearly have no solutions since the right-hand side of (6.2.9) is always negative. For $k = +1$ a solution does exist, and in the conformal gauge ($g = fr$) it is given by
  $$f(r) = \left( 1 - \frac{||v||^2}{8(D-1)(D-2)} r^{-2(D-2)} \right)^{\frac{1}{D-2}},$$
  (6.2.15)
  where indeed only $||v||^2 < 0$ is valid. This geometry is smooth everywhere and describes a wormhole, since there is a $\mathbb{Z}_2$-symmetry that acts as follows
  $$r^{D-2} \rightarrow \frac{-||v||^2}{8(D-1)(D-2)} r^{-(D-2)},$$
  (6.2.16)
  and interchanges the two asymptotic regions. The harmonic function is given by
  $$h(r) = \sqrt{\frac{8(D-1)}{(D-2)||v||^2}} \arctan\left( \sqrt{\frac{-||v||^2}{8(D-1)(D-2)}} r^{-(D-2)} \right) + b.$$
  (6.2.17)
  For a very nice definition of a wormhole and its geometrical structure we refer the reader to [161].
6.3 Geodesic Curves

Our method to understanding all geodesic curves, traced out by scalars falling into $G/H$ or $G/H^*$ nonlinear $\sigma$-models which arise in pure Kaluza-Klein theories and maximally extended supergravities, is by constructing the generating solution. This is what we call a solution-generating technique. By definition, a generating solution is a geodesic with the minimal number of arbitrary integration constants so that the action of the isometry group $G$ generates all other geodesics from the generating solution. In [E] we proved a theorem states that for $G/H$, where $G$ is a maximally non-compact real form of a complex semi-simple group and $H$ is the maximal compact subgroup, the generating solution can be taken to be the straight line through the origin carried only by the dilaton fields.

\textbf{Proof}

We have seen in chapter 5 that the Riemannian symmetric spaces $G/H$ can be parameterized by the Borel coordinate system, namely by scalar fields which are divided into dilatons $\phi^I$ and axions $\chi^\alpha$. This can be done by choosing the coset representative to be

$$L = \Pi_I \exp \left[ \frac{1}{2} \phi^I H_I \right] \Pi_\alpha \exp [\chi^\alpha E_\alpha],$$

where $H_I$ are the generators spanning the Cartan subalgebra CSA of the Lie algebra of $G$, and the $E_\alpha$ are the positive root generators. We know that the dimension of the CSA is the rank $r$ of $G$ and for the symmetric spaces $G/H$ (moduli spaces) listed in the table 5.4.1, the rank is $r = 11 - D$. Since the Lie algebra of $H$ is spanned by the combinations $E_\alpha - E_{-\alpha}$, the number of axions equals the dimensions of $H$.

In the Borel gauge the geodesic equation takes on the form

$$\ddot{\phi}^I + \Gamma^I_{JK} \dot{\phi}^J \dot{\phi}^K + \Gamma^I_{\alpha J} \dot{\chi}^\alpha \dot{\phi}^J + \Gamma^I_{\alpha \beta} \dot{\chi}^\alpha \dot{\chi}^\beta = 0,$$

$$\ddot{\chi}^\alpha + \Gamma^\alpha_{JK} \dot{\phi}^J \dot{\phi}^K + \Gamma^\alpha_{\beta J} \dot{\chi}^\beta \dot{\phi}^J + \Gamma^\alpha_{\beta \gamma} \dot{\chi}^\beta \dot{\chi}^\gamma = 0.$$  

At points for which $\chi^\alpha = 0$, the components $\Gamma^I_{JK}$ and $\Gamma^\alpha_{JK}$ of the Christoffel symbol on the moduli space $G/H$ vanish. Therefore a trivial solution can be found

$$\phi^I = v^I t, \quad \chi^\alpha = 0,$$

for some parameter $t$. How many other solutions are there? A first thing we notice is that every global $G$-transformation $\Phi \rightarrow \tilde{\Phi}$ brings us from one solution to another solution. Since $G$ generically mixes dilatons and axions we can construct solutions with non-trivial axions in this way.

Let us now prove that in this way all geodesics are obtained and that depends on the fact that $G$ is maximally non-compact with $H$ the maximal compact subgroup of $G$. Consider an arbitrary geodesic curve $\Phi(t)$ on $G/H$. The point $\Phi(0)$ can be mapped
to the origin $L[\Phi(t)] = 1$ using $G$-transformation (in the fundamental representation), since we can identify $\Phi(0)$ with an element of $G$ and then we multiply the geodesic curve $\Phi(t)$ with $\Phi(0)^{-1}$, generating a new geodesic curve $\Phi_2(t) = \Phi(0)^{-1}\Phi(t)$ that goes through the origin. The origin is invariant under $H$-rotations but the tangent space at the origin transforms under the adjoint representation $(\text{Adj})$ of $H$. One can prove that there always exists an element $k \in H$, such that $\text{Adj}_k \Phi_2(0) \in CSA$ [167]. Therefore $\dot{\chi}^\alpha = 0$ and this solution must be straight line. In short, we started out with a general curve $\Phi(t)$ and proved that the curve $\Phi_3(t) = k\Phi(0)^{-1}\Phi(t)$ is a straight line

$$\phi^I(t) = v^I h(t), \quad \chi^\alpha = 0, \quad (6.3.5)$$

where $h(t)$ is the harmonic function obtained by the reparametrization 6.2.3 for $\rho = t$ (time-dependent S-brane solutions are carried by the scalars $\Phi(t)$). This is the end of the proof. ■

For the sake of illustration, let us give a counting argument to advocate the theorem. The number of integration constants in the geodesic equation of motion equals $2 \times (\dim Borel)$ since for every scalar field $\phi^I, \chi_\alpha$ we have to specify the initial speed and place. If we classify geodesic $\tau$ by the couple

$$\tau = (\vec{v}, g), \quad \vec{v} \in CSA, \quad g \in G, \quad (6.3.6)$$

then the number of parameters is indeed $\dim CSA + \dim G = 2 \times (\dim Borel)$ under the conditions on $G/H$ stated. Thus the number of dilatons is given by $r = \text{rank}G$, and the number of axions by $\dim H$. Nonetheless, one can not forget about the theorem proved above and consider this counting as a proof since it may be that the action of $G$ does not create independent integration constants.

