Chapter 4

Duality Symmetries for Interacting Fields

In chapter 2 we have pointed out that one of the indirect methods for constructing the higher derivative terms of Born-Infeld theory is to maximize the usefulness of duality symmetries. It has been shown a long time ago that most of the four dimensional nonlinear electrodynamics models, including Born-Infeld theory, are electromagnetic dual invariant (selfdual) theories. In this chapter we will start our discussion with introducing and defining electromagnetic duality in (non)-linear electromagnetism. Then we will try to make contact with the previous chapter by including higher derivative corrections of the field strength $F_{ab}$. The investigation of electromagnetic duality invariance in the presence of scalar fields is of great interest for us, since such symmetries are going to play a crucial role in constructing the nonlinear sigma models that we will study in chapter 5. We also find it useful to derive the Noether charges associated with the duality symmetries since those charges will form the cornerstone in chapter 6 for classifying solutions in (super)gravity theories. Selfdual invariant quantities will be briefly mentioned, and the chapter will be closed with a discussion.

4.1 Electro-Magnetic Duality: Overview

It has been established a long time ago that the Hodge duality operation $\sim$ (defined in A.2.2) is the duality symmetry of the four dimensional Maxwell theory. In terms of the electric field $E$ and magnetic field $B$, Hodge duality symmetry is a $SO(2)^1$

\footnote{All the group theory notions and definitions that will be used in this thesis are reviewed in appendix C.}
rotation written in matrix form as
\[
\begin{pmatrix} E \\ B \end{pmatrix} \to \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} E \\ B \end{pmatrix}.
\] (4.1.1)

In differential form notation on Minkowski spacetime\(^2\), the transformation is expressed in terms of a field strength \(F_{ab} = \partial_a A_b - \partial_b A_a\) so that the duality symmetry \(\sim\) takes on the form
\[
F_{ab} \to \cos \alpha F_{ab} + \sin \alpha \tilde{F}_{ab}.
\] (4.1.2)

The sourceless linear Maxwell’s equations read
\[
\begin{align*}
\nabla \cdot B &= 0, \\
\nabla \times E &= \frac{\partial B}{\partial t}, \\
\nabla \cdot D &= 0, \\
\nabla \times H &= +\frac{\partial D}{\partial t},
\end{align*}
\] (4.1.3) (4.1.4)

where \(D\) is the electric induction and \(H\) the magnetic intensity, which are simply equivalent in forms notation to the combined field equations system of the Bianchi identity and the equations of motion
\[
\partial_a F^{ab} = 0, \quad \partial_a \tilde{F}^{ab} = 0.
\] (4.1.5)

What is meant by electro-magnetic E-M duality symmetry is the symmetry of the Maxwell equations not of their corresponding Lagrangian which is expressed by
\[
\mathcal{L}_{Maxwell} = -\frac{1}{4} F_{ab} F^{ab} = -\frac{1}{2} (B^2 - E^2).
\] (4.1.6)

Notice that only for the case of Maxwell theory in vacuum for which \(E = D\) and \(B = H\), E-M duality breaks down to Hodge duality. For more general cases, namely the nonlinear electrodynamics models, E-M duality has a slightly different interpretation which is the topic of the next section.

### 4.2 (Self)duality Rotations: the Gaillard-Zumino Model

In this section we review some results of Gaillard and Zumino [104–106] from the early 80’s and developed further by Gibbons and Rasheed later in the 90’s [107, 108].

First of all, it is well-known that E-M duality transformation is implemented via

\footnote{We adopt in this chapter the following conventions; Minkowski metric \(\eta_{ab}\) with diag(\(-, +, +, +\)) signature, where \(\tilde{F} = -F\), and often we make use of the notation \(\text{tr}FG = -F_{ab}G^{ab}\).}
the transformation of the field strength $F_{ab}$ rather than the fundamental variable, the vector gauge potential $A_a$. Therefore E-M duality can be realized only on the level of equations of motion. The reason is that the duality transformations are consistent only on-shell. Since the independent variable of the theory is $A_a$, $\delta F_{ab}$ in eq.4.1.2 should be derived from $\delta A_a$:

$$\partial_a \delta A_b - \partial_b \delta A_a = \cos \alpha F_{ab} + \sin \alpha \tilde{F}_{ab}. \quad (4.2.1)$$

The integrability of this equation requires $\partial_a (\cos \alpha \tilde{F} + \sin \alpha \tilde{F}) = 0$, i.e., $\partial_a F_{ab} = 0$. Thus the equations of motion must be satisfied. Even if we ignore this point and formally consider the transformation 4.1.2 off-shell, the lagrangian 4.1.6 is not invariant. Therefore in order to construct theories invariant under E-M duality transformations, it is easier to study the covariance of equations of motion.

The off-shell realization of manifestly E-M duality invariance on the level of the action has been exhaustively investigated by a handful of researchers. We mention in particular results by [109], in which they found a formalism that helps with uplifting duality invariance to the action. In such avenues, E-M duality transformations are basically defined through the off-shell gauge potential $A_a$. This thus generates non-local terms in the action in the light of the relation $F = dA$. However, in order to circumvent the locality violation, all one needs is to double the number of gauge fields [109](and references therein)- Sen-Schwarz model- in the sense that the gauge fields and their duals appear on a par in the action. In doing so, there is a price one should pay; doubling gauge fields actually ruins Lorentz covariance which Pasti et al. [110] have managed to restore afterwards. However, the manifest E-M duality invariant actions are out of the scope of this thesis. In what follows we are going to focus on some aspects of E-M dualities which are only symmetries of the equations of motions.

To end this short introduction we remind the reader that all our considerations are classical. The systems we study should be regarded as effective theories, in accordance with the appearance of the Born-Infeld action as the worldvolume action of the D-brane (see chapter 2).

### 4.2.1 Gaillard-Zumino Condition: Selfduality Condition

In order to gain some insights into the significance of duality invariance\(^3\) (selfduality), we start our analysis by considering nonlinear extensions of Maxwell theory, i.e., nonlinear electrodynamics models. The nonlinearity of such a model can be obtained by adding polynomial higher order terms of the field strength $F_{ab}$ to the Lagrangian of the Maxwell theory. For physical reasons we restrict ourselves to nonlinear models

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\(^3\)Throughout this chapter notions like duality invariance, selfduality and duality symmetry all refer to the same concept.
which coincide with the Maxwell model at weak field limits, i.e. \( \mathcal{L}(F) = -\frac{1}{4} F_{ab} F^{ab} + \mathcal{O}(F^4) \). The equations of motion and the Bianchi identity are

\[
\partial_a G^{ab} = 0, \quad \partial_a \tilde{F}^{ab} = 0,
\]

where \( G \) is the dual\(^4\) of \( F \) defined by the following constitutive relation

\[
G^{ab}(F) = -\partial \mathcal{L}(F) \partial F^{ab}, \quad \partial F_{ab} \partial F_{cd} = (\delta^c_a \delta^d_b - \delta^c_b \delta^d_a).
\]

Note that \( F_{ab} \) and \( G_{ab} \) are not independent of one another but nonlinearly related by 4.2.3. A pair \( (G, F) \) can be mapped to \( (G', F') \) via transformations \( S \) such that

\[
\left( G'(F') \right) \frac{\partial \mathcal{L}(F')}{\partial F'} = S \left( G(F) \right) \frac{\partial \mathcal{L}(F)}{\partial F},
\]

with \( S \in \text{GL}(2, \mathbb{R}) \). This means that one can solve 4.2.4 for \( (G, F) \) in terms of \( (G', F') \) and then use the equations 4.2.2-4.2.3 to find the transformed version of the field equations and the Lagrangian. Thus, the transformed Lagrangian \( \mathcal{L}' \) does exist and must satisfy \( G_{ab}(F') = -\partial \mathcal{L}'(F') \partial F_{ab} \). In general, the Lagrangian \( \mathcal{L}'(F) \) differs from \( -\frac{1}{4} F_{ab} F^{ab} + \mathcal{O}(F^4) \). Therefore the shift in the functional form of the Lagrangian behaves as

\[
\Delta \mathcal{L}(F) = \mathcal{L}'(F) - \mathcal{L}(F) = \delta \mathcal{L}(F) - \text{tr} \left( \delta F \frac{\partial \mathcal{L}}{\partial F} \right),
\]

where \( \delta \) is the infinitesimal form of 4.2.4, and the variation of the Lagrangian under such a transformation is defined by \( \delta \mathcal{L}(F) = \mathcal{L}'(F) - \mathcal{L}(F) \).

The requirement of selfduality or the E-M duality invariance of the equations of motion comes down to setting \( \Delta \) to zero. Consequently, the possible functional form of \( \mathcal{L} \) is severely constrained

\[
\mathcal{L}'(F) = \mathcal{L}(F).
\]

Models which obey 4.2.6 are called selfdual models.

The selfduality requirement has many implications which follow from 4.2.6:

- The invariance of the constitutive relation 4.2.3.
- The Lagrangian \( \mathcal{L} \) solves a second order partial differential equation in the six variables of \( F_{ab} \)

\[
\text{tr} G\tilde{G} = \text{tr} F\tilde{F}.
\]

\(^4\)From the viewpoint of Hamiltonian formalism, there is a geometric interpretation for \( G, F \) together with \( G \) form respectively the coordinates and the dual coordinates of the 2-form space \( V = \Lambda^2(\mathbb{R}^4) \) and its dual \( V^* \). From this one can obtain the symplectic phase space \( V \oplus V^* \) where the Lagrangian \( \mathcal{L} \) plays the role of the generating functional. That is where the word dual comes from.
This is the necessary and sufficient condition on the Lagrangian \( L(F) \) so that its corresponding field equations admit E-M duality invariance. For the derivation of 4.2.7 we refer the reader to the ensuing sections where we shall do the calculations for more general cases.

- For purely nonlinear electrodynamics models, the E-M duality symmetry is described and represented by the compact abelian group \( U(1) \sim SO(2) \).

- The combination \( L(F) - \frac{1}{4} tr FG \) is duality invariant.

As mentioned before, we do not impose the invariance of the Lagrangian itself; we shall see that the system of the equations of motion can be invariant only if \( \delta L \) does not vanish. Instead the variation of \( L \) is required to have a specific form.

### 4.2.2 Solutions of Selfduality Condition

An interesting solution of 4.2.7 is the Born-Infeld Lagrangian defined in chapter 2. It is given in four dimensions by the Lagrangian\(^5\)

\[
L(F)_{BI} = (-D^\frac{1}{2}(F) + 1),
\]

(4.2.8)

where

\[
D(F) = -\det(\eta_{ab} + F_{ab}) = 1 - \frac{1}{2}\text{tr}F^2 - \frac{1}{16}(\text{tr}F\tilde{F})^2
\]

(4.2.9)

\[
= 1 + \frac{1}{2}P - \frac{1}{16}Q^2,
\]

with \( P = -\text{tr}FF \) and \( Q = -\text{tr}F\tilde{F} \) are the only two independent Lorentz invariants of electromagnetism in four dimensions.

Then

\[
\frac{\partial D}{\partial F} = 2F - \frac{1}{2}\tilde{F}, \quad G = -\frac{\partial L}{\partial F} = D^{-\frac{1}{2}}(-F + \frac{1}{4}\tilde{F}).
\]

(4.2.10)

Using 4.2.10, one can verify that \( \text{tr}G\tilde{G} = \text{tr}F\tilde{F} \), and hence the Born-Infeld theory is duality invariant [111].

It is quite natural to ask whether the Born-Infeld theory is the most general physically acceptable solution of 4.2.7. This has been investigated intensively in [104, 105, 108, 112] where a negative answer has been reached.

More general solutions of the differential equation 4.2.7 have been studied in [105, 113] and a prescription to obtain solutions, corresponding to selfdual models, has been

\(^5\)We set the fundamental (scale)^2, \( T^{-1} = 2\pi\alpha' \) in string theory context, equal to one.
presented. The derivation of such solutions goes as follows: first we consider \( \mathcal{L} \) to be a function of \((P, Q)\), namely \( \mathcal{L}(P, Q) \). It follows that the dual tensor \( G \) is written as

\[
G = -4(\mathcal{L}_P F + \mathcal{L}_Q \tilde{F}), \quad \tilde{G} = -4(\mathcal{L}_P \tilde{F} - \mathcal{L}_Q F)
\]

where \( \mathcal{L}_P \) and \( \mathcal{L}_Q \) are the derivatives of \( \mathcal{L} \) w.r.t \( P \) and \( Q \) respectively. Substituting 4.2.11 into 4.2.7 leads to

\[
[16((\mathcal{L}_Q)^2 - (\mathcal{L}_P)^2) + 1] Q + 32P \mathcal{L}_P \mathcal{L}_Q = 0.
\]

(4.2.12)

This may be simplified further by considering another change of variables

\[
U = \frac{1}{8}(P + \sqrt{P^2 + Q^2}), \quad V = \frac{1}{8}(P - \sqrt{P^2 + Q^2}).
\]

(4.2.13)

Then 4.2.12 is reduced to [108]

\[
\mathcal{L}_U \mathcal{L}_V = 16.
\]

(4.2.14)

This is a familiar nonlinear differential equation which has been studied extensively in mathematics. In our case we must also impose the boundary condition which makes \( \mathcal{L}(U, V) \) approaches the Maxwell Lagrangian \( \mathcal{L}_M(U, V) = -P/4 = -U - V \) when the field strength \( F \) is small. According to [114], the general solution solving 4.2.14 is expressed explicitly in terms of an arbitrary function \( \mathcal{F}(T) \) determined by the initial values \( \mathcal{L}(0, V) = \mathcal{F}(V) \) and \( \mathcal{L}_U(0, V) = 1/\mathcal{F}'(V) \), where the prime is the derivative of \( \mathcal{F} \) with respect to \( T \). The general solution thus reads

\[
\frac{1}{4} \mathcal{L}(U, V) = \frac{2U}{\mathcal{F}'(T)} + \mathcal{F}(T), \quad \text{with } V = \frac{U}{[\mathcal{F}'(T)]^2 + T}.
\]

(4.2.15)

Solving the second equation of 4.2.15 for \( T(U, V) \) results in determining the corresponding \( \mathcal{L}(U, V) \). It is worth noting that in [115] it has been verified that indeed 4.2.15 solves 4.2.14 with \( \mathcal{L}_U = 4/\mathcal{F}' \) and \( \mathcal{L}_V = 4\mathcal{F}' \), and moreover the condition that \( \mathcal{L} \) should approach Maxwell Lagrangian for a small field strength implies that \( \mathcal{F}(T) = \mathcal{L}(U = 0, T) \equiv -T \) for a small \( T \).

There have been a few explicit and exact solutions of the selfduality condition [113]. As a crosscheck one might reconsider the Born-Infeld theory example given above. In terms of \((U, V)\), 4.2.8 takes on the form

\[
\mathcal{L}_{BI}(U, V) = -\sqrt{(1 + 2U)(1 + 2V)} + 1.
\]

(4.2.16)

The associated function \( \mathcal{F}(T) \) to \( \mathcal{L}_{BI} \) is determined by setting \( U = 0 \). This yields

\[
\mathcal{F}(T) = -(1 + 2T)^{1/2} + 1, \quad \text{with } T(U, V) = \frac{V - 4U}{1 + 8U}.
\]

(4.2.17)
Thus roughly speaking, there are as many selfdual deformations of Maxwell theory as there are real analytic functions $F(T)$ of one real argument.

In order to have a better understanding of what is exactly going on, we present the following illustration:

We actually know that approximate solutions to Eq.4.2.7 can also be obtained by a power series expansion in $F$ of Lagrangians whose leading order is Maxwell term $-1/4\text{tr}F^2$. It is convenient to write down the Lagrangian in terms of $P$ and $Q$. Up to fourth order in $F$, using the identities

$$\bar{F}F^b_a = \frac{1}{4} Q^b_a, \quad (\bar{F}^2 - F^2)_a^b = \frac{1}{2} P^b_a, \quad (4.2.18)$$

the Lagrangian can be expressed as

$$\mathcal{L}(P, Q) = \frac{1}{4} P + a_1 P^2 + a_2 Q^2 + a_3 PQ + O(P^3, Q^3), \quad (4.2.19)$$

where $a_1$, $a_2$ and $a_3$ are arbitrary coefficients. If we substitute 4.2.19 in 4.2.7, then one can find, up to the same order, that $a_1 = 0$ and a free parameter $a = a_2 = a_3$. However, the coefficient $a$ can be yet appropriately fixed by rescaling the field strength $F$. It turns out that, up to order four in $F$, $\mathcal{L}$ coincides with $\mathcal{L}_{BI}$. One can push this calculation an order further by adding the most general Lagrangian terms through order six in $F$, and again requiring that the resulting Lagrangian solves 4.2.7. What comes out is that, up to this given order, all the coefficients but one are determined.

The question now is how many free parameters one has in the power expansion of duality invariant Lagrangians. The power expansion for arbitrarily large $m$ behaves as

$$\mathcal{L} = \frac{P}{4} + a \mathcal{L}_1 + a^2 \mathcal{L}_2 + \cdots + a^{2m-1} \mathcal{L}_{2m-1} + a^{2m} \mathcal{L}_{2m}, \quad (4.2.20)$$

where $\mathcal{L}_{2m-1}$ and $\mathcal{L}_{2m}$ are written as:

$$\mathcal{L}_{2m-1} = \sum_{n=0}^{m} a_n P^{2(m-n)} Q^{2n} \quad (4.2.21a)$$

$$\mathcal{L}_{2m} = \sum_{n=0}^{m} a_n P^{2(m-n)+1} Q^{2n}. \quad (4.2.21b)$$

In [116] a theorem has been reached which states that in the power expansion 4.2.20 of duality invariant Lagrangians, i.e. Lagrangians that satisfy 4.2.7, there will occur a free parameter in each $\mathcal{L}_{2m-1}$ as defined in 4.2.21a and there will occur no free parameter in each $\mathcal{L}_{2m}$ as defined in 4.2.21b.

We conclude that for specific choices of the free parameters, the power expansion 4.2.20, satisfying duality condition, coincides with the power expansion of the
Born-Infeld Lagrangian. Actually there are infinitely many free parameters, since for arbitrary large odd powers in the expansion there will still occur new free parameters in the general expansion. So, there are indeed more Lagrangians which admit selfduality next to the Born-Infeld theory.

### 4.2.3 Selfduality of Nonlinear Models with Derivative Corrections

The extension of the E-M selfduality principle to models which involve higher derivative corrections is of great interest. In particular, this is relevant for applications in string theory, where it is known (chapter 2) that the open superstring effective action, which for slowly varying fields coincides with the Born-Infeld theory, also comprises derivative corrections. It is natural to ask what happens if the Lagrangian also depends on derivatives of the field strength, i.e., $L(F, \partial F)$. At first sight, it seems that the analysis presented in 4.2.1 is no more valid, and hence should be modified. However, most of the discussion in 4.2.1 can be taken over if one works with the action $S$ rather than the Lagrangian, and uses differentiation of functionals [112][B].

#### Derivation of Selfduality Condition

We start out with the definitions

$$ G_{ab} = -\frac{\delta}{\delta F_{ab}} S[F], \quad \frac{\delta}{\delta F_{ab}(x)} F_{cd}(y) = 2\delta_{cd} \delta(y - x). $$  \hspace{1cm} (4.2.22)

The selfduality rotations 4.2.4 have the same structure, yet $G'$ must be consistently obtained from an action $S'$. The selfduality condition, written in the integral form, then reads

$$ S'[F] = S[F] \Rightarrow \int d^4 x tr G\tilde{G} = \int d^4 x tr F\tilde{F}. $$  \hspace{1cm} (4.2.23)

#### Proof:

We resort to the infinitesimal version of 4.2.4 associated to SO(2), then one has

$$ G'^{ab}[F'] = G^{ab}[F] + \lambda \tilde{F}^{ab}, \quad \tilde{F}'^{ab} = \tilde{F}^{ab} - \lambda G^{ab}[F], $$  \hspace{1cm} (4.2.24)

where $\lambda$ is the infinitesimal parameter of 4.2.4, and $G'^{ab}[F'] = \delta S'[F']/\delta F'_{ab}(x)$. Using the selfduality condition $S'[F] = S[F]$ one finds

$$ G'^{ab}[F', x] = -\frac{\delta S[F']}{\delta F'_{ab}(x)} = -\left( \frac{\delta S[F]}{\delta F_{ab}(x)} + \frac{\delta}{\delta F_{ab}(x)} \delta S[F] \right), $$  \hspace{1cm} (4.2.25)

where we have made use of

$$ \delta S[F] = S[F'] - S[F], \quad \text{and} \quad \frac{\delta}{\delta F_{ab}(x)} \delta S[F] = \frac{\delta}{\delta F_{ab}(x)} \delta S[F] + O(\lambda^2). $$  \hspace{1cm} (4.2.26)
The evaluation of $\frac{\delta S[F]}{\delta F'_{ab}(x)}$ results in

$$\frac{\delta S[F]}{\delta F'_{ab}(x)} = -G^{ab}[F, x] + \frac{\lambda}{4}\frac{\delta}{\delta F_{ab}(x)} \left( \int d^4y G^{cd}[F, y] \tilde{G}_{cd}[F, y] \right). \quad (4.2.27)$$

The substitution of 4.2.27 into 4.2.25 gives rise to

$$G'^{ab}[F', x] = G^{ab}[F, x] - \frac{\delta}{\delta F_{ab}(x)} \left( \frac{\lambda}{4} \int d^4y G^{cd}[F, y] \tilde{G}_{cd}[F, y] \right). \quad (4.2.28)$$

On the other hand, from the variation 4.2.24 of $G$ it follows

$$G'^{ab}[F', x] = G^{ab}[F, x] - \frac{\delta}{\delta F_{ab}(x)} \left( -\frac{\lambda}{4} \int d^4y F^{cd}(y) \tilde{F}_{cd}(y) \right). \quad (4.2.29)$$

Inserting the variation $\delta S = -\frac{\lambda}{4} \int d^4y G^{cd}[F, y] \tilde{G}_{cd}[F, y]$ into 4.2.28, and comparing the resulting expression to 4.2.29, one obtains the integrated form of the consistency condition 4.2.23. Here is the end of the proof.

**Application**

The terms we will consider are the terms 2.5.13 discussed in chapter 2

$$L_{(m,n)} = \alpha'^m \partial^m F^p, \quad \text{for } p = m + 2 - n/2, \quad (4.2.30)$$

$$L_{m} = \alpha'^m F^{m+2}, \quad \text{for } n = 0. \quad (4.2.31)$$

Now, we need to establish the E-M duality invariance of the $p = 4$ terms. Of course, superstring corrections are obtained in $d = 10$, while E-M selfduality, that has been discussed so far in this chapter, is realized in $d = 4$. Therefore we will examine the validity of E-M selfduality of the contributions of the type 4.2.30, setting all other ten-dimensional fields to zero, and truncating the resulting expression to $d = 4$, by restricting the Lorentz index to run from one to four. Moreover, the result can hold order-by-order in $\alpha'$ so that for each order of $\alpha'$ the corresponding $p = 4$ contribution to the Lagrangian satisfies, together with Maxwell $m = 0$ term, E-M selfduality to order $m$ in $\alpha'$.

We start the analysis with the $m = 4$ terms. Then the four-derivative terms 2.5.25 stated in chapter 2 are

$$L_{(4,4)} = a_{(4,4)} \alpha'^4 \xi_8^{abcdfg} \partial_k F_{ab} \partial^k F_{cd} \partial^f F_{ef} \partial_h F_{gb}, \quad (4.2.32)$$

where $t_8^{abcdfg}$ has been defined by 2.5.23. The combination $L_0 + L_{(4,4)}$ generates

$$G^{ab} = F^{ab} + G^{ab}_{(4,4)}, \quad G^{ab}_{(4,4)} = \alpha'^4 \xi_8^{abcdfg} \partial_k (\partial^k F_{cd} \partial^f F_{ef} \partial_h F_{gb}). \quad (4.2.33)$$
To establish E-M selfduality we have to verify whether $L_0 + L_{(4,4)}$ solves 4.2.23. The verification only makes sense to order four in $\alpha'$, since in higher orders other contributions to the effective action would interfere. Since the zero order term in 4.2.23 cancel it remains to verify that

$$I = \int d^4x \text{tr} \tilde{F}G_{(4,4)} = 0.$$  \hspace{1cm} (4.2.34)

Integrating 4.2.34 by parts implies

$$I = \int d^4x t_8^{abcdefgh} \partial_k \tilde{F}_{ab} \partial^k F_{cd} \partial F_{ef} \partial^l F_{gh}.$$  \hspace{1cm} (4.2.35)

The crucial property, which in fact holds to all orders in $\alpha'$, is that in $L_{(m,2m-4)}$ the indices of the field strengths $F$ are all contracted amongst each other, and therefore also the derivatives are contracted (see section 2.5.5). The complete symmetry of $t_8$ in combination with

$$(\tilde{F}_k F_l + \tilde{F}_l F_k)_{a^b} = -\frac{1}{2} \delta_a^b \text{tr} \tilde{F}_k F_l; \quad \text{with } F_k = \partial F,$$  \hspace{1cm} (4.2.36)

allows us to express all the traces over the four matrices resulting from the expansion of 4.2.35 in terms of products of traces over two matrices. Thus the cancellation of 4.2.34 has been verified [B].

For higher orders in $\alpha'$ the $p = 4$ terms contain more derivatives, but again these are all contracted, while the tensor structure of the field strengths remains the same. Essentially one has to show that

$$t_8^{abcdefgh} \left( \tilde{F}_1^{ab} F_2^{cd} F_3^{ef} F_4^{gh} + F_1^{ab} \tilde{F}_2^{cd} F_3^{ef} F_4^{gh} + F_1^{ab} F_2^{cd} \tilde{F}_3^{ef} F_4^{gh} + F_1^{ab} F_2^{cd} F_3^{ef} \tilde{F}_4^{gh} \right),$$  \hspace{1cm} (4.2.37)

where the subscripts 1, 2, 3, 4 indicate the derivative structure, vanishes. Using again 4.2.36 and the symmetry of $t_8$ one establishes that 4.2.37 vanishes independently of the precise way the derivatives are contracted.

This gives the desired result: E-M selfduality survives, up to this order in $\alpha'$, the addition of derivative corrections.

**Selfduality and Higher Orders Terms**

It would be of interest to use electromagnetic selfduality to constrain, or to determine, the derivative corrections to the Born-Infeld action that are not known explicitly. However, it is well-known that already the Born-Infeld action itself is not the only selfdual deformation of the Maxwell action, the ambiguity can be parametrized by a real function of one variable. From what precedes it is clear that $L_{(m,2m-4)}$ is not the only $p = 4$ action with derivative corrections that satisfies 4.2.23 to order $\alpha'^4$. Indeed,
we found that the result depends only on the presence of the tensor $t_8$ and on the fact that there are no contractions between derivatives and field strengths. The result is independent of the precise way the derivatives are placed.

Given these ambiguities, it is clear that E-M selfduality can only constrain but not determine the derivative corrections to the terms related to the six-point function, $p = 6$. For the four-derivative terms $n = 4$ we do have the result of [57]. The method used above is however not applicable, because the property of having no contractions between field strengths and derivatives no longer holds. Nevertheless, it would be interesting to extend the analysis of selfduality to those terms.

**Selfduality and Field Redefinitions**

It could happen that some of the Lagrangians in question contain terms which are proportional to the lowest order equations of motion $\partial_a G^{ab}_0$; for instance, the Lagrangian 4.2.32 is equivalent modulo lowest order equations of motion to the terms found in [56], for $m = 4$ and $p = 4$, written in a different basis. Now, we argue that those terms, found in [56], satisfy 4.2.23 modulo terms proportional to the lowest orders equations of motion. For the sake of simplicity, we shall prove this statement for a more general case:

Given an action $S$ written as

$$S = S_0 + S_1, \quad S_1 = \int d^4x V_b[F, x] \partial_a G^{ab}_0.$$

The equations of motion derived from 4.2.38 contain

$$G^{ab}(x) = G^{ab}_0(x) - \int d^4y \left( \frac{\delta V_d(y)}{\delta F_{ab}(x)} \partial_c G^{cd}_0(y) - \partial_c V_d(y) \frac{\delta G^{cd}_0(y)}{\delta F_{ab}(x)} \right).$$

Using the fact that $S_0$ is selfdual, the only remaining terms in the selfduality condition are

$$0 = \int d^4x d^4y \left( \delta G^{ab}_0(x) \frac{\delta V_d(y)}{\delta F_{ab}(x)} \partial_c G^{cd}_0(y) - \partial_c V_d(y) \frac{\delta G^{cd}_0(y)}{\delta F_{ab}(x)} \delta G^{ab}_0(x) \right).$$

By virtue of the identity $\frac{\delta G^{ab}_0(y)}{\delta F_{ab}(x)} = \frac{\delta G^{ab}_0(x)}{\delta F_{ab}(y)}$ and again selfduality of $S_0$, the second term in 4.2.40 can be expressed in terms of a combination involving $\partial_\mu F^{ab}$ which vanishes due to the Bianchi identity. The remaining term is proportional to $\partial_\mu G^{cd}_0(y)$ which must disappear as a result of a field redefinition on the vector potential. This implies that we should allow 4.2.23 to hold up to terms containing $\partial_\mu G^{ab}_0$, the equation of motion of $S_0$.

So, if we have an action $S_0$ satisfying the condition of selfduality, then of course any action related to that action by a field redefinition should also be considered to be electromagnetically selfdual.
4.3 Selfduality of Nonlinear Models Coupling to Matter Fields

A natural step forward is to couple the model of the previous section, being generalized to \( n \) abelian gauge fields, to matter fields, e.g. scalars, fermions and forms \([117]\). For these models the consistency of selfduality requires, besides the invariance of the field equations 4.2.2, the covariance of the matter fields equations of motion under selfduality transformations. Therefore the selfduality condition 4.2.23 has to be amended to include the effect of the matter fields couplings.

For the sake of generality, we denote the action of such a model by\(^6\)

\[
S[F^i, \Phi^\alpha] = \int d^4x \mathcal{L}(F^i, \partial_\mu F^i, \Phi^\alpha, \partial_\mu \Phi^\alpha),
\]

where \( i \) and \( \alpha \) label the number of gauge fields and matter fields, respectively. Then we consider a linear infinitesimal transformation of the form

\[
\begin{align*}
G'_{ab}[F, \Phi] &= (A + 1)G_{ab}[F, \Phi] + B\tilde{F}_{ab}, \\
\tilde{F}_{ab}' &= CG_{ab}[F, \Phi] + (1 + D)\tilde{F}_{ab}, \\
\Phi'^\alpha &= \Phi^\alpha + \zeta^\alpha[\Phi],
\end{align*}
\]

where \( A, B, C \) and \( D \) are arbitrary real \( n \times n \) matrices, and \( \zeta^\alpha \) some unspecified functions of the matter fields.

The derivation of a selfduality consistency condition for those models is straightforward, and it is to some extent similar to the derivation which has been done in section 4.2.3. Using the fact that 4.3.1 satisfies \( S'[F, \Phi] = S[F, \Phi] \), the transformed dual tensor \( G' \), arising in the equations of motion of the gauge fields, is expressed as

\[
G'_{ab}[F', \Phi', x] = -\frac{\delta}{\delta F'^{ab}(x)} S'[F', \Phi']
\]

\[
= -\frac{\delta}{\delta F^{ab}(x)} S[F, \Phi] - \frac{\delta}{\delta F^{ab}(x)} \delta S[F, \Phi],
\]

where \( \delta S[F, \Phi] = S[F', \Phi'] - S[F, \Phi] \). By means of definitions 4.3.2b and 4.3.2c, one can express 4.3.3 solely in terms of the original fields. The variation \( \delta G \) then reads

\[
\delta G_{ab}[F, \Phi, x] = \int d^4y C^{jk} G_{cd}(y) \delta G_{jcd}(y) + D^{ji} G_{ab}(F, \Phi, x) - \frac{\delta}{\delta F^{ab}(x)} \delta S[F, \Phi].
\]

\(^6\)Cases, where there are no derivatives for the field strength, have been thoroughly worked out in \([104, 106, 107, 112]\).
4.3 Selfduality of Nonlinear Models Coupling to Matter Fields

Now, Eq. 4.3.4 must coincide with the variation of $G$ following from 4.3.2. One therefore obtains the following constraints on the parameters of the transformation

\[ D^{ij} + A^{ji} = \epsilon \delta^{ij}, \quad B^{ij} = B^{ji}, \quad C^{ij} = C^{ji}, \quad (4.3.5) \]

from which it follows that

\[ \frac{\delta}{\delta F^i_{ab}(x)} \left[ \delta S + \epsilon S + \frac{1}{4} \int d^4y \left( B^{kj} \tilde{F}_{cd}^k(y) F^j_{cd}(y) - C^{jk} G^j_{cd}(y) \tilde{G}^k_{cd}(y) \right) \right] = 0. \quad (4.3.6) \]

This is the condition that $S$ should satisfy in order that the equations of motion of the gauge fields, combined with the Bianchi identity, are invariant under 4.3.2.

It is clear that the parameter $\epsilon$ might still be determined. We have not actually used up all the selfduality requirements. One still has the demand that for selfdual models the matter field equations of motion should transform covariantly under 4.3.2.

Given the matter equations of motion

\[ \Sigma^\alpha[F, \Phi, x] = \frac{\delta}{\delta \Phi^\alpha(x)} S[F, \Phi] = 0, \quad (4.3.7) \]

the transformed equations read

\[ \Sigma'^\alpha[F', \Phi', x] = \frac{\delta}{\delta \Phi^\alpha(x)} S[F, \Phi] + \int d^4y \frac{\delta S}{\delta \Phi^\beta(y)} \frac{\delta \Phi^\beta(y)}{\delta \Phi^\alpha(x)} + \int d^4y \frac{\delta S}{\delta F^i_{ab}(y)} \frac{\delta F^i_{ab}(y)}{\delta \Phi^\alpha(x)}. \quad (4.3.8) \]

Again, one might make use of 4.3.2 so as to obtain the variation $\delta \Sigma$ in terms of the original fields

\[ \delta \Sigma^\alpha[F, \Phi, x] = \frac{\delta}{\delta \Phi^\alpha(x)} \left[ \delta S + \frac{1}{4} \int C^{ij} d^4y \tilde{G}^i_{ab}(y) G^j_{cd}(y) \right] - \frac{\partial}{\partial \Phi^\alpha} \Sigma^\beta[F, \Phi]. \quad (4.3.9) \]

The requirement that 4.3.7 is covariant under 4.3.2 results in

\[ \frac{\delta}{\delta \Phi^\alpha(x)} \left[ \delta S + \frac{1}{4} \int C^{ij} d^4y \tilde{G}^i_{ab}(y) G^j_{cd}(y) \right] = 0. \quad (4.3.10) \]

The two relations 4.3.6 and 4.3.10 are compatible with one another provided that $\epsilon = 0$, and therefore for a selfdual model the action should vary in a specific way under 4.3.2

\[ \delta S[F, \Phi] = \frac{1}{4} \int d^4y \left( -B^{kj} \tilde{F}_{cd}^k(y) F^j_{cd}(y) + C^{jk} G^j_{cd}(y) \tilde{G}^k_{cd}(y) \right). \quad (4.3.11) \]

Notice that in the absence of $\partial F$, the functional form of expression 4.3.6 breaks to the normal form.
Moreover, one can easily realize that
\[
\delta \left( S - \frac{1}{4} \int d^4 y \text{tr} F^i(y) G^i(y) \right) = 0.
\] (4.3.12)

Finally, the selfduality consistency equation is summarized as
\[
\delta_\Phi S[F, \Phi] = \Sigma^\alpha [F, \Phi] \delta \Phi^\alpha = -\frac{1}{4} \int d^4 y \left( B_{ij} F^i_{ab}(y) \tilde{F}^j_{ab}(y) + C_{ij} G^i_{ab}(y) \tilde{G}^j_{ab}(y) + 2D_{ij} G^i_{ab}(y) F^j_{ab}(y) \right).
\] (4.3.13)

The combination of 4.3.5 and the condition \( \epsilon = 0 \) lead to the fact that the selfduality transformations 4.3.2 are described by the real non-compact group \( \text{Sp}(2n, \mathbb{R}) \); non-compact real form or slice of the complex \( \text{Sp}(2n) \) in a real basis of the \( 2n \)-dimensional representation.\(^8\) The group \( \text{Sp}(2n, \mathbb{R}) \) is the maximal group of duality transformations, although in specific models the group of selfduality transformations \( G \), leaving the field equations invariant, may be actually smaller. It should be pointed out that \( \text{Sp}(2n, \mathbb{R}) \) or its subgroup \( G \) may appear as the group of duality symmetries if the set of matter fields \( \Phi^\alpha \) include scalar fields parameterizing the coset space \( G/H \), with \( H \) is the maximal compact subgroup of \( U(n) \) (see chapter 5 for more detailed discussion). Any selfdual model without matter fields, \( \mathcal{L}(F) \), can be viewed as a selfdual model \( \mathcal{L}(F, \Phi, \partial \Phi) \) with the matter fields frozen, \( \Phi^\alpha(x) = \Phi^\alpha_0 \in G/H \). The duality transformation preserving this background has thus to be a subgroup of \( U(n) \), the maximal compact subgroup of \( \text{Sp}(2n, \mathbb{R}) \). Strictly speaking, in the absence of matter fields the \( \text{Sp}(2n, \mathbb{R}) \) breaks down to its maximal subgroup \( U(n) \).\(^9\) If one treats the matter fields \( \Phi^\alpha \) as coupling constants, then non-compact duality transformations relate models with different coupling constants. We stress in the end that the formalism that has been developed may be applied directly to cases in which the fields \( F \) and \( \Phi \) interact with an external gravitational field, described by \( g_{ab} \) or by a vierbein; that is actually because those fields are inert under the action of duality symmetries.

It is worth mentioning that the symplectic group \( \text{Sp}(2n, \mathbb{R}) \) preserves an antisymmetric bilinear form \( \eta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) such that
\[
L \begin{pmatrix} \tilde{K} \\ F \end{pmatrix} = \begin{pmatrix} 0 & -I_{n \times n} \\ I_{n \times n} & 0 \end{pmatrix} \begin{pmatrix} G \\ E \end{pmatrix} \text{ is invariant.}
\] (4.3.14)

Therefore the only duality invariant which can be constructed from vectors in the fundamental \( 2n \)-dimensional representation is an antisymmetric bilinear 4.3.14.

\(^8\) The symplectic group \( \text{Sp}(2n, \mathbb{R}) \) is isomorphic to \( \text{SL}(2, \mathbb{R}) \) for \( n = 1 \).

\(^9\)For the maximal compact subgroup \( U(n) \), the relations 4.3.5 are reduced to \( D = A, \ C = -B, \ A^T = -A, \ B^T = B \).
4.3 Selfduality of Nonlinear Models Coupling to Matter Fields

4.3.1 Coupling to Axion and Dilaton

As a well-known particular example [104, 105, 107], we consider a model $\mathcal{L}(F, \chi, \partial \chi)$ in which one gauge field ($n = 1$) is coupling to a complex scalar field $\chi = \chi_1 + i\chi_2$ (i.e. $\Phi^\alpha = (\chi, \bar{\chi})$), with $\bar{\chi}$ is the complex conjugate of $\chi$. Consequently, the compact duality symmetry SO(2) (for pure electrodynamics) should be enhanced to a duality symmetry under a larger non-compact group SL(2, $\mathbb{C}$) whose finite realization on the fields reads

$$
\left( \frac{G'}{F'} \right) = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \left( \frac{G}{F} \right), \quad \chi' = \frac{a\chi + b}{c\chi + d}.
$$

(4.3.15)

The corresponding infinitesimal transformations 4.3.2 become

$$
G_{ab}' = (A + 1)G_{ab} + B\tilde{F}_{ab},
$$

(4.3.16a)

$$
\tilde{F}_{ab}' = CG_{ab} + (1 - A)\tilde{F}_{ab},
$$

(4.3.16b)

$$
\delta \chi = B + 2A\chi - C\chi^2,
$$

(4.3.16c)

where the scalar field $\chi$ transforms nonlinearly in consistency with the shift invariance. The only manifest SL(2, $\mathbb{R}$) invariant term of $\mathcal{L}(F, \chi, \partial \chi)$ is the kinetic term of the scalar field

$$
\mathcal{L}_S(\chi, \partial \chi) = \frac{\partial \chi \partial \bar{\chi}}{(\chi - \bar{\chi})^2}.
$$

(4.3.17)

Accordingly, we assume that the total Lagrangian decomposes into two parts

$$
\mathcal{L}(F, \chi, \partial \chi) = \mathcal{L}_S(\chi, \partial \chi) + \hat{\mathcal{L}}(F, \chi).
$$

(4.3.18)

Thus the problem of finding the most general Lagrangian $\mathcal{L}$, whose equations of motion are invariant under SL(2, $\mathbb{R}$), boils down to a problem of solving equation 4.3.13 for $\mathcal{L}$. For this case the consistency condition 4.3.13 becomes

$$
\frac{1}{4} B \text{tr} F \tilde{F} + \frac{1}{4} C \text{tr} G \tilde{G} - \frac{1}{2} A \text{tr} G F = \frac{\partial \hat{\mathcal{L}}}{\partial \chi} \delta \chi + \frac{\partial \hat{\mathcal{L}}}{\partial \bar{\chi}} \bar{\delta} \chi.
$$

(4.3.19)

Now, substitute 4.3.16c in 4.3.19 and then make use of the fact that $A, B$ and $C$ are arbitrary parameters, the selfduality equation 4.3.19 leads to the following three equations

$$
\frac{\partial \hat{\mathcal{L}}}{\partial \chi} + \frac{\partial \hat{\mathcal{L}}}{\partial \bar{\chi}} = \frac{1}{4} \text{tr} F \tilde{F},
$$

(4.3.20)

$$
\chi \frac{\partial \hat{\mathcal{L}}}{\partial \chi} + \bar{\chi} \frac{\partial \hat{\mathcal{L}}}{\partial \bar{\chi}} = \frac{1}{4} \text{tr} G F,
$$

(4.3.21)

$$
\chi^2 \frac{\partial \hat{\mathcal{L}}}{\partial \chi} + \bar{\chi} \frac{\partial \hat{\mathcal{L}}}{\partial \bar{\chi}} = -\frac{1}{4} \text{tr} G \tilde{G}.
$$

(4.3.22)
It has been shown in [107, 108] that the solution of these equations is
\[ \hat{\mathcal{L}}(F, \chi) = \mathcal{L}_0(\sqrt{\chi_2} F) - \frac{1}{4} \chi_1 \text{tr} F \tilde{F}, \]
(4.3.23)
where \( \mathcal{L}_0(\sqrt{\chi_2} F) \) solves the selfduality equation 4.2.7 of purely (non)-linear electrodynamics models. In other words, if we redefine \( F \) by \( F = \sqrt{\chi_2} F \), one then sees that \( F \) and its dual \( G = -\partial \mathcal{L} / \partial F \) are scale invariant and have the very simple transformation law
\[ \delta \tilde{F} = -\chi_2 C G, \quad \delta G = \chi_2 C \tilde{F}, \]
(4.3.24)
i.e., they transform according to the SO(2) transformation whose infinitesimal parameter \( \lambda \) redefined as \( \lambda = \chi_2 C \).

If we replace \( \chi_1 \) and \( \chi_2 \) respectively by axion and dilaton, namely \( \chi = a + i e^{-\varphi} \), then the most general selfdual Lagrangian \( \mathcal{L} \) is schematically written as
\[ \mathcal{L}(F, a, \varphi, \partial a, \partial \varphi) \sim -\frac{1}{2} (\partial \varphi)^2 - \frac{1}{2} e^{2\varphi} (\partial a)^2 - \text{tr} F \tilde{F} + \mathcal{L}_0(e^{-\frac{1}{2} \varphi} F). \]
(4.3.25)
But wait there is more! in [107] it has been established that there is a unique generalization of the Born-Infeld theory, i.e. \( \mathcal{L}_0 = \mathcal{L}_{BI} \), admitting SL(2, \( \mathbb{R} \)) invariant equations of motion. The reader should not be confused with an analogous effective Lagrangian [108] stemming from closed superstring theory, which also contains the Born-Infeld sector. For such a Lagrangian the SL(2, \( \mathbb{R} \)) selfduality is completely lost. However it continues to hold once one truncates the Born-Infeld sector to its low-energy limit, i.e. keeping only the quadratic terms in \( F \).

### 4.3.2 Coupling to Type IIB Supergravity Backgrounds

As seen in chapter 2 one of the nice features of type IIB string theory is that it contains a four-dimensional effective gauge field theory living on the D3-brane, which actually motivates the study of selfduality. Indeed, in [35, 118] it has been proven that the worldvolume theory of the D3-brane admits SL(2, \( \mathbb{R} \)) as a duality group symmetry. It has actually been widely believed that selfduality of D3-brane is inherited from the SL(2, \( \mathbb{R} \)) symmetry of type IIB supergravity. Alternatively, we are going to invoke the machinery that has been developed in this chapter in order to establish the selfduality of D3-brane worldvolume theory [34, 117, 119].

We know from section 2.5.2 that the D3-brane worldvolume action is divided into two pieces, namely the Dirac-Born Infeld sector and the Wess-Zumino sector. In type IIB on-shell supergravity backgrounds the action is
\[ \mathcal{L} = \mathcal{L}_{DBI} + \mathcal{L}_{WZ}. \]
(4.3.26)
The two pieces of 4.3.26 are written in terms of component fields as
\[ L_{DBI} = -\sqrt{-g} \sqrt{1 + \frac{e^{-\varphi}}{2} F_{ab} F^{ab} - \frac{e^{-2\varphi}}{16} (F_{ab} \tilde{F}^{ab})^2}, \]
\[ L_{WZ} = e^{abcd} \left( \frac{1}{24} C_{abcd} + \frac{1}{4} C_{ab} F_{cd} + \frac{1}{8} e^{ab} F_{cd} \right), \]
where \( g = \det g_{ab} \), and \( \Phi^\alpha = (a, \varphi, B, C_2, C_4) \) are the possible bosonic background fields of type IIB with \( F_{ab} = F_{ab} - B_{ab} \).

Now let us see whether the selfduality condition 4.3.13 is satisfied for the above action. First of all, the dual tensor \( G \) following from 4.3.26 can be derived as
\[ G^{ab} = -\frac{L_{DBI}}{\partial F_{ab}} - 2\tilde{C}_2^{ab} - a\tilde{F}^{ab}, \]
\[ \tilde{G}^{ab} = \left( \frac{L_{DBI}}{\partial F} \right)^{ab} + C_4^{ab} + aF^{ab}. \]

The \( SL(2, \mathbb{R}) \) infinitesimal transformations of various fields in our theory are given by 4.3.16a, 4.3.16b and
\[ \delta a = 2Aa - Ca^2 + B + Ce^{-2\varphi}; \quad \delta \varphi = -2A + 2aC, \]
\[ \delta C_2^{ab} = AC_2^{ab} + BB_2^{ab}; \quad \delta B^{ab} = CC_2^{ab} - AB^{ab}, \]
\[ \delta C_4^{abcd} = \frac{B}{2} B^{ab} B^{cd} + \frac{C}{2} C_2^{ab} C_2^{cd}. \]

The argument that the transformation of \( C_4 \) provides a nonlinear representation of \( SL(2, \mathbb{R}) \), is traced back to the consistency that one should maintain between the duality transformations and the standard gauge transformations associated with forms \( B, C_2 \) and \( C_4 \).

As a next step we evaluate the variation of the Lagrangian \( L \) w.r.t its argument \( \Phi \) in the left-hand side of 4.3.13
\[ \delta \phi \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \varphi} \delta \varphi + \frac{\partial \mathcal{L}}{\partial a} \delta a + \frac{1}{2} \frac{\partial \mathcal{L}}{\partial B^{ab}} \delta B^{ab} + \frac{1}{2} \frac{\partial \mathcal{L}}{\partial C_2^{ab}} \delta C_2^{ab} + \frac{1}{4} \frac{\partial \mathcal{L}}{\partial C_4^{abcd}} \delta C_4^{abcd}, \]
then substitute 4.3.31a-4.3.31c into 4.3.32 and the resulting expression into 4.3.13. The demand that the equation 4.3.13 must hold for arbitrary values of \( A, B \) and \( C \) leads to the fact that all the coefficients of those parameters should vanish identically. The vanishing of those coefficients results in the following three equations
\[ -\frac{1}{2} G_{ab} F^{ab} - 2a \frac{\partial \mathcal{L}}{\partial \varphi} + 2a \frac{\partial \mathcal{L}}{\partial a} - \frac{1}{2} \frac{\partial \mathcal{L}}{\partial B