Chapter 5

Nonlinear $\sigma$-models and Toroidal Reductions

In this chapter we shall study some aspects of nonlinear sigma models on Riemannian and pseudo-Riemannian symmetric spaces. Definition of the nonlinear sigma model will be given, exhibiting how such models arise in Kaluza-Klein theories and extended supergravity theories.

It is known that dimensional reduction has been used to make a connection to lower dimensional theories, in particular to our four-dimensional universe. But dimensional reduction will be studied here not only for this reason, but also to show in the next chapter that reducing a theory over some of its dimensions leads to a theory which is easier to solve than the original one. Via uplifting back to the original dimensions we have generated a solution of the higher dimensional system. In chapter 6 we will be also interested in reducing brane solutions over their worldvolumes. Therefore we will restrict in this chapter to torus reductions, distinguishing between what we will call spacelike and timelike reductions. In the end of this chapter we will outline the reduction of maximally extended supergravity theories over a torus.

5.1 Introduction

In chapter 4 we have found that the most general duality group that can be realized with a 4-dimensional theory describing among the other fields a set of $n$ abelian vector fields and $m$ scalar fields is the non-compact real symplectic group Sp($2n,\mathbb{R}$), which has U($n$) as its maximal compact subgroup. In addition we have seen that in the absence of scalar matter fields, U($n$) is the largest group of duality transformations. In specific examples the actual group of duality transformations can be smaller, namely
a non-compact subgroup $G$ of $\text{Sp}(2n, \mathbb{R})$, having $H$ as its maximal compact subgroup. In those examples the scalar sector is described by a 4-dimensional nonlinear sigma model, which means that the scalar fields are local coordinates of a non-compact Riemannian manifold $G/H$ and the scalar action is required to be invariant with respect to the isometries of $G/H$. For example, in $N = 8$, $D = 4$ supergravity there are 28 field strengths. The non-compact invariance is the $E_7$ subgroup of $\text{Sp}(56, \mathbb{R})$ and its maximal compact subgroup is the $\text{SU}(8)$ subgroup of $\text{U}(28)$. The scalar fields thus take values in the target space $E_{7(7)}/\text{SU}(8)$, describing $E_{7(7)}/\text{SU}(8)$ nonlinear $\sigma$-model.

Nonlinear $\sigma$-models of our interest are those typically arise through dimensional reduction of gravitational theories ((super)gravity theories), where the scalar fields form coset manifolds $G/H$ exhibiting explicitly larger and larger symmetries as one goes down in dimensions. In the case of eleven-dimensional supergravity for example, reduction on a $n$-torus, $\mathbb{T}^n$, reveals a chain of exceptional symmetries $G = E_{n(n)}$. In these theories, generically the fields transform as follows. The metric is invariant, the (abelian) gauge fields transform linearly under $G$ and the fermions transform linearly under the group $H$. However, in some dimensions the $G$-invariance is not realized at the level of the action, but at the level of the combined field equations and Bianchi identities. For example, in the 4-dimensional example given above the 28 vector fields do not constitute a representation of the group $E_{7(7)}$. As we have seen in chapter 4 the group $G$ in this case is realized by electromagnetic duality and acts on the field strengths, rather on the vector fields.

We begin our study by describing some general aspects of nonlinear sigma models for finite-dimensional coset spaces. In this chapter and the one that follows we will try to minimize the geometrical and group theoretical technicality, which we review in appendices A and C.

### 5.2 Nonlinear Sigma Model Based on Symmetric Spaces

A nonlinear sigma model describes maps $\Phi$ from one (pseudo)-Riemannian space $\Sigma$ equipped with a metric $g$ to another (pseudo)-Riemannian space, “the target space” $M$, with metric $G_{ij}$. Let $x^\mu (\mu = 1, \cdots D)$ be coordinates on $\Sigma$ and $\Phi^i (i = 1, \cdots \dim M)$ be coordinates on $M$. Then the standard action for this sigma model is

$$S = \int_{\Sigma} d^D x \sqrt{|g|} g^{\mu\nu} \partial_\mu \Phi^i(x) \partial_\nu \Phi^j(x) G_{ij}(\Phi(x)).$$

Solutions to the equations of motion resulting from this action will describe the maps $\Phi^i$ as functions of $x^\mu$.

In what follows, we shall be concerned with sigma models on non-compact Rie-
mannian symmetric spaces $M = G/H$ where $G$ is a non-compact Lie group generated by the semi-simple real Lie algebra $\mathfrak{g}$ and $H$ is its maximal compact subgroup generated by the real Lie algebra $\mathfrak{h}$. Since elements of the coset are obtained by quotienting out $H$, this subgroup is referred to as the “local gauge symmetry” (see below). Our aim is to provide an algebraic construction of the metric $G_{ij}$ on the coset and of the Lagrangian [106, 124–126].

### 5.2.1 Symmetric Spaces and Nonlinear Realizations

Suppose $G$ is a group and $H$ is a subgroup of $G$. The coset space $G/H$ is defined as the set of equivalence classes $[g]$ of $G$ defined by the equivalence relation

$$g \sim g' \iff gg'^{-1} \in H, \text{ and } g, g' \in G,$$

i.e.,

$$[g] = \{hg | h \in H\}. \tag{5.2.3}$$

If $G$ is a Lie group and $H$ is any Lie subgroup of $G$, the coset $G/H$ is a manifold and thus can be described by local coordinates. Since any two points $p$ and $p'$ on $G/H$ can by construction be connected by an action of $G$, the manifold is a homogeneous space with $G$ being the isometry group and $H$ the isotropy group.

We have investigated the non-compact real forms $G$ of a complex semi-simple group $G^c$ in appendix C.2.2 and have found that the involutive automorphism $\theta$ of $G$ induces a Cartan decomposition of $\mathfrak{g}$ into even and odd eigenspaces:

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{f}, \tag{5.2.4}$$

where

$$\mathfrak{h} = \{j \in \mathfrak{g} | \theta(j) = j\}, \quad \mathfrak{f} = \{k \in \mathfrak{g} | \theta(k) = -k\} \tag{5.2.5}$$

play central roles. The decomposition 5.2.4 is orthogonal, in the sense that $\mathfrak{f}$ is the orthogonal complement of $\mathfrak{h}$ with respect to the invariant inner product $\langle \cdot | \cdot \rangle$ induced by the Killing metric $B(\cdot, \cdot)$,

$$\mathfrak{f} = \{k \in \mathfrak{g} | \forall j \in \mathfrak{h} : \langle k | j \rangle = 0\}. \tag{5.2.6}$$

The commutator relations split in a way characteristic for symmetric spaces,

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{f}] \subset \mathfrak{f}, \quad [\mathfrak{f}, \mathfrak{f}] \subset \mathfrak{h}. \tag{5.2.7}$$

The subspace $\mathfrak{f}$ is not a subalgebra. Elements $\mathfrak{f}$ transform in some representation of $\mathfrak{h}$, which depends on the Lie algebra $\mathfrak{g}$. We stress that if the commutator $[\mathfrak{f}, \mathfrak{f}]$ had also contained elements in $\mathfrak{f}$ itself, this would not have been a symmetric space.

If the so-called involution $\theta$ taken to be the involution $\theta_c$ defined in appendix C.2.2,
then it will have the effect of reversing the sign of every non-compact generator in the Lie algebra $\mathfrak{g}$, while leaving the sign of every compact generator unchanged (see appendix C.2.2). If we denote the positive root generators, negative root generators and Cartan generators of $\mathfrak{g}$ by $(E_{\bar{\alpha}}, E_{-\bar{\alpha}}, \tilde{H})$, where $\bar{\alpha}$ ranges over all the positive roots, then for our algebra $\theta_c$ effects the mapping
\[ \theta_c : (E_{\bar{\alpha}}, E_{-\bar{\alpha}}, \tilde{H}) \rightarrow (-E_{-\bar{\alpha}}, -E_{\bar{\alpha}}, -\tilde{H}). \] (5.2.8)

The real form corresponding to $\theta_c$ is called the maximally non-compact (split) real form $G_{\theta_c}$ of $G^c$. It is always, in any real Lie group $G_{\theta_c}$, the generator combinations $(E_{\alpha} - E_{-\alpha})$ are compact while the combinations $(E_{\alpha} + E_{-\alpha})$ are non-compact.

The Cartan split of the generators into compact and non-compact parts reveals the fact that the Killing metric of a real form $G$ may have an indefinite signature. However, the metric $G_{ij}$ on the symmetric space $G/H$ has a definite sign. This can be seen from the fact that all the generators that span $G/H$ are non-compact, therefore the Killing metric restricted to $\mathfrak{g}$ is positive definite, and hence the symmetric space $G/H$ is Riemannian. The restriction of the Killing metric to $\mathfrak{h}$ gives rise to a negative definite metric. Note that had we started with a different real form of $G$ with a different involutive automorphism, we would have obtained a different symmetric space with a different signature for the metric. For more about the classification of real forms of a complex Lie algebra using involutive automorphisms we refer to [127].

The group $G$ naturally acts through (here, right) multiplication on the coset space $G/H$ as
\[ [k] \rightarrow [kg]. \] (5.2.9)
This definition makes sense because if $k \sim k'$, i.e. $k' = hk$ for some $h \in H$, then $k'g \sim kg$ since $k'g = (hk)g = h(kg)$. Note that left and right multiplications commute.

In order to describe a dynamical theory on the coset space $G/H$, it is convenient to introduce as dynamical variable the group element $L(x) = L(\Phi(x)) \in G$ ($\Phi^i$ are the local coordinates parameterizing $G/H$) and to construct the action for $L(x)$ in such a way that the equivalence relation
\[ \forall h(x) \in H : L(x) \sim h(x)L(x) \] (5.2.10)
corresponds to a gauge symmetry. The physical (gauge invariant) degrees of freedom are then parameterized indeed by points of the coset spaces. We want also to impose 5.2.9 as a rigid symmetry. Thus, the action should be invariant under
\[ L(x) \rightarrow h(x)L(x)g, \quad h(x) \in H, \ g \in G. \] (5.2.11)

One may develop the formalism without fixing the $H$-gauge symmetry, or one may instead fix the gauge symmetry by choosing a specific coset representative $L(x) \in G/H$. When $H$ is a maximal compact subgroup of $G$, i.e. $G/H$ is a Riemannian
5.2 Nonlinear Sigma Model Based on Symmetric Spaces

In a symmetric space, there are no topological obstructions, and a standard choice which is always available, is to take \( L(x) \) to be upper triangular form as allowed by Iwasawa decomposition (discussed in section 5.2.3). This gauge choice is called the solvable gauge. Given such a gauge choice (or any other one), the global action of an arbitrary element \( g \in G \), on \( L(x) \) will lead to a non-standard representative \( L'(x) \) and must therefore be accompanied by a compensating local transformation \( h(L(x), g) \in H \) in order to maintain the gauge choice. The total transformation is thus

\[
L(x) \mapsto L''(x) = h(L(x), g)L(x)g, \quad h(L(x), g) \in H, \; g \in G. \tag{5.2.12}
\]

where \( L''(x) \) is again in the upper triangular gauge. Since now \( h(L(x), g) \) depends nonlinearly on \( L(x) \), this is called a nonlinear realization of \( G \).

5.2.2 Nonlinear Sigma Model Coupled to Gravity

Given the field \( L(x) \), we can form the Lie algebra valued one-form

\[
dLL^{-1} = dx^b \partial_\mu LL^{-1}. \tag{5.2.13}
\]

Under the Cartan decomposition, this element splits according to 5.2.4,

\[
dL(x)L^{-1}(x) = (\Omega_\mu(x) + E_\mu(x))dx^\mu, \tag{5.2.14}
\]

where \( \Omega_\mu \in \mathfrak{h} \) and \( E_\mu \in \mathfrak{f} \). In virtue of the involutive automorphism \( \theta \) one can write these explicitly as projections onto the odd and even eigenspaces as follows:

\[
E = \frac{1}{2}[dLL^{-1} - \theta(dLL^{-1})],
\]

\[
\Omega = \frac{1}{2}[dLL^{-1} + \theta(dLL^{-1})]. \tag{5.2.15}
\]

Now, if we define a generalized transpose \( \overset{\nabla}{\cdot} \) [128] (see also [126]) by

\[
(\cdot)^\nabla = -\theta(\cdot), \tag{5.2.16}
\]

then \( E \) and \( \Omega \) correspond to symmetric and antisymmetric elements, respectively,

\[
E^\nabla(x) = E(x), \quad \Omega^\nabla(x) = -\Omega(x). \tag{5.2.17}
\]

The Lie algebra valued one-forms with components \( dLL^{-1} \), \( \Omega \) and \( E \) are invariant under the rigid right multiplication, \( L(x) \mapsto L(x)g \).

Being an element of the Lie algebra of the gauge group, \( \Omega_\mu(x) \) can be interpreted as a gauge connection for the local symmetry \( H \). Under local transformation \( h(x) \in H \), \( \Omega_\mu(x) \) transforms as

\[
H : \; \Omega_\mu(x) \mapsto h(x)\Omega_\mu(x)h^{-1}(x) + \partial_\mu h(x)h^{-1}(x), \tag{5.2.18}
\]
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which indeed is the characteristic transformation property of a gauge connection. On the other hand, $E_\mu(x)$ transforms covariantly,

$$H : E_\mu(x) \longrightarrow h(x)E_\mu(x)h^{-1}(x). \quad (5.2.19)$$

Making use of the scalar product $\langle . | . \rangle$, induced by the Killing metric $B$ written in a $R$ representation of the group $G$, we can now form a manifestly $H$ and $G$ invariant expression by simply squared $E_\mu(x)$. Thus the D-dimensional nonlinear sigma model action coupled to gravity takes the form

$$S_{G/H} = \int_{\Sigma} d^Dx \sqrt{|g|} \left[ R - g^{\mu\nu}(E_\mu(x)|E_\nu(x)) \right] \quad (5.2.20)$$

$$= \int_{\Sigma} d^Dx \sqrt{|g|} \left[ R - C_Rg^{\mu\nu}\text{tr}[E_\mu(x)E_\nu(x)] \right]. \quad (5.2.21)$$

$$= \int_{\Sigma} d^Dx \sqrt{|g|} \left[ R - \frac{1}{2}g^{\mu\nu}\partial_\mu\Phi^i(x)\partial_\nu\Phi^j(x)G_{ij}(\Phi(x)) \right], \quad (5.2.22)$$

where $R$ is the Ricci scalar and $C_R$ is a positive constant depending on the representation $R$ of $G$ with $\langle . | . \rangle = C_R\text{tr}[ ]$.

There is an alternative way of writing the above action. We can also form a generalized “metric”$^1$ $\mathcal{M}$ that does not transform at all under the local symmetry, but only transforms under the rigid $G$-transformations. This can be done by using the generalized transpose, extended from the algebra to the group through the exponential map. Actually the main advantage behind using the automorphism $\theta$ is that it provides us with an embedding of $G/H$ in $G$ [129]

$$L \mapsto \mathcal{M} = \theta(L^{-1})L = L^2L, \quad \theta(\mathcal{M}) = \mathcal{M}^{-1}, \quad (5.2.23)$$

where $\theta(L^{-1}) = L^2$. The matrix $\mathcal{M}$ transforms as follows under global transformations on $L(x)$ from the right

$$G : \mathcal{M}(x) \mapsto g^4\mathcal{M}(x)g, \quad g \in G. \quad (5.2.24)$$

A short calculation shows that the relation between $\mathcal{M}(x) \in G$ and $L(x) \in G/H = \exp\mathcal{F}$ is given by

$$\frac{1}{2}\mathcal{M}^{-1}(x)\partial_\mu\mathcal{M}(x) = L^{-1}(x)E_\mu(x)L(x). \quad (5.2.25)$$

$^1$We call $\mathcal{M}$ a “generalized metric” because in the Kaluza-Klein reduction of Einstein gravity over a torus, it corresponds to the metric of the torus, the field $L(x)$ being the “vielbein” (see section 5.3.2).
This relation provides another way to write the $G$- and $H$-invariant action, completely in terms of the generalized metric $M(x)$,

$$
S_{G/H} = \int_{\Sigma} d^D x \sqrt{|g|} \left[ R - \frac{1}{4} g^{\mu\nu} \langle M^{-1} \partial_\mu M | M^{-1} \partial_\nu M \rangle \right]
$$

(5.2.26)

$$
= \int_{\Sigma} d^D x \sqrt{|g|} \left[ R + \frac{C_R}{4} \text{tr} [\partial_\mu M \partial^\mu M^{-1}] \right].
$$

(5.2.27)

The equations of motion following from the nonlinear $\sigma$-model coupled to gravity action read

$$
R_{\mu\nu} - \frac{1}{2} \partial_\mu \Phi^i \partial_\nu \Phi^j G_{ij}(\Phi) = 0,
$$

(5.2.28)

$$
D^\mu \partial_\mu \Phi^i(x) = 0,
$$

(5.2.29)

where $D$ is the covariant derivative associated with the local transformation $H$. The second equation is equivalent to

$$
D^\mu E_\mu(x) = 0,
$$

(5.2.30)

with $D_\mu E_\mu(x) = \nabla_\mu E_\mu(x) - [\Omega_\mu(x), E_\mu(x)]$, where $\nabla^\mu$ is a covariant derivative on $\Sigma$ compatible with the Levi-Civita connection. Equations 5.2.30 simply express the covariant conservation of $E_\mu(x)$.

It is also interesting to examine the dynamics in terms of the generalized metric $M$. The equations of motion for $M$ are

$$
\nabla^\mu (M^{-1} \partial_\mu M) = 0.
$$

(5.2.31)

These equations ensure the conservation of the current

$$
J^\mu = \frac{1}{2} M^{-1} \partial_\mu M = L^{-1} E_\mu L,
$$

(5.2.32)

i.e., $\nabla^\mu J_\mu = 0$. This is the conserved Noether current associated with the rigid $G$-invariance of the action. Using $\theta(M) = M^{-1}$, one can derive that the currents $J$ obey the identity $\theta(J) = -M J M^{-1}$.

The incorporation of form fields into the action 5.2.26 is straightforward as long as one does not want to turn on duality invariance (selfduality of section 4.4). For example in $D = 4$ the addition of vectors fields might be accompanied by the duality action of $G$ exactly the way we saw in the previous chapter. In this case the vector fields should be added to gravity and scalar fields forming a $G/H \sigma$-model in such a way that the resulting equations of motion are invariant under the action of $G$ if the field strengths together with their duals are transformed suitably with a $R$ representation of $G$. In [130] it has been shown that the matrix $M$ parameterized...
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by the scalar $\Phi^i$ is intimately related to the symmetric matrix defined in section 4.4. Furthermore it has been argued that the coupling of the vector fields should be such that the resulting energy density is positive, and hence the symmetric matrix $\mathcal{M}$ is positive definite.

The action describing a nonlinear $\sigma$-model coupled to gravity and vector fields arises typically in Kaluza-Klein theories and extended supergravity theories, in particular from the torus reduction of pure Einstein gravity\(^2\) to $D$ dimensions which will be discussed in section 5.3.

### 5.2.3 Iwasawa Decomposition: Borel Gauge

We have seen above that there exists a nice gauge or coordinate frame, namely the solvable gauge. It has been shown that the idea behind such a gauge originates from the Iwasawa decomposition. Given a real form $G$ of a semisimple complex Lie group $G^\mathbb{C}$, the associated Lie algebra $\mathfrak{g}$ can be decomposed as $\mathfrak{g} = \mathfrak{h} + \mathfrak{s}$ where $\mathfrak{h}$ is the maximal compact subalgebra, and $\mathfrak{s}$ is the solvable subalgebra of $\mathfrak{g}$. The solvable Lie algebra $\mathfrak{s}$ can itself split up as $\mathfrak{s} = \mathfrak{c} \oplus \mathfrak{n}$ where $\mathfrak{c}$ is the maximal set of commuting non-compact generators, i.e. the non-compact part of the Cartan subalgebra CSA $\mathfrak{h}$ of $G$, and $\mathfrak{n}$ is the lie algebra consists of the generators which have the positive roots with respect to $\mathfrak{c}$. This is the Iwasawa decomposition, a description of which can be found in reference [131]. One of the nice properties of a solvable Lie algebra is that the matrix representation can be chosen such that all elements of $\mathfrak{s}$ are upper triangular. Due to this decomposition one can easily notice that the coset space $G/H$ is globally isometrical to a group $G_s$, a subgroup of $G$ associated to the solvable Lie algebra, namely

$$
\frac{G}{H} \cong G_s, \quad \text{with } G_s = \exp[\mathfrak{s}]. \quad (5.2.33)
$$

This reflects the fact that the solvable parametrization of $G/H$ holds globally.

We denote the Cartan subalgebra generators by $H_I$ with $I = 1, \cdots, r = \text{rank}(G)$ and the positive root generators with $E_{\vec{\alpha}}$. The commutation relations read

$$
[H_I, H_J] = 0, \quad [H_I, E_{\vec{\alpha}}] = \alpha_I E_{\vec{\alpha}}, \quad (5.2.34a)$$

$$
[E_{\vec{\alpha}}, E_{\vec{\beta}}] = \begin{cases} 
0 & \text{if } \alpha + \beta \text{ is not a root} \\
N(\alpha, \beta)E_{\vec{\alpha}+\vec{\beta}} & \text{otherwise},
\end{cases} \quad (5.2.34b)
$$

where $\vec{\alpha}$ is the root vector of the Lie algebra $\mathfrak{g}$.

Now let us be a little more precise and distinguish two cases:

- If $\mathfrak{c}$ coincides with the whole Cartan subalgebra $\mathfrak{h}$ and $\mathfrak{n}$ is the subspace generated by all the positive root generators, then $\mathfrak{g}$ is $\mathfrak{g}_{\mathfrak{c}}$ the \textit{maximally non-compact}
real form of a complex semisimple algebra $\mathfrak{g}^\mathbb{C}$ (see C.2.2). This means that the difference between the number of non-compact generators and the number of compact ones is the rank $r = \text{rank}G_{\theta_c}$. In this case the solvable algebra $\mathfrak{s}$ is isomorphic to the Borel subalgebra of $\mathfrak{g}_{\theta_c}$. The Borel subalgebra of any Lie algebra $\mathfrak{g}_{\theta_c}$ is the subalgebra generated by the positive root generators and the Cartan generators, namely

$$L \in \frac{G_{\theta_c}}{H} \equiv \exp[Borel\mathfrak{g}_{\theta_c}], \quad L = \prod_I \exp[-\frac{1}{2} \phi^I H_I] \Pi_\alpha \exp[\chi^\alpha E_\alpha]. \quad (5.2.35)$$

In fact for the nonlinear sigma model action of the type discussed above, the scalars might come in two disguises: either they appear with derivatives or they also appear in exponential couplings to other fields. The scalars of the first kind are called axions $\chi^\alpha$ and the scalars of the second kind are called dilatons $\phi^I$. This means that the coset representative $L$ is written in the Borel coordinate system.

$$H_{ij} = (\beta_i) \delta_{ij}. \quad (5.2.36)$$

The tracelessness of the special linear group SL generators implies $\sum_i \beta_{il} = 0$. Moreover, a convenient normalization of the Cartan killing form (metric of the Lie algebra

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3 Apart from the fact that these coordinates can be useful for practical purposes, it is also the coordinate system that is naturally obtained after torus dimensional reduction defined later in this chapter. The dilatons $\phi^I$ then correspond to the radii of the internal tori and the axions $\chi^\alpha$ are the various off-diagonal internal gravitational degrees of freedom and the internal components of the $p$-form gauge potentials.
in the fundamental representation) leads to two additional identities obeyed by the weight vectors
\[ \sum_i \beta_i \tilde{\beta}_i = 2 \delta_{1J}, \quad \tilde{\beta}_i \cdot \tilde{\beta}_j = 2 \delta_{ij} - \frac{2}{n}. \]  
(5.2.37)

The positive step operators \( E_{ij} \) are all upper triangular and a handy basis is one in which they have only non-zero entry \( (E_{ij})_{ij} = 1 \). Note that the negative step operators are the transpose of the positive ones.

The SO\((n)\) subalgebra is spanned by the combinations
\[ \frac{1}{\sqrt{2}} (E_\beta - E_{-\beta}). \]  
(5.2.38)

If we work in the fundamental representation of SL, the Lie algebra of SO\((n)\) is then the vectorspace of anti-symmetric matrices. Now, use the fact that for SL the generalised transpose \( \sharp \) defined above coincides with the ordinary matrix transpose "T", namely \( \theta(L) = (L^\sharp)^{-1} = (L^T)^{-1} \), the relations 5.2.15 become
\[ E = \frac{1}{2} [dLL^{-1} + (dLL^{-1})^T] \quad \Omega = \frac{1}{2} [dLL^{-1} - (dLL^{-1})^T]. \]  
(5.2.39)

Thus the hermitian positive definite matrix \( M \) becomes \( M = L^T \delta L = L^T L \), i.e. symmetric matrix, with \( \delta \) is the Euclidean metric invariant under the action of SO\((n)\).

A calculation exhibits that the SL\((n, \mathbb{R})/SO(n)\) nonlinear \( \sigma \)-model Lagrangian reads
\[ \mathcal{L}_{\text{scalar}} = -\sqrt{|g|} \text{tr}[E^2] = \frac{1}{4} \sqrt{|g|} \text{tr}[\partial \mathcal{M} \partial \mathcal{M}^{-1}]. \]  
(5.2.40)

The action will generically look complicated but when all axions are set to zero, \( L \) is diagonal \( L = \text{diag}[\exp(\frac{1}{2} \beta_i \cdot \phi)] \) and the action becomes
\[ \frac{1}{4} \text{tr} \partial \mathcal{M} \partial \mathcal{M}^{-1} = \frac{1}{4} \sum_i \beta_i \beta_i \partial \phi^i \partial \phi^i = -\frac{1}{2} \tilde{\beta}_i \tilde{\beta}_i \partial \phi^i \partial \phi^i. \]  
(5.2.41)

This action describes a truncated \( \mathbb{R}^{n-1} \) nonlinear \( \sigma \)-model, where the \( n - 1 \) dilatons are parameterizing the flat scalar space \( \mathbb{R}^{n-1} \).

This brings us to the issue of consistent truncations. According to [130], a nonlinear \( \sigma \)-model with target space \( G_1/H_1 \) is a consistent truncation of another \( \sigma \)-model with target space \( G_2/H_2 \) if \( G_1/H_1 \subset G_2/H_2 \) and if every solution of the field equations for the \( G_1/H_1 \) \( \sigma \)-model is a solution of the field equations for the \( G_2/H_2 \) \( \sigma \)-model as well. In other words, the truncation is consistent if and only if \( G_1/H_1 \) is totally geodesic\(^4\) subspace of \( G_2/H_2 \). Note that if \( G/H \) is a Riemannian symmetric

\(^4\)Totally geodesic submanifold is a submanifold such that all geodesics in the submanifold are also geodesics of the surrounding manifold.
space then every totally geodesic sub-space of $G/H$ is again a Riemannian symmetric space. Similar statements hold true if we are dealing with a pseudo-Riemannian $G/H^*$ symmetric space that will be defined in a minute. Thus in the above example, putting all the axions to zero turns out to be consistent with the equations of motion

$$\partial_\mu (\sqrt{|g|} M^{-1} \partial^\mu M) = 0.$$  \hfill (5.2.42)

Let us consider the example of the coset space for $n = 2$, which, although very simple, is nonetheless quite illustrative. The Lie algebra of $\text{SL}(2, \mathbb{R})/\text{SO}(2)$ has the following standard commutation relations

$$[H, E_2] = 2E_2, \quad [H, E_{-2}] = -2E_{-2}, \quad [E_2, E_{-2}] = H.$$ \hfill (5.2.43)

The $\text{SL}(2, \mathbb{R})$ fundamental realization takes on the form

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_{-2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$ \hfill (5.2.44)

The coset representative, written in the Borel gauge (upper-triangular), behaves as

$$L = \exp \left[ \frac{1}{2} \phi H \right] \exp \left[ \chi E_2 \right] = \begin{pmatrix} e^{\phi/2} & e^{-\phi/2} \chi \\ 0 & e^{-\phi/2} \end{pmatrix},$$ \hfill (5.2.45)

where $\phi(x)$ and $\chi(x)$ represent coordinates on the coset space, i.e. they describe the sigma model map

$$x \in \Sigma \mapsto (\phi, \chi) \in \text{SL}(2, \mathbb{R})/\text{SO}(2).$$ \hfill (5.2.46)

We saw that an arbitrary transformation on $L(x)$ behaves as

$$L(x) \rightarrow h(x)L(x)g, \quad h(x) \in \text{SO}(2), \quad g \in \text{SL}(2, \mathbb{R}),$$ \hfill (5.2.47)

which infinitesimally becomes

$$\delta_{h,g} L = \delta h(x)L + L \delta g,$$ \hfill (5.2.48)

where $\delta h(x)$, and $\delta g$ are respectively elements of the $\text{SO}(2)$ and $\text{SL}(2, \mathbb{R})$ Lie algebras. Let us now check how $L(x)$ transforms under the generators $\delta g = \{H, E_2, E_{-2}\}$. As expected, the Borel generators $H$ and $E_2$ preserve the upper triangular structure

$$\delta_{E_2} L = LE_2 = \begin{pmatrix} 0 & e^{\phi/2} \\ 0 & 0 \end{pmatrix}, \quad \delta_H L = LH = \begin{pmatrix} e^{\phi/2} & -e^{\phi/2} \chi \\ 0 & -e^{\phi/2} \end{pmatrix},$$ \hfill (5.2.49)

whereas the negative root generator $E_{-2}$ does not respect the form of $L(x)$,

$$\delta_{E_{-2}} L = LE_{-2} = \begin{pmatrix} e^{\phi/2} \chi & 0 \\ e^{-\phi/2} & 0 \end{pmatrix}.$$ \hfill (5.2.50)
Thus in this case we need a compensating transformation to restore the upper triangular form. This transformation needs to switch off the entry $e^{-\phi/2}$ in the lower left corner of the matrix 5.2.50 and therefore it must necessarily depend on $\phi(x)$. The transformation that can do this job is a SO(2) Lie algebra element

$$\delta h(x) = \begin{pmatrix} 0 & e^{-\phi} \\ -e^{-\phi} & 0 \end{pmatrix}, \quad \text{(5.2.51)}$$

and we find

$$\delta_{h,E_{-2}} L = \delta h(x) L + L E_{-2} = \begin{pmatrix} e^{\phi/2} \chi & e^{-3\phi/2} \\ 0 & -\chi e^{-\phi/2} \end{pmatrix} \in \text{SL}(2,\mathbb{R})/\text{SO}(2). \quad \text{(5.2.52)}$$

Finally, from 5.2.45 one can calculate $M = L^T L$ and hence the SL(2,\mathbb{R})/SO(2) nonlinear $\sigma$-model Lagrangian is

$$L_{\text{scalar}} = -\frac{1}{2} \sqrt{|g|} \left[ (\partial \phi)^2 + e^{2\phi}(\partial \chi)^2 \right]. \quad \text{(5.2.53)}$$

In type IIB supergravity the scalar coset is SL(2,\mathbb{R})/SO(2) where the Borel gauge has the interpretation that the dilaton $\phi$ is in the NS sector and determines the string coupling and $\chi$ is the RR field $C(0)$ that couples electrically to D-instanton ($-1$-brane) and magnetically to 7-brane.

This is somewhat a trivial example so let us consider SL(3,\mathbb{R})/SO(3). The Borel algebra is

$$[H_1, E_{12}] = 2E_{12}, \quad [H_1, E_{13}] = E_{13}, \quad [H_1, E_{23}] = -E_{23},$$

$$[H_2, E_{12}] = 0, \quad [H_2, E_{13}] = \sqrt{3} E_{13}, \quad [H_2, E_{23}] = \sqrt{3} E_{23}, \quad \text{(5.2.54)}$$

$$[E_{12}, E_{13}] = 0, \quad [E_{13}, E_{23}] = 0, \quad [E_{12}, E_{23}] = E_{13}.$$ 

The Cartan generators in the 3-dimensional fundamental representation are written as

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad \text{(5.2.55)}$$

and the three positive step operators are

$$E_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad \text{(5.2.56)}$$
The three negative step operators can be found by taking the transpose of the positive ones. The coset representative is expressed as

\[
L = \exp[\chi_{12} E_{12}] \exp[\chi_{13} E_{13}] \exp[\chi_{23} E_{23}] \exp\left[\frac{1}{2} \phi_1 H_1 + \frac{1}{2} \phi_2 H_2\right]
\]

\[
= \begin{pmatrix}
  e^{\frac{1}{2}(\phi_1 + \frac{\phi_2}{\sqrt{3}})} & e^{\frac{1}{2}(-\phi_1 + \frac{\phi_2}{\sqrt{3}})} e^{-\frac{\phi_2}{\sqrt{3}}} (\chi_{12} \chi_{23} + \chi_{13}) & e^{-\frac{\phi_2}{\sqrt{3}}} \\
  0 & e^{\frac{1}{2}(-\phi_1 + \frac{\phi_2}{\sqrt{3}})} & e^{-\frac{\phi_2}{\sqrt{3}}} \\
  0 & 0 & e^{-\frac{\phi_2}{\sqrt{3}}}
\end{pmatrix}.
\]

(5.2.57)

The \( \text{SL}(3,\mathbb{R})/\text{SO}(3) \) nonlinear \( \sigma \)-model Lagrangian is expressed as

\[
\mathcal{L}_{\text{scalar}} = -\sqrt{|g|} \left\{ -\frac{1}{2} (\partial \phi_1)^2 - \frac{1}{2} (\partial \phi_2)^2 - \frac{1}{2} (e^{-2\phi_1} + e^{-\phi_1 - \sqrt{3} \phi_2}) (\partial \chi_{12})^2 \\
- \frac{1}{2} e^{-\phi_1 - \sqrt{3} \phi_2} (\partial \chi_{13})^2 - \frac{1}{2} e^{\phi_1 - \sqrt{3} \phi_2} (\partial \chi_{23})^2 - e^{-\phi_1 - \sqrt{3} \phi_2} \chi_{23} \partial \chi_{12} \partial \chi_{13}\right\}.
\]

(5.2.58)

5.2.5 \( \sigma \)-model on Pseudo-Riemannian Symmetric Spaces \( G/H^* \)

The theories that we have seen so far are Minkowskian theories in \( D \) dimensions where the scalars form \( G/H \) nonlinear \( \sigma \)-model with \( G/H \) is a Riemannian target space. However, there exist also non linear \( \sigma \)-models which might arise from Euclidean Kaluza-Klein (reduction over timelike killing vector) and extended supergravity theories where the scalars take values in a pseudo-Riemannian space \( G/H^* \), with \( G \) is the maximally non-compact real form of a complex semisimple Lie algebra and \( H^* \) is a non-compact version of \( H \). The Borel coordinates \( \phi^I, \chi^\alpha \) are no longer valid for non-Riemannian symmetric spaces since this coordinate system does not cover the whole manifold\(^5\) (see [134]). Nevertheless, one can still work on the level of the \( H^* \)-invariant matrix, \( \mathcal{M} \), for which the choice of the gauge does not play any role.

The basis for a representation \( R \) will be chosen such that

\[
\theta(j) = -\eta(j)^T \eta, \quad \forall \ j \in \mathfrak{g}.
\]

(5.2.59)

The matrix \( \eta \) is \( H^* \)-invariant matrix having a Lorentzian signature. It can be restricted to the metric on \( G/H^* \) manifold or to the Killing metric of \( H^* \). If we define \( \mathcal{M} = \eta \mathcal{M} = L^T \eta L \), the matrix \( \mathcal{M} \) will again be hermitian (symmetric) and \( \mathcal{M} = \eta L^T \eta L \) is an element of \( G \).

Due to the non-compactness of \( H^* \) the matrix \( \eta \) and thus \( \mathcal{M} \) and \( G_{ij} \) will not

\(^5\)The solvable (Borel) parametrization for \( G/H^* \), in contrast to the \( G/H \) case in which \( H \) is the maximal compact subgroup of \( G \), holds only locally. To understand this issue, one can think of the simple case of \( dS_2 = \text{SO}(1,2)/\text{SO}(1,1) \), in which the solvable parametrization describes the stationary univers and thus covers only half of the hyperboloid.
be positive definite. Therefore there exist kinetic terms with the ‘wrong sign’ in the \( G/H^* \) nonlinear \( \sigma \)-model Lagrangians. Scalar fields with the wrong sign of kinetic terms are sometimes called *ghosts* in analogy with ghosts in the quantization of Yang-Mills theory. For example the nonlinear \( \sigma \)-model of Euclidean type IIB supergravity has the following Lagrangian

\[
L_{\text{scalar}} = \sqrt{|g|} \left[ -\frac{1}{2} (\partial \phi)^2 + \frac{1}{2} e^{-2\phi} (\partial \chi)^2 \right].
\]  

The only difference with Lorentzian IIB is the sign of \((\partial \chi)^2\). An explanation of this sign difference can for instance be found in [135]. The different sign does not ruin the \( \text{SL}(2, \mathbb{R}) \)-invariance but does change the scalar coset to \( \text{SL}(2, \mathbb{R})/\text{SO}(1, 1) \). Thus the metric on the scalar coset is indefinite, and hence as we expected the metric \( G_{ij} \) fixes the kinetic terms of the scalar fields. As we have mentioned before for such a case the isotropy group \( \text{SO}(1, 1) = H^* \) is not the maximal compact subgroup, therefore the Iwasawa gauge (Borel) fails. This can be seen as follows. Let \( k \in \text{SO}(1, 1) \) (the 1+1 Lorentz group, consisting only of boosts), and \( V \in \text{SL}(2, \mathbb{R}) \), then the gauge action \( \text{SO}(1, 1) \cdot V \) is a boost of both columns. If a column is lightlike then you can not boost it to become spacelike or timelike. In other words for \( V \) one can not find an upper-triangular representative. There are however other general gauges one can think of. One can see for instance [134] for a “good gauge”.

It is worth recalling that the \( G/H \) and \( G/H^* \) nonlinear \( \sigma \)-models arising in the pure Kaluza-Klein theories and maximally extended supergravities follow basically from the dimensional reduction over tori of pure Einstein-gravity theory and higher dimensional supergravities, respectively. Therefore the coming sections will be devoted to studying the dimensional reduction over tori, focusing mainly on the reduction of pure Einstein gravity over Euclidean and Lorentzian tori.

### 5.3 Dimensional Reduction

Dimensional reduction of a theory, by definition, consists of an expansion over an internal space and subsequent truncation to the lightest modes. In order to see that let’s have an instructive illustration:

We consider complex scalar field \( \hat{\phi} \) living in \( \hat{D} \) dimensional spacetime parameterized by the coordinates \( x^\mu = (x^\alpha, y) \). The fourier transformation of \( \hat{\phi} \) with respect to the coordinate \( y \) can be performed as

\[
\hat{\phi}(x, y) = \int dk \phi_k(x)e^{iky},
\]  

where \( k \) represents the momentum of modes \( \phi_k \). Now one can first compactify the \( y \) direction to have the length \( 2\pi L \), then impose the boundary condition

\[
\hat{\phi}(x, 0) = \hat{\phi}(x, 2\pi L).
\]
It implies that the integral is converted to a sum due to the fact that over compact direction the momentum takes discrete values

\[ \hat{\phi}(x, y) = \sum_n \phi_n(x)e^{i\frac{n}{L}y}. \]  
(5.3.3)

Assume the scalar field \( \hat{\phi} \) solves the Klein-Gordon equation such that

\[ \Box_D \hat{\phi} = 0, \quad \text{with} \quad \Box_D = \partial_\mu \partial^\mu + \partial_y \partial^y. \] 
(5.3.4)

In order to identify the fields of \( D = \hat{D} - 1 \) dimensions, we insert 5.3.3 in 5.3.4, implying infinite number of separate equations for every mode \( \phi_n \) with different mass

\[ \Box_D \phi_n(x) - \left( \frac{n}{L} \right)^2 \phi_n(x) = 0, \] 
(5.3.5)

where \( \Box_D = \partial_\mu \partial^\mu \). As a result a spectrum of fields, which are called the Kaluza-Klein fields, arises so that \( \phi_0 \) is effectively a massless field and \( \phi_n \) are massive fields with mass \( m = \frac{|n|}{L} \).

In fact, this holds for a general compact internal space and fields \( \hat{\phi} \). From the \( D \)-dimensional viewpoint we always have a massless sector and a massive sector with mass inversely proportional to the size of the extra dimensions (internal directions). Since we live in an effectively four dimensional world, we take the radius of the internal direction to be small in order for it to be unobservable; the fields \( \phi_n \) become extremely massive. Therefore these modes are too massive to be physically important and are usually decoupled, namely keep only \( \phi_0 \) and truncate the other modes. When the massive fields are truncated the field \( \hat{\phi} \) is independent of the internal dimensions. However, the lower dimensional degrees of freedom are not always massless. It may happen that the \( D \)-dimensional spectrum does not contain massless fields. In this case we truncate to the lightest modes of the fields.

**Consistency of Dimensional Reduction**

One can always obtain lower dimensional theories through a dimensional reduction over compact internal spaces. However one can not reduce over any compact space since there are consistency conditions. Consistency of dimensional reduction is nothing else than that every lower dimensional solution can be uplifted to a higher dimensional solution. In practice, consistency of dimensional reduction is the consistency of the truncation of the massive modes. To check consistency, one can consider a general internal space and a set of eigenfunction \( E^{\lambda_j} \) of the Laplacian operator defined above, namely \( \Box E^{\lambda_j} = \lambda_j E^{\lambda_j} \). Again the Fourier decomposition over the general internal space gives

\[ \hat{\phi}(x, y) = \sum_j \phi_{\lambda_j}(x)E^{\lambda_j}. \] 
(5.3.6)
As before we have massless fields for which $\lambda_0 = 0$ and massive fields. Substituting 5.3.6 into higher dimensional equations leads to

$$□_D \phi_{\lambda_j}(x) = S_{\lambda_j}. \quad (5.3.7)$$

$S_{\lambda_j}$ is a function which depends on both massless and massive fields. If $S$ is such that it vanishes when all massive $\phi_{\lambda_j}$ are constant, then the equation 5.3.7 is solved, hence the truncation to the massless fields are consistent. In other words, we can truncate the massive modes if the the massless fields do not form a source for massive ones. This implies that one has a restrictive number of internal spaces where one can reduce. However the non-consistency of the truncation of the massive fields does not mean that reduction is useless. From a physical point of view it is probable that the massive modes have negligible interactions with the massless sector because they are heavy. So even if the massive modes can not be truncated consistently, leaving them out will not be too much problem at low energies. However, dimensional reduction is not only used here for obtaining effective lower-dimensional theories. The exact result often matters, for example if a dimensional reduction is used as a solution generating technique (see chapter 6) when lower-dimensional solutions are lifted to higher-dimensional solutions or vice-versa, the reduction has to be consistent.

This way of performing dimensional reduction—an expansion over an internal space and truncation to the lightest sector—is unrealistic. Actually this procedure amounts to writing down an Ansatz which relates the higher dimensional fields to lower dimensional ones, i.e. to the lightest modes of the expansion. Dimensional reduction then consists in substituting the reduction Ansatz in the field equations or the action (Lagrangian). In most cases the reduction Ansatz depends on the internal space coordinates. This dependence should cancel at the end of the day in order to get the field equations corresponding to lower dimensional theory. This requirement is equivalent to the consistency of the truncation to a finite number of lower dimensional fields discussed above. From now on, we can just make the higher dimensional fields independent of the internal dimensions in order to perform dimensional reduction. Note that the number of degrees of freedom is unchanged under dimensional reduction.

The compactness of the internal manifold is not a necessary demand for having a dimensional reduction, what does really matter is to have a consistent truncation. It might happen that the internal space is non-compact, in this case the resulting Kaluza-Klein spectrum is continuous, and nonetheless the truncation to massless fields is still plausible. Such a reduction is called non-compactification while the reduction over compact spaces is called compactification.

5.3.1 Circle Reduction of Gravity

We will now consider the dimensional reduction of Einstein gravity in $\hat{D}$ dimensions over a circle to $D = \hat{D} - 1$ dimensions. In general, bosonic solution in $D$ dimensions
is specified by the metric, describing the geometry and other fields such as scalar and general $p$-forms. Regarding the metric field, the solution and the geometric structure must be compatible in a sense that the $\hat{D}$-dimensional space $\mathcal{M}_\hat{D}$ can be split up in a $D$-dimensional space and a $S^1$ as an internal manifold, that is $\mathcal{M}_\hat{D} = \mathcal{M}_D \times S^1$. The coordinates are split up according to $x^\hat{\mu} = (x^\mu, y)$. In this way we write the metric

$$
\begin{align*}
\hat{g}^{\mu\nu} = (g_{\mu\nu} A^{\mu} A_{\nu}, \varphi).
\end{align*}
$$

(5.3.9)

Thus the vector field $A^\mu$ gives rise to a vector field and a scalar field (axion) in lower dimensions, and the reduction of the metric generates a metric $g_{\mu\nu}$, scalar $\varphi$ (dilaton) and the so-called Kaluza-Klein $A^\mu$. This example reflects the fact that dimensional reduction of a higher-dimensional theory generates a lower-dimensional theory with more fields. This means that a 4-dimensional theory that looks complex can have a simple higher dimensional origin.

The Ansatz 5.3.9 for the metric is a correct Ansatz, but if we plug it into $\hat{D}$-dimensional Einstein equations, then the $D$-dimensional equations would take on an odd form which rather unfamiliar. For instance, the lower dimensional theory would not have the standard Einstein-Hilbert terms. Therefore the $D$-dimensional metric can not be interpreted as a solution of the standard Einstein equation. In order to get around this problem, one may redefine the metric $(g_D)_{\mu\nu} \rightarrow e^{2\varphi}(g_D)_{\mu\nu}$ so that the metric $(g_D)_{\mu\nu}$ solves the lower dimensional Einstein equation. The Ansatz becomes

$$
\begin{align*}
\hat{g}^{\mu\nu} = e^{2\varphi} g^{\mu\nu} + e^{2\beta\varphi}(dy + A)^2,
\end{align*}
$$

(5.3.10)

where we have redefined the Kaluza-Klein vector such that its corresponding kinetic term is as close to the standard Maxwell-form $1/4F^2$. The $\alpha$ and $\beta$ that parameterize the metric must be

$$
\begin{align*}
\alpha^2 = \frac{1}{2(D - 1)(D - 2)}, \quad \beta = -(D - 2)\alpha.
\end{align*}
$$

(5.3.11)
Nonlinear $\sigma$-models and Toroidal Reductions

The $D$-dimensional Einstein-Hilbert action can be obtained by inserting the Ansatz 5.3.10 into $\hat{D}$-dimensional equations, i.e. in form notation we get

$$\mathcal{L} = \ast R_D - \frac{1}{2} \ast d\varphi \wedge d\varphi - \frac{1}{2} e^{-2(D-1)\alpha\varphi} \ast dA \wedge dA. \quad (5.3.12)$$

The coupling between $\varphi$ and $A_{\mu}$ indicates that one cannot truncate $\varphi$ and maintain $A$ in order to obtain Einstein-Maxwell theory. If this would have been possible it would have implied the unification of electromagnetism with gravity. A truncation of the vector without truncation of the scalar is possible.

The consistency of the metric Ansatz can be understood as follows: imagine we take the size of $S^1$, $e^\sigma$ defined above, to be non-dynamical; namely the function $\sigma$ is constant. Therefore one would not be able to find a solution to the equations of motion. This can be seen from the fact that the more general and correct Ansatz 5.3.9 gives equation for $\varphi$ that does not have constant $\varphi = e^\sigma$ as a solution. In other words, the details of the interactions between various lower-dimensional fields prevent the truncation of the scalar $\varphi$.

5.3.2 Torus Reduction of Gravity

The circle reduction explained above can be repeated on a series of circles leading to what is called reduction over a torus $\mathbb{T}^n = S^1 \times \cdots \times S^1$.

The reduction of gravity over $\mathbb{T}^n$ generates $n$ vector fields $A^n$ with $m = 1, \cdots, n$, $n$ dilatons $\phi^m$ (they correspond to the radii of the circles), and $n(n-1)/2$ axions $\chi_{\alpha}$. The dilatons and axions scalar fields parameterize the coset $GL(n, \mathbb{R})/SO(n) = \mathbb{R}^+ \times SL(n, \mathbb{R})/SO(n)$. Thus the reduction Ansatz of $\hat{D}$-dimensional gravity over $n$-torus to $D = \hat{D} - n$ dimensions reads (with coordinates $x^{\bar{\mu}} = (x^\mu, y^m)$)

$$ds_D^2 = e^{2\alpha\varphi} ds_D^2 + e^{23\varphi} M_{mn}(dy^m + A^m_{\mu} dx^{\mu})(dy^n + A^n_{\nu} dx^{\nu}). \quad (5.3.13)$$

The kinetic term for $M_{mn}$ reveals that $M$ parameterizes the coset $SL(n, \mathbb{R})/SO(n)$ and since the “breathing mode” $\varphi$ decouples from the scalars in $M_{mn}$ in the kinetic term, the scalar manifold gets an extra factor $\mathbb{R}^+$ and thus becomes the coset $GL(n, \mathbb{R})/SO(n)$. The resulting scalar manifold plays the role of the moduli space of the torus $\mathbb{T}^n$, where the scalars in $M$ can be interpreted as shape-moduli of the torus. The scalar matrix $M_{mn}$ is a strictly positive definite symmetric matrix defined.

---

6 The original motivation for dimensional reduction was unification of the forces in nature from the dimensional reduction of pure gravity in some higher dimension. Unfortunately this is not possible since the dilaton field can not be stabilized.

7 This is associated with the constant shift symmetry $SO(1, 1) \cong \mathbb{R}^+$ of the dilaton $\varphi$.

8 The moduli are parameters that change the shape of the torus, at fixed volume, while keeping it flat. One can see, as an example, for $T^2$ that as $\phi$ (not the breathing one) varies, the relative radii of the two circles change, while as $\chi$ varies, the angle between the two circles changes.
5.3 Dimensional Reduction

by \( \mathcal{M} = L^T L \), \( \det \mathcal{M} = 1 \) and plays the role of an internal metric of the torus \( \mathbb{T}^n \), where \( L \) is the corresponding vielbein. We call \( \varphi \) the breathing mode since it describes the overall volume of the torus, and \( \text{GL}(n, \mathbb{R}) \) is the symmetry of \( \mathbb{T}^n \). Now if we plug 5.3.13 into the \( D \)-dimensional Einstein-Hilbert Lagrangian, it yields

\[
\mathcal{L} = \star R_D - \frac{1}{2} \star d \varphi \wedge d \varphi + \frac{1}{4} \star d \mathcal{M}^{mn} \wedge d (\mathcal{M}^{-1})^{mn} - \frac{1}{2} e^{2(\beta - \alpha)\varphi} \mathcal{M}^{mn} \star d A^n \wedge d A^m, \tag{5.3.14}
\]

with

\[
\alpha^2 = \frac{n}{2(D + n - 2)(D - 2)}, \quad \beta = -\frac{(D - 2)\alpha}{n}. \tag{5.3.15}
\]

The Lagrangian 5.3.14 is the Lagrangian of a pure Kaluza-Klein theory in \( D \) dimensions, which describes the coupling of \( \text{GL}(n, \mathbb{R})/\text{SO}(n) \) nonlinear \( \sigma \)-model to gravity and \( n \) vector gauge fields. For future use it is worthy noting that the matrix \( \mathcal{M} \) might be combined with the breathing \( \varphi \) into a matrix \( \hat{\mathcal{M}} \) which parameterizes the coset \( \text{GL}(n, \mathbb{R})/\text{SO}(n) \), namely

\[
\hat{\mathcal{M}} = (|\det \hat{\mathcal{M}}|)^{\frac{1}{n}} \mathcal{M}, \quad |\det \hat{\mathcal{M}}| = \exp \sqrt{2n\varphi}. \tag{5.3.16}
\]

We close this part with a word of caution: in the above reduction we reduced the action in order to find a new lower-dimensional action. This is not without danger since it is known that filling in on-shell information (the Ansatz) in an action and then performing Euler-Lagrange variation with respect to the remaining unfixed degrees of freedom can be inconsistent. To avoid this problem one should actually do more work and reduce the field equations instead of the action. But we are lucky as this problem will not arise in the reductions in this thesis.

5.3.3 Spacelike and Timelike Toroidal Reductions

So far we have considered dimensional reductions of pure Einstein gravity over Euclidean tori, giving rise to Minkowskian pure Kaluza-Klein theories in \( D \) dimensions. We call such reductions spacelike reductions, reductions over spacelike isometries (the killing vectors are spacelike). However, one can also reduce over a \( n \)-torus with a Lorentzian signature \( \mathbb{T}^{n-1,1} \) and obtain an Euclidean theory. This means that the time is included in the dimensional reductions, and hence the reduction over the time-circle is named timelike reduction.

What we have done for spacelike reductions turns out to be valid for timelike reductions. For instance the reduction Ansatz 5.3.13 continues to hold for timelike case. The only difference between the two reductions is encoded in the definition of the internal metric \( \mathcal{M} \) of the Lorentzian torus \( \mathbb{T}^{n-1,1} \), i.e.

\[
\mathcal{M} = L^T \eta L, \quad \det \mathcal{M} = -1, \quad \eta = \text{diag}(-1, 1, \cdots), \tag{5.3.17}
\]
where $\eta$ here is the tangent space metric of the torus. Surprisingly enough, timelike reductions give rise to the Lagrangian 5.3.14 but now for Euclidean Kaluza-Klein theories where the scalars form $\text{GL}(n,\mathbb{R})/\text{SO}(n-1,1)$ nonlinear $\sigma$-models. That is indeed an example of pseudo-Riemannian $\sigma$-models that we have discussed in section 5.2.5.

Reducing to three dimensions makes things look special. One can dualize all the gauge potentials $A^m$ to scalars as follows: consider in $D=3$ the last term in 5.3.14

$$ S_A \sim \int -\mathcal{M}_{mn} \star F^m \wedge F^n. \quad (5.3.18) $$

To dualize the vectors we need to be sure that the $F^m$ are closed and therefore locally exact, i.e. $F^m = dA^m$. Therefore we enforce this using the Lagrange multipliers $\chi_m$

$$ S_{F,\chi} \sim \int -\mathcal{M}_{mn} \star F^m \wedge F^n - \chi_m dF^m. \quad (5.3.19) $$

Variation with respect to $\chi$ indeed gives us that $F$ is closed two-form. Now we can treat the action as if $F$ is a fundamental field. The equation of motion for $F$ is

$$ \mathcal{M}_{mn} \star F^n = d\chi_m \Rightarrow F^n = (-)^s(\mathcal{M}^{-1})^{mn} \star d\chi_m, \quad (5.3.20) $$

where we have made use of the relation A.2.32 for a $p$-form. If we plug the result for $F$, perform partial integration and reshuffle the terms, then we get

$$ S_\chi \sim \int (-1)^s(\mathcal{M}^{-1})^{mn} \star d\chi_m \wedge d\chi_n - (-)^s d(\chi_n(\mathcal{M}^{-1})^{mn} \star d\chi_m), \quad (5.3.21) $$

where the total derivative can of course be dropped. We therefore notice that in the case of the three-dimensional Euclidean theory (reduction over a timelike isometry) $(s = 0)$ the axion kinetic terms appear with the opposite sign of their related vector kinetic terms. Consequently, there is a symmetry enhancement in $D = 3$ since it can be shown that the extra scalars combine with the existing scalars into the coset $\text{SL}(n+1,\mathbb{R})/\text{SO}(n-1,2)^9$. In this case there is no decoupled $\mathbb{R}$. Note that for the reduction over an Euclidean torus $(s = 1)$ from $3+n$ to three dimensions we obtain the coset $\text{SL}(n+1,\mathbb{R})/\text{SO}(n+1)$.

### 5.4 Torus Reductions of Maximal Supergravities

We start our discussion with emphasizing the fact that torus reductions do not break supersymmetry. Therefore the dimensional reductions of type IIB and type IIA on an $n$-torus and 11-dimensional supergravity on $n+1$-torus lead to the unique maximal

---

9This means that $\text{SO}(n-1,2)$-invariant matrix $\eta$ has as a signature $\text{diag}(-1, -1, +1, +1, \cdots, +1)$. 


5.5 From $G/H$ to $G/H^*$: the Wick Rotation

Table 5.4.1: Cosets for maximal supergravities in Minkowskian and Euclidean signatures.

<table>
<thead>
<tr>
<th>$D$</th>
<th>$G/H$-Minkowskian SUGRA</th>
<th>$G/H^*$-Euclidean SUGRA</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>$SO(1,1)$</td>
<td>$SO(1,1)$</td>
</tr>
<tr>
<td>9</td>
<td>$\frac{GL(2,\mathbb{R})}{SO(2)}$</td>
<td>$\frac{GL(2,\mathbb{R})}{SO(1,1)}$</td>
</tr>
<tr>
<td>8</td>
<td>$\frac{SL(3,\mathbb{R}) \times SL(2,\mathbb{R})}{SO(3) \times SO(2)}$</td>
<td>$\frac{SL(3,\mathbb{R}) \times SL(2,\mathbb{R})}{SO(2) \times SO(1,1)}$</td>
</tr>
<tr>
<td>7</td>
<td>$SL(5,\mathbb{R})$</td>
<td>$SL(5,\mathbb{R})$</td>
</tr>
<tr>
<td>6</td>
<td>$SO(5,\mathbb{R})$</td>
<td>$SO(5,\mathbb{R})$</td>
</tr>
<tr>
<td>5</td>
<td>$E_{6(+6)} \times Sp(8)$</td>
<td>$E_{6(+6)} \times Sp(4,4)$</td>
</tr>
<tr>
<td>4</td>
<td>$E_{7(+7)} \times SU(8)$</td>
<td>$E_{7(+7)} \times SU^*(8)$</td>
</tr>
<tr>
<td>3</td>
<td>$E_{8(+8)} \times SO(16)$</td>
<td>$E_{8(+8)} \times SO^*(16)$</td>
</tr>
</tbody>
</table>

supergavities in $D < 11$.

The (maximal) supergravity theories can be classified according to the nonlinear sigma models describing the scalar field interactions. This means that the geometry of (pseudo)-Riemannian symmetric spaces fixes the scalar field interactions terms in supergravity Lagrangian. We summarize the scalar manifolds of the maximally extended supergravities that appear after dimensional reduction of 11-dimensional supergravity on a torus in table 5.4.1 [136]. For future use we show it both for Minkowskian and Euclidean maximal supergravities. The cosets $G/H$ in the left column are all maximally non-compact since $G$ is the maximally non-compact real form of a semisimple complex Lie group and $H$ is its maximal compact subgroup. Since $H$ is compact the metric is strictly positive definite and the coset is Riemannian. The cosets $G/H^*$ in the right column only differ in the isotropy group $H^*$ which is some non-compact version of $H$ and as a result $G/H^*$ is pseudo-Riemannian. In appendix D of [G] we perform the reductions of type II theories and we give in particular the precise group theoretical characterization of the ten-dimensional origin of the bosonic fields in $D = 3$.

Let us end this chapter by defining a generalized Wick rotation which maps a geodesic on $G/H$ in a Minkowskian theory, into geodesic on $G/H^*$ in its Euclidean version [G].
In order to map a compactification on a spatial circle into one on a time-circle, we need to analytically continue the internal radius: \( R \rightarrow iR_0 \). This transformation can be viewed as the action of a complexified \( O(1,1) \) transformation

\[
\mathcal{O} = i^{H_0},
\]  

(5.5.1)
on the Minkowskian \( D \)-dimensional theory in which the scalar fields span the target space \( G/H \). The generator \( H_0 \) is a suitable combination of the Cartan generators \( H_I \) of \( \mathfrak{g} \) with \( I = 1, \cdots \) rank\( \mathfrak{g} \). Consider the following action on the generators \( t_n \) of \( \mathfrak{g} \) taking values in some representation of \( \mathfrak{g} \):

\[
t_n \rightarrow O_n^m [Ot_m\mathcal{O}^{-1}].
\]  

(5.5.2)
The action of \( \mathcal{O} \) on the Cartan generators is trivial \( \mathcal{O}^I = \delta^I_I \), while it has the following action on the shift generators:

\[
\mathcal{O}_\alpha^\sigma = i^{-\alpha(H_0)}\delta^\sigma_\alpha,
\]  

(5.5.3)
where \( \alpha(H_0) \) is the scalar product defined via the commutation relation \([H_0, E_{\alpha}] = \alpha(H_0)E_{\alpha}\). We see that a generic shift generator \( E_\alpha \) is mapped into itself by 5.5.2.

\[
E_\alpha \rightarrow \mathcal{O}_\alpha^\prime [i^{H_0}E_\alpha i^{-H_0}] = \mathcal{O}_\alpha^\prime \alpha^*(H_0)E_{\alpha'} = E_{\alpha'}.
\]  

(5.5.4)
Therefore the transformation 5.5.2 maps \( \mathfrak{g} \) into itself. According to the Cartan decomposition 5.2.4 the compact subalgebra \( \mathfrak{h} \) and the non-compact space \( \tilde{\mathfrak{g}} \) are generated by

\[
\mathfrak{h} = \{ \tilde{J}_I \} = \{ E_\alpha - E_{-\alpha} \}, \quad \tilde{\mathfrak{g}} = \{ H_I, \tilde{K}_\alpha \} = \{ H_I, E_\alpha + E_{-\alpha} \}.
\]  

(5.5.5)
On a generic element \( g \) of \( G \) the above transformation amounts to a combination of a change of basis for the matrix representation and a redefinition of the group parameters. Indeed if we write an element of \( G \) as the product of a coset representative \( \tilde{L} \in \exp(\tilde{\mathfrak{g}}) \) times an element \( \tilde{h} \) of \( H \) we have

\[
g = \tilde{L}(\varphi, \phi)\tilde{h}(\xi) = e^{\phi\tilde{K}_\alpha}e^{\varphi^I H_I}e^{\xi^\alpha \tilde{J}_\alpha} \rightarrow O e^{\phi\tilde{K}_\alpha}e^{\varphi^I H_I}e^{\xi^\alpha \tilde{J}_\alpha} O^{-1},
\]  

(5.5.6)
where the redefined parameters are

\[
\varphi^I = \varphi^I; \quad \phi^\alpha = \phi^\sigma O_\sigma^\alpha; \quad \xi^\alpha = \xi^\sigma O_\sigma^\alpha.
\]  

(5.5.7)
Let us consider the effect of this transformation on the generators of the coset representative and on the compact factor

\[
\phi^\alpha [O \tilde{K}_\alpha O^{-1}] = \phi^\alpha i^{-\alpha(H_0)}[i^{\alpha(H_0)}E_\alpha + i^{-\alpha(H_0)}E_{-\alpha}] = \phi^\alpha (E_\alpha + (-1)^{\alpha(H_0)}E_{-\alpha}) = \phi^\alpha K_\alpha,
\]  

(5.5.8)
\[
\xi^\alpha [O \tilde{J}_\alpha O^{-1}] = \xi^\alpha i^{-\alpha(H_0)}[i^{\alpha(H_0)}E_\alpha - i^{-\alpha(H_0)}E_{-\alpha}] = \xi^\alpha (E_\alpha - (-1)^{\alpha(H_0)}E_{-\alpha}) = \xi^\alpha J_\alpha,
\]  

(5.5.8)
where \( J_\alpha \) and \( K_\alpha \) differ from \( \tilde{J}_\alpha \) and \( \tilde{K}_\alpha \) only for \( \alpha = \gamma \), for which \( J_\gamma = E_\gamma + E_{-\gamma} \) and \( K_\gamma = E_\gamma - E_{-\gamma} \). \( J_\alpha \) are therefore generators of \( H^* \) and \( K_\alpha \), together with \( H_I \) are in \( G/H^* \). The Wick rotation defines therefore a mapping between two different representations of a same element \( g \) of \( G \): one as the product of a coset representative \( \tilde{L} \) in \( G/H \) and an element \( \tilde{h} \) of \( H \) and the other as a product of a coset representative \( L \) in \( G/H^* \) times an element \( h \) in \( H^* \). The matrix \( \hat{M}(\phi^I, \phi^\alpha) = \tilde{L}^* \tilde{L} \), defined in subsection 5.2.2, which describes the scalar fields on \( G/H \) transforms as follows:

\[
\hat{M}(\phi^I, \phi^\alpha) \rightarrow \mathcal{O}^2 \hat{M}(\phi'^I, \phi'^\alpha) \mathcal{O} = L^* \eta L = \hat{M}(\phi^I, \phi^\alpha),
\]

(5.5.9)

where \( \eta = \mathcal{O}^2 \mathcal{O} \) and \( \hat{M} \) is the matrix describing the scalars on \( G/H^* \). For example in \( D = 3 \) maximal supergravity, the effect of the transformation \( \mathcal{O} \) is to map the \( E_8(8)_{/SO(16)} \) coset in the last row of table 5.4.1 to \( E_8(8)_{/SO^*(16)} \) coset of the same row. In other words, 56 compact generators of \( SO(16) \) are mapped into 56 non-compact generators \( J_\gamma \) in \( SO^*(16) \).