Unfortunately, due to the fact that the Borel gauge is not a valid gauge on pseudo-Riemannian spaces $G/H^*$, the derivation of the generating geodesic is no longer feasible by working on the level of the coset representative $L$ (unless one can find a “good” gauge choice). Instead one can work on the level of the matrix $M$ (see section 5.2.5) so that one does not need to be bothered with the subtleties regarding the gauge choice. Note that the counting argument given above is not a trustful proof for $G/H^*$ either since it might happen that the solutions lie in disconnected areas of the moduli spaces. There the straight line solution is not generating since the affine velocity is positive:

$$||v||^2 = \sum_I (v^I)^2 > 0. \quad (6.3.7)$$

The affine velocity is invariant under $G$-transformation and by transforming the straight line we only generate spacelike geodesics ($||v||^2 > 0$). But cosets with non-compact $H^*$ (see table 5.4.1 for examples in maximal supergravity) have metrics with an indefinite signature and therefore allow for lightlike, spacelike and timelike geodesics.
In the following we will work directly with the matrix $\mathcal{M}$ or \hat{$\mathcal{M}$} (defined in 5.3.16), restricting to pure Kaluza-Klein theories, i.e. \( G \) will be the group $\text{GL}(n, \mathbb{R})$ for $D > 3$ or $\text{SL}(n + 1, \mathbb{R})$ for $D = 3$ \([G]\). The approach that we will follow allows us to rederive the straight line generating geodesic of $\text{GL}(n, \mathbb{R})/\text{SO}(n)$ moduli space but also allows for the generalisation to $\text{GL}(s + r = n, \mathbb{R})/\text{SO}(r, s)$. The extension of our approach to cover the symmetric spaces $G/H^*$ of the right column of table 5.4.1, namely of those of the maximally-extended Euclidean supergravities, has been worked out in \([G]\).

### 6.3.1 The Geodesic Curves of Pure Kaluza-Klein Theory

As mentioned before the \((-1)\)-brane solutions are carried by the metric and the scalar fields and therefore we truncate the Kaluza-Klein vectors in $D > 3$ and dualize them in $D = 3$. The \((-1)\)-brane geometries have just been described above. Let us therefore focus on the geodesic motion that comes about.

The $\text{SL}(n, \mathbb{R})/\hat{H}$ nonlinear $\sigma$-model action of the geodesic curves can be compactly written in terms of the symmetric coset matrix $\mathcal{M}$

\[
S = \int dh \, \text{tr}[\partial_h \mathcal{M} \partial_h \mathcal{M}^{-1}], \quad \mathcal{M} = L^T \eta L, \quad (6.3.8)
\]

where $\hat{H}$ can be $H = \text{SO}(n)$ or $H^* = \text{SO}(n-1,1)$, $\text{SO}(n-2,2)$. The matrix $\eta$ is the $\hat{H}$-invariant matrix. The corresponding equations of motion can compactly be written as

\[
[\mathcal{M}^{-1} \mathcal{M}']' = 0, \quad (6.3.9)
\]

where the prime denotes the derivative with respect to the affine parameter $h(\rho)$. This implies that $\mathcal{M}^{-1} \mathcal{M}' = K$ with $K$ a constant matrix, which can be seen as the matrix of Noether charges of the group $\text{SL}(n, \mathbb{R})$ (see section 5.2.2). The affine velocity squared of the geodesic curves is $||v||^2 = \frac{1}{4} \text{tr}[K^2]$. Because of the identity $\mathcal{M}^{-1} \mathcal{M}' = K$, the problem is integrable and a general solution is found to be

\[
\mathcal{M}(h) = \mathcal{M}(0)e^{Kh(\rho)}. \quad (6.3.10)
\]

By virtue of the transitive action of the isometry group $\text{SL}(n, \mathbb{R})$ on the symmetric space, we can restrict ourselves to geodesics that go through the origin. Since one has the freedom of affine reparametrization of $h$ we can assume that $L(0) = 1$. The matrix of Noether charges is not completely arbitrary, it is actually subject to a constraint which can be derived from the properties of $\mathcal{M}$

\[
\eta K = K^T \eta, \quad \text{tr}[K] = 0. \quad (6.3.11)
\]

\(K\) is an element of the Lie algebra of $\text{SL}(n, \mathbb{R})$ and accordingly it transforms in the adjoint of $\text{SL}(n, \mathbb{R})$

\[
K \to \Omega K \Omega^{-1}, \quad (6.3.12)
\]
under which the \((n - 1)\) quantities

\[
I_k = \text{tr}[K^{k+1}], \quad \text{with } k = 1, \cdots, n - 1
\]

(6.3.13)

are invariant, i.e. are Casimirs. Notice that the constraints 6.3.11 are not invariant under the total isometry group but under the smaller isotropy group \(\tilde{H}\).

We here stress that for a pure Kaluza-Klein theory in \(D > 3\) all geodesics that are related through a \(\text{GL}(n, \mathbb{R})\)-transformation lift up to exactly the same vacuum (pure gravity) solution in \(D + n\) dimensions since the \(\text{GL}(n, \mathbb{R})\) corresponds to rigid coordinate transformations from a \((D + n)\)-dimensional point of view. So, in this sense it is absolutely necessary to understand the generating geodesic since it classifies higher dimensional solutions modulo coordinate transformations. Of course, this is not true for \(D = 3\) where \(\text{SL}(n + 1, \mathbb{R})\) maps higher dimensional solutions to each other that are not necessarily related by coordinate transformations.

Due to the non-compactness of the isotropy group such as \(\text{SO}(n - 1, 1)\), the theory will contain ghosts. A ghost, as previously mentioned, is an axion field with the opposite sign for the kinetic term in the Lagrangian. For the sake of generality, let us discuss the ghost content for a theory with scalar coset \(\text{GL}(r + s, \mathbb{R})/\text{SO}(r, s)\).

For a general symmetric space \(\text{GL}(s + r)/\text{SO}(r, s)\) the number of axion fields with the opposite sign for the kinetic term (ghosts) is \(r \times s\). For the pure Kaluza-Klein moduli spaces this can be seen as follows. When one considers a reduction over time then there are two possible origins for ghosts. Ghost fields \(\chi^\Lambda\) appear as the zero-component of a one-form \(\hat{A}^\Lambda\) in the higher dimension, that is \(\hat{A}^\Lambda = \chi^\Lambda dt + A^\Lambda\). Alternatively, extra ghost fields appear in three dimensions upon dualisation of the one-form\(^6\). Therefore, imagine we reduce Einstein gravity in \(D + n\) to dimensions to \(D + 1\) dimensions over a spacelike torus and then perform a subsequent reduction over a timelike circle, then the \(n - 1\) Kaluza-Klein vectors in \(D + 1\) dimensions give \(n - 1\) ghostlike axions. This fits with the fact that the scalar coset is \(\text{GL}(n, \mathbb{R})/\text{SO}(n - 1, 1)\). If \(D = 3\) then we can further dualise those \(n - 1\) descendants of the Kaluza-Klein vectors to \(n - 1\) ghostlike axions, thereby doubling the number of ghosts. The Kaluza-Klein vector that appears from the last timelike reduction does not dualise to a ghost but to a normal axion since that vector appeared with a wrong sign in three dimensions. This indeed explains why there are \(2(n - 1)\) ghosts in \(\text{SL}(n + 1)/\text{SO}(n - 1, 2)\).

Since the matrix \(K\) determines all geodesics through the origin, and by transitivity all geodesics on \(G/H\) we will look for the normal form \(K_N\) of \(K\) under \((H \subset G)\)-transformations. As a result the geodesics will be determined by the “integration constants” in \(K_N\), generating all geodesics through a rigid \(G\)-transformation\(^7\). A matrix normal form or matrix canonical form describes the transformation of a matrix

\(^6\)This is due to the fact that the three-dimensional theory is Euclidean.

\(^7\)In the discussion section we will briefly mention another approach to generate geodesic solutions, this approach uses the local isotropy \(H\).
to another with special properties. For instance the normal form of a symmetric matrix is the diagonal matrix which can be realized by the action of the orthogonal groups. Since we have restricted to geometries that go through the origin- a point which is invariant under the action of the isotropy $H$- we see that one can always find matrices $K_N$ having all possible combinations of values of the invariants $I_k$, namely one can restrict the $n-1$ invariants of $K$ on $K_N$ and hence any $H$-orbit has one element of the form $K_N$. In other words we can always transform a generic $K$ into $K_N$ through an $H$ transformation. Below we derive the normal forms of the matrix $K$ associated to the generating geodesic curves on the $\text{GL}(n,\mathbb{R})/\text{SO}(n)$, $\text{GL}(n,\mathbb{R})/\text{SO}(n-1,1)$, and $\text{GL}(n+1,\mathbb{R})/\text{SO}(n-1,2)$ moduli spaces. We are able to give a compact proof for the general case $\text{GL}(r+s,\mathbb{R})/\text{SO}(r,s)$, instead of a case by case discussion. It is worth recalling that GL-cosets are parameterized by the matrix $M$ (see section 5.3.16) rather than $M$.

The Normal Form of $\text{gl}(r+s)/\text{so}(r,s)$

Consider $K \in \text{gl}(r+s)/\text{so}(r,s)$, where $\text{gl}$ and $\text{so}$ are respectively the Lie algebras of GL and SO. By definition $K$ obeys

$$\eta K = K^T \eta, \quad \text{with} \quad \eta = (-I_r, +I_s). \quad (6.3.14)$$

Two eigenvectors of $K$, $v_1$ and $v_2$, that belong to different eigenvalues $\lambda_1$ and $\lambda_2$ are necessarily orthogonal with respect to the inner product $(,)$ defined with the bilinear form $\eta$, because $(v_2, K v_1) = (K v_2, v_1)$ and thus $\lambda_1 (v_1, v_2) = \lambda_2 (v_1, v_2)$. Now if $\lambda_1 \neq \lambda_2$ then this is only consistent if $(v_1, v_2) = 0$. We will say that $v_1$ and $v_2$ are pseudo-orthogonal. If two eigenvectors have the same eigenvalue we can always perform a generalized Gramm-Schmidt procedure so that they become pseudo-orthogonal with respect to $\eta$. In general $K$ may not be diagonalizable. This is the case for instance if $K$ is nilpotent, namely if $K^k = 0$ for some $k > 1$. Or proof applies for diagonalizable matrices only. By definition $M$, see eq. 6.3.10, and therefore $K$, should always be real matrices and thus if $\lambda$ is a complex eigenvalue of $K$ also $\bar{\lambda}$ is. Let $v$ and $\bar{v}$ be the corresponding eigenvectors. If we write $v = v_1 + iv_2$ and $\lambda = \lambda_1 + i\lambda_2$ then this means that

$$K v_1 = \lambda_1 v_1 - \lambda_2 v_2, \quad K v_2 = \lambda_2 v_1 + \lambda_1 v_2, \quad (6.3.15)$$

equation of pseudo-orthogonality between $v$ and $\bar{v}$ implies $(v_1, v_1) = -(v_2, v_2)$.

In what follows we shall consider the cases in which $K$ is nilpotent as singular limits of diagonalizable matrices, construct a normal form $K_N$ of $K$ and then show that the resulting normal form also encodes the most general nilpotent case. In a

---

8Here the $\text{GL}(n,\mathbb{R}) = \mathbb{R}^+ \times \text{SL}(n,\mathbb{R})$ with $\mathbb{R}^+$ is associated with the breathing mode.

9A nilpotent matrix is an $n \times n$ square matrix $K$ such that $K^m = 0$ for some positive integer $m$. 
suitable basis constructed out of the real and imaginary parts of $\mathbf{v}$, the matrix $K$ is represented by a $2 \times 2$ real block. We shall construct a basis out of the eigenvectors of a diagonalizable $K$ in which $K$ has a block diagonal form $K_N$, with a real $2 \times 2$ block for each couple of $\lambda$, $\bar{\lambda}$ eigenvalues, and single diagonal entry for each real eigenvalue. $K_N$ will satisfy the same property 6.3.14 as $K$ and can be obtained from it through an SO$(r, s)$ conjugation. The general form of $K_N$ is thus characterized by the maximal number of complex eigenvalues. In the following we shall construct $K_N$ for the coset $GL(r + s, \mathbb{R})/SO(r, s)$ and show that the maximal number of complex eigenvalues is $\min(r, s)^{10}$.  

**Construction of $K_N$**

Let $K \in gl(n, \mathbb{R})/so(r, s)$, with $n = r + s$ and $r \leq s$, be an $n \times n$ real matrix satisfying eq. 6.3.14. We want to show that $K$ can have at most $r$ complex distinct eigenvalues.

Let $\mathbf{v} = (\vec{v}, \vec{w})$ denote a vector in $\mathbb{R}^{r,s}$ with $\vec{v} \in \mathbb{R}^r$ and $\vec{w} \in \mathbb{R}^s$. We start showing that a set $\mathbf{v}(i) = (\vec{v}(i), \vec{w}(i))$, $i = 1, \ldots, \ell + 1$, of mutually pseudo-orthogonal, null-vectors, are linearly independent iff the vectors $(\vec{v}(i))$ are. Suppose $(\vec{v}(i))_{i=1,\ldots,\ell}$ are linearly independent but that $\vec{v}(\ell+1)$ can be expressed as a linear combination of the first $\ell$: $\vec{v}(\ell+1) = \sum_{i=1}^{\ell} a^i \vec{v}(i)$. Let us show that $\mathbf{v}(\ell+1) = \sum_{i=1}^{\ell} a^i \mathbf{v}(i)$. By hypothesis $(\mathbf{v}(i))_{i=1,\ldots,\ell+1}$ are light-like and mutually pseudo-orthogonal:

$$
(\mathbf{v}^{(i)}, \mathbf{v}^{(j)}) = 0 \iff \vec{v}^{(i)} \cdot \vec{v}^{(j)} = \vec{w}^{(i)} \cdot \vec{w}^{(j)}, \; \forall i, j = 1, \ldots, \ell + 1. \quad (6.3.16)
$$

Being $(\vec{v}^{(i)})_{i=1,\ldots,\ell}$ linearly independent, we can define the following non singular matrix $h_{ij}$:

$$
h_{ij} = \vec{v}^{(i)} \cdot \vec{v}^{(j)} = \vec{w}^{(i)} \cdot \vec{w}^{(j)}, \; i, j = 1, \ldots, \ell. \quad (6.3.17)
$$

The vector $\vec{w}^{(\ell+1)}$ will have the general form: $\vec{w}^{(\ell+1)} = \sum_{i=1}^{\ell} c^i \vec{w}^{(i)} + \vec{w}_\perp$, where $\vec{w}^{(i)} \cdot \vec{w}_\perp = 0$, $\forall i = 1, \ldots, \ell$. Then from the pseudo-orthogonality condition 6.3.16 we find that $\vec{v} h_{ij} = \vec{v}^{(i)} \cdot \vec{w}^{(\ell+1)} = \vec{v}^{(i)} \cdot \vec{v}^{(\ell+1)} = a^j h_{ij}$, from which we conclude that $c^i = a^i$, $i = 1, \ldots, \ell$. From this it follows that $\mathbf{v}(\ell+1) = \sum_{i=1}^{\ell} a^i \mathbf{v}(i) + (0, \vec{w}_\perp)$. Requiring $\mathbf{v}(\ell+1)$ to be null, we find $\vec{w}_\perp \cdot \vec{w}_\perp = 0$, which implies $\vec{w}_\perp = 0$ and thus $\mathbf{v}(\ell+1) = \sum_{i=1}^{\ell} a^i \mathbf{v}(i)$. Therefore if $(\vec{v}^{(i)})_{i=1,\ldots,\ell+1}$ are not linearly independent, the same is true for $(\mathbf{v}^{(i)})_{i=1,\ldots,\ell+1}$. It is straightforward to show the reverse, namely that if $(\vec{v}^{(i)})_{i=1,\ldots,\ell+1}$ are linearly independent, also $(\mathbf{v}^{(i)})_{i=1,\ldots,\ell+1}$ are.

Let us now suppose that $K$ has $r+1$ distinct complex eigenvalues $\lambda_i \neq \bar{\lambda}_i$, $i = 1, \ldots, r + 1$, and let $\mathbf{v}(i) = \mathbf{v}_1^{(i)} + i \mathbf{v}_2^{(i)}$ be the corresponding linearly independent eigenvectors. Let us show that $\mathbf{v}^{(r+1)}$ is either zero or a linear combination of the

---

10This number of distinct complex eigenvalues of $K_N$ will turn out to be equal to the number of ghostlike scalar fields that will be existing in the geodesics generating solution $\mathcal{M}_N$. 

first $r$ eigenvectors, contradicting thus the hypothesis. From the general analysis we know that
\[ (v_{1,2}^{(i)}, v_{1,2}^{(j)}) = 0, \quad i \neq j, \quad (v_1^{(i)}, v_1^{(j)}) = -(v_2^{(i)}, v_2^{(i)}). \quad (6.3.18) \]

We shall consider the case in which if $v_1^{(i)}$ is null for some $i$, $v_1^{(i)} \cdot v_2^{(i)} \neq 0 \quad 11$. Under these assumptions the linear independence of $v^{(i)}$ implies the linear independence of $v_1^{(i)}$. If all $v_1^{(i)}$ (and thus $v_2^{(i)}$) were null-vectors, then we would have $r+1$ independent, mutually pseudo-orthogonal null vectors. This would imply that the $r+1$ components $\vec{v}_1^{(i)}$ were independent, which can not be, being them $r$-dimensional vectors. Consider now the case in which some of the $v_1^{(i)}$ are timelike, or spacelike. We can define the eigenvectors in such a way that $v_1^{(a)} = (\vec{v}^{(a)}, w^{(a)})$, $a = 1, \ldots, \ell$, are null-vectors, while $v_r^{(r)}, r = \ell + 1, \ldots, r + 1$, are timelike. Moreover we can use $SO(r, s)$ to write the timelike vectors in the form $v_r^{(r)} = (\vec{v}^{(r)}, 0)$, $(\vec{v}^{(r)})$ being mutually orthogonal. Therefore the matrices $h_{rs} = \vec{v}^{(r)} \cdot \vec{v}^{(s)}$ and $h_{ab} = \vec{v}^{(a)} \cdot \vec{v}^{(b)}$ are non–singular matrices. Suppose $v_1^{(r+1)} = (\vec{v}^{(r+1)}, 0)$, and thus $\vec{v}^{(r+1)}$, depends linearly on the first $r$ vectors:
\[ \vec{v}^{(r+1)} = c^a \vec{v}^{(a)} + c^r \vec{v}^{(r)}, \quad (6.3.19) \]

from the pseudo–orthogonality condition we find:
\[ 0 = v_1^{(r)} \cdot v_1^{(r+1)} = \vec{v}^{(r)} \cdot \vec{v}^{(r+1)} = h_{rs} c^s \Rightarrow c^r = 0, \]
\[ 0 = v_1^{(a)} \cdot v_1^{(r+1)} = \vec{v}^{(a)} \cdot \vec{v}^{(r+1)} = h_{ab} c^b \Rightarrow c^a = 0, \quad (6.3.20) \]
which implies that $v_1^{(r+1)} = 0$.

Consider then a matrix with $2r$ distinct complex eigenvalues and $s - r$ real. We can construct the following basis of pseudo-orthonormal vectors $u_1^{(i)}$, defined as follows
\[ v_1^{(i)} \cdot v_1^{(i)} = 1: \quad u_1^{(i)} = v_1^{(i)}, \quad u_2^{(i)} = \sin \alpha_i v_1^{(i)} + \cos \alpha_i v_2^{(i)}, \]
\[ |v_1^{(i)}|^2 = 0: \quad u_1^{(i)} = \frac{v_1^{(i)} \pm v_2^{(i)}}{\sqrt{2 |v_1^{(i)} \cdot v_2^{(i)}|}}, \quad u_2^{(i)} = \frac{v_1^{(i)} \mp v_2^{(i)}}{\sqrt{2 |v_1^{(i)} \cdot v_2^{(i)}|}}, \quad (6.3.21) \]

where we have denoted by $\tan(\alpha_i) = -v_1^{(i)} \cdot v_2^{(i)}$. In the last of the above equations the upper and lower signs refer to the cases in which $v_1^{(i)} \cdot v_2^{(i)}$ is positive and negative respectively. Define now the following matrix
\[ T = (u_1^{(i)}, u_2^{(i)}, \ldots, u_1^{(r)}, u_2^{(r)}, u_1^{(k)}), \quad (6.3.22) \]

---

11 If this were not the case one can show that the corresponding eigenvalue would be degenerate and the matrix not diagonalizable.
where $u^{(k)}$ are the $s - r$ spacelike (normalized) eigenvectors corresponding to the real eigenvalues $\lambda^k$. The matrix $T$ is in $SO(r, s)$:

$$ T^T \eta' T = \eta', \quad \eta' = \text{diag}(+, -, \ldots, +, -, \ldots, -). $$

(6.3.23)

Upon action of $T$, the matrix $K$ will acquire the following normal form $K_N$:

$$ K_N = T^{-1} K T = \begin{pmatrix}
A_1 & 0 & \ldots & 0 \\
0 & \ddots & \ddots & \\
\vdots & \ddots & A_r & 0 \\
0 & \ldots & \ldots & \lambda_s
\end{pmatrix}, $$

(6.3.24)

where $A_1, \ldots, A_r$ are $2 \times 2$ blocks corresponding to each complex eigenvalue $\lambda_1, \ldots, \lambda_r$ whose for is

$$ v_1^{(i)} \cdot v_1^{(i)} = 1 : \quad A_i = \begin{pmatrix}
\lambda_1^{(i)} + \lambda_2^{(i)} \tan(\alpha_i) & -\lambda_2^{(i)} \cos^{-1}(\alpha_i) \\
\lambda_2^{(i)} \cos^{-1}(\alpha_i) & -\lambda_2^{(i)} \tan(\alpha_i) + \lambda_1^{(i)}
\end{pmatrix}, $$

$$ v_1^{(i)} \cdot v_1^{(i)} = 0 : \quad A_i = \begin{pmatrix}
\lambda_1^{(i)} + \pm \lambda_2^{(i)} \\
\mp \lambda_2^{(i)}
\end{pmatrix}. $$

(6.3.25)

The remaining eigenvalues $\lambda_{r+1}, \ldots, \lambda_s$ in $K_N$ are real. On each block $A_i$ we can still act by means of $r$-independent $O(1, 1)$ transformations which may set $\alpha_i$ either to 0 or to $\pi$. Although $K_N$ also describes non diagonalizable matrices, like for instance in the case $\lambda_1 = 0, \lambda_2 = a \cos(\alpha)$ and $\alpha = \pi/2$, in which case the block $A$ is nilpotent, we the above theorem holds for diagonalizable matrices only. If $K$ is not diagonalizable, its normal form can be expressed by the normal form $K_N^{(0)}$ of the diagonalizable matrix $K^{(0)}$ having the same spectrum as $K$, plus a constant nilpotent matrix which interpolates between the blocks corresponding to the same degenerate eigenvalue, the spectrum being encoded in $KK^{(0)}$. For instance an example of the normal form on a non-diagonalizable matrix in $\mathfrak{sl}(4, \mathbb{R})/\mathfrak{so}(2, 2)$ with a 2 times degenerate complex eigenvalue $\lambda \neq \bar{\lambda}$ is:

$$ K_N = K_N^{(0)} + \text{Nil}, $$

$$ K_N^{(0)} = \begin{pmatrix} A & 0 \\
0 & A \end{pmatrix}, \quad \text{Nil} = \begin{pmatrix} \text{Id}_2 & \text{Id}_2 \\
-\text{Id}_2 & -\text{Id}_2 \end{pmatrix}, \quad A = \begin{pmatrix} \lambda_1 & \lambda_2 \\
-\lambda_2 & \lambda_1 \end{pmatrix}. $$

(6.3.26)

We come to the conclusion that there is a subset of $K$’s (of smaller dimension than the whole space of $K$ matrices) that is not “diagonalizable” when the eigenvalues of the charge matrix are degenerate. In the following, we will restrict to the diagonalizable $K$: the absence of the constant nilpotent part $\text{Nil}$.
6.4 Uplift to Einstein Vacuum Solutions

In order to uplift the solutions from $D > 3$ dimensions to $D + n$ ($n=p+1$, the word-volume dimensions of $p$-brane) dimensions one uses the Kaluza–Klein Ansatz 5.3.13 where with Kaluza–Klein vectors put to zero

$$ds^2_{D+p+1} = e^{2\alpha \varphi} ds^2_D + e^{2\beta \varphi} M_{mn} dz^m dz^n.$$  \hfill (6.4.1)

Consider the symmetric coset matrix $\hat{M}(h) = \eta \exp K_N h$ with $K_N$ the normal form of $K$ that generates all other geodesics and $h$ the harmonic function defined in 6.2.3. The relation between $\hat{M}$ and the moduli $\varphi$ and $M$ is as follows

$$\hat{M} = (|\det \hat{M}|)^{\frac{1}{n}} M, \quad |\det \hat{M}| = \exp \sqrt{2n} \varphi.$$  \hfill (6.4.2)

In the following we will present the vacuum solutions obtained from uplifting the $(-1)$-brane solutions of pure Kaluza–Klein theory. This is a nice illustration of the power of the sigma-model technique since we construct the solutions in a purely algebraic way. Solving the second-order differential equations for such a vacuum Ansatz with this degree of complexity is highly non-trivial. It is worth recalling that we will not make use of a coordinate system on the cosets, so in the case of a non-compact isotropy group $H^*$ we do not need to be bothered with subtleties regarding the Borel gauge. Note that for uplifting in $D = 3$ one has to take into account the Kaluza-Klein vectors.

6.4.1 Time-dependent Solutions

Here we consider the time-dependent $(-1)$-brane solutions in $D$ dimensions and their uplifts over a $p + 1$-torus to $S^p$-brane solutions of $D + p + 1 = d$-dimensional pure gravity.

$S^p$-brane from $GL(p + 1 = n, \mathbb{R})/SO(n)$

In section 6.3 we showed that the most general geodesic on $GL(p + 1 = n, \mathbb{R})/SO(n)$ is given by the most general $GL(p + 1, \mathbb{R})$-transformation of the generating straight line solution through the origin. Alternatively, by using the relation 6.3.10 and the normal form $K_N$ (defined above) one can obtain the same result, namely

$$\hat{M}(h) = \begin{pmatrix} e^{\lambda_1 h} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & e^{\lambda_n h} \end{pmatrix},$$  \hfill (6.4.3)

with $h$ given by 6.2.13.

The scalar field matrix transforms as $M \rightarrow \Omega^T M \Omega$ with $\Omega \in SL(p + 1, \mathbb{R})$. 

Therefore all we need is to uplift the straight line geodesic since all the other geodesics are just $\Omega$-transformations which can be absorbed by redefining the torus coordinates $d\vec{z}' = \Omega d\vec{z}$. The higher-dimensional geometries we find depend on the curvature $k$ of the lower-dimensional FLRW-space.

If we take the $(-1)$-brane geometry with $k = 0$, then the generating solution lifts up to the Kasner solutions with $|E|

$$\mathrm{ds}^2 = -\tau^{2p_0} \, d\tau^2 + \sum_b \tau^{2p_b} \, dx^2_b,$$  

$b = 1, \ldots, D + n - 1.$  \hfill (6.4.4)

where the power-laws are defined by

$$p_0 = (D - 2) + \frac{\alpha \sum_i \lambda_i}{\sqrt{2an}}, \quad p_1 = \ldots = p_{D-1} = 1 + \frac{\alpha \sum_i \lambda_i}{\sqrt{2an}},$$  \hfill (6.4.5)

$$p_{D+i-1} = \frac{\sum_i \lambda_i \left( \frac{2\beta}{\sqrt{2n}} - \frac{1}{n} \right)}{2\sqrt{a}} + \frac{\lambda_i}{2\sqrt{a}},$$  \hfill (6.4.6)

where we defined $a$ in equation 6.2.12 and use that $||v||^2 = \frac{1}{2} \sum_i \lambda_i^2$. The numbers $p$ obey the Kasner constraints

$$p_0 + 1 = \sum_{b>0} p_b, \quad (p_0 + 1)^2 = \sum_{b>0} p_b^2.$$  \hfill (6.4.7)

When we take slicing with $k = -1$, we obtain a generalization of the fluxless S-brane solutions [147, 153, 168]. For $k = +1$ the solutions are not given a special name. Uplifting the straight line gives a generalization of the fluxless solutions considered in for instance [169]. Those solutions are the familiar Kasner solutions. We present the solutions with $k = \pm 1$.

$$\mathrm{ds}^2 = W^{p_0} \left( \frac{d\ell^2}{at^{2(D-2)} - k} + t^2 \mathrm{d}^2 \Sigma_k \right) + \sum_{i=1}^n W^{p_i} \left( dz^i \right)^2,$$  \hfill (6.4.8)

where the function $W(t)$ is defined as

$$W(t) = \sqrt{at^{2-D} + \sqrt{at^{2(D-2)} - k}},$$  \hfill (6.4.9)

and the various constants $p_0, p_i$ are defined as

$$p_0 = -\frac{\sum_i \lambda_i}{||v||(D-2)\sqrt{\frac{2(D-1)}{(D+n-2)}}}, \quad p_i = -\frac{D - 2}{n} p_0 + \frac{\left( \sum_j \lambda_j - n \lambda_i \right)}{n||v||\sqrt{\frac{2(D-1)}{D-2}}},$$  

and the affine velocity is given by $||v||^2 = \frac{1}{2} \sum_i \lambda_i^2$.

Note that the $k = -1$ solutions (S-branes) approach flat Minkowski space in Milne coordinates for $t \to \infty$. 

---

6.4 Uplift to Einstein Vacuum Solutions 121
Time-dependent Solutions from $\text{SL}(n+1, \mathbb{R})/\text{SO}(n+1)$

If we reduce to three dimensions a symmetry-enhancement of the coset takes place. The dualisation of the three-dimensional KK vectors generate the coset $\text{SL}(n+1, \mathbb{R})/\text{SO}(n+1)$ instead of the expected $\text{GL}(n, \mathbb{R})/\text{SO}(n)$. However the generating solution of the $\text{SL}(n+1, \mathbb{R})/\text{SO}(n+1)$-coset has only non-trivial dilatons and is therefore the same as the generating solution of $\text{GL}(n, \mathbb{R})/\text{SO}(n)$. Nonetheless, there is an important difference with the time-dependent solutions from $\text{GL}(n, \mathbb{R})/\text{SO}(n)$.

In that case a solution-generating transformation $\in \text{GL}(n, \mathbb{R})$ can be interpreted as a coordinate transformation in $D+n$ dimensions and therefore maps the vacuum solution to the same vacuum solution in different coordinates. In the case of symmetry enhancement to $\text{SL}(n+1, \mathbb{R})$ a solution-generating transformation is not necessarily a coordinate transformation in $D+n$ dimensions. Instead the time-dependent vacuum solution transforms into a ”twisted” vacuum solution. Where the twist indicates off-diagonal terms that cannot be redefined away. Such twisted solutions with $k=-1$ have received considerable interest since they can be regular [156, 157].

6.4.2 Stationary Solutions from $\text{GL}(n, \mathbb{R})/\text{SO}(n-1, 1)$

The normal form is given by

$$K_N = \begin{pmatrix} \lambda_a & \omega & 0 & \cdots & 0 \\ -\omega & -\lambda_a & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} + \begin{pmatrix} \lambda_b & 0 & 0 & \cdots & 0 \\ 0 & \lambda_b & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \lambda_n \end{pmatrix}. \tag{6.4.11}$$

We exponentiate this to

$$\hat{\mathcal{M}}(h(r)) = \eta e^{K_N h(r)} =
\begin{pmatrix}
-\omega e^{\lambda h(r)} f_+(r) & -\omega e^{\lambda h(r)} \Lambda^{-1} \sinh(\Lambda h(r)) & 0 & \cdots & 0 \\
-\omega e^{\lambda h(r)} \Lambda^{-1} \cosh(\Lambda h(r)) & e^{\lambda h(r)} f_-(r) & 0 & \cdots & 0 \\
0 & 0 & e^{\lambda h} & \cdots & 0 \\
0 & 0 & 0 & \cdots & e^{\lambda h(r)} \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix}, \tag{6.4.12}
$$

with

$$f_{\pm}(r) = e^{\lambda h(r)} \left( \cosh(\Lambda h(r)) \pm \lambda_a \frac{\sinh(\Lambda h(r))}{\Lambda} \right), \quad \tag{6.4.13}$$

and where we define the $\text{SO}(1, 1)$ invariant quantity $\Lambda$ as

$$\Lambda = \sqrt{\lambda_a^2 - \omega^2}. \quad \tag{6.4.14}$$
There exist three distinctive cases depending on the character of $\Lambda$. If $\Lambda$ is real the above expression does not need rewriting but we can put $\omega$ to zero using a SO(1, 1)-boost and then the generating solution is just the straight line solution. If $\Lambda = i\tilde{\Lambda}$ with $\tilde{\Lambda}$ real then the terms with $\cosh(\Lambda h)$ become $\cos \tilde{\Lambda}$ and $\Lambda^{-1} \sinh \Lambda h$ become $\tilde{\Lambda}^{-1} \sin \tilde{\Lambda} h$. Finally, if $\Lambda = 0$ then the term $\Lambda^{-1} \sinh \Lambda h$ becomes just $h$ and the term with $\cosh \Lambda h$ becomes equal to one.

To discuss the zoo of solutions one should make a classification in terms of the different signs for $k$, $||v||^2$ and $\Lambda^2$. We restrict to solutions in spherical coordinates which have $k = +1$. The other solutions can similarly be found. The solutions with spherical symmetry have the more interesting properties that they lift up to vacuum solutions that can be asymptotically flat. These solutions can be found in [G].

6.5 Discussion

In this chapter we studied the $(-1)$-brane solutions of the Kaluza-Klein theory (KK vectors truncated) that can be obtained from reducing pure gravity over a torus. We introduced the concept of a generating solution. A generating solution is a geodesic with the minimal number of arbitrary integration constants such that the action of isometry group $G$ generates all other geodesics from it. We then presented the theorem that for maximally non-compact Riemannian cosets $G/H$, with $H$ its maximal compact subgroup, the generating solution can be constructed from the Cartan subalgebra CSA only. In [G] we have shown that the analysis of the generating solution can be extended to $G/H^\ast$, where $H^\ast$ is a non-compact version of $H$. In this chapter we focused only on the GL($r + s$, $\mathbb{R}$)/SO($r, s$) coset and showed that the number of complex eigenvalues that one needs in order to construct the normal form $K_N$ of the generating solution is at most $\min(r, s)$.

We first illustrated this technique for $Sp$-branes. That is we studied a Lagrangian contains only Einstein gravity. Reducing this over the worldvolume of the $Sp$-brane gives the moduli space GL($p + 1$, $\mathbb{R}$)/SO($p + 1$). Using the above mentioned theorem we presented the generating geodesic solution, a straight line, for the $S(-1)$ belonging to this coset. We oxidized the time-dependent geodesic solution back to the original higher dimensional theory where we obtained a fluxless $Sp$-brane. This led to $S(-1)$-brane /$Sp$-brane map. If we reduce to three dimensions a symmetry enhancement of the coset takes place. The dualisation of the three-dimensional Kaluza-Klein vectors generates the coset SL($p + 2$, $\mathbb{R}$)/SO($p + 2$). The uplifting back to pure gravity leads to time-dependent vacuum solution transforming into a twisted vacuum solution.

We like to mention another closely related way to classify geodesics on symmetric spaces $G/H$ when $G$ and $H$ obey the same conditions as above. This mechanism is called the compensator algorithm and is developed by Fré et al. [158, 170]. The compensator algorithm offers a way to write down exact solutions for the different scalar fields in an iterative manner that illustrates nicely the integrability of the geodesic
equations of motion. More precisely, this method constructs geodesic curves from the straight line by performing a $H$-local transformation on the tangent space at each point on the straight curve.

Similarly, we used the same technique for the timelike $p$-brane case. This actually has led to different classes of instantons, labelled by the sign of $||v||^2$. Reducing gravity over the worldvolume of timelike $p$-brane gives rise to $GL(p+1, \mathbb{R})/SO(p,1)$ $\sigma$-model for $D > 3$ and $SL(p+2, \mathbb{R})/SO(p+1,1)$ for $D = 3$, which can be considered as two extensions of the prototype Lorentzian scalar coset $SL(2, \mathbb{R})/SO(1,1)$ of Euclidean type IIB supergravity. The geodesic generating solution on $GL(p+1, \mathbb{R})/SO(p,1)$ is constructed using our approach. Uplifting this stationary solution back to higher dimensional theory provided us with new solutions which still have to be analyzed. The generating solution for $SL(p+2, \mathbb{R})/SO(p+1,1)$ case and its uplifting to vacuum solutions are still under investigation. However, the upplings of 3D instanton solutions to intermediate dimensions, in particular to four-dimensional extremal black hole solutions, have been pointed out first by Breitenlohner et al. [130], and worked out recently by Gaiotto et al. [163]. This is known as instanton /black hole correspondence. The authors of [163] have actually derived extremal solutions for a variety of four-dimensional models which, after Kaluza-Klein reduction, admit a description in terms of 3D gravity coupled to a $\sigma$-model with symmetric target space. The solutions are found to be in correspondence with certain nilpotent generators $K$ of the isometry group $G$. In particular, they provide the exact solution for a non-BPS black hole with generic charges and asymptotic moduli in $N = 2$ supergravity coupled to two vector fields (one vector multiplet). In [G] we extend the analysis of [163], applying our generating solution approach, to obtain extremal and non-extremal black holes in $D = 4, N = 8$ supergravity.

Although in this chapter our analysis has been restricted to the cosets of pure Kaluza-Klein theory following from the reduction of Einstein-gravity over a torus, our mechanism for obtaining the generating solution can be straightforwardly extended to theories with $GL(r + s = n, \mathbb{R})/SO(r, s)$ cosets and also to the cosets in the right column of table 5.4.1.

Let us finish the discussion by making one surprising observation. It is known that a geodesic motion sometimes occurs in the presence of a scalar potential and for time dependent solutions, this can happen for scaling cosmologies. In [E] we studied such a solution in the context of fake/pseudo-supersymmetry (see for definition [151]) for multi-field systems whose first-order equations we derived using the Bogomol'nyi-like method. In particular we showed that scaling solutions that are pseudo-BPS must describe geodesic curves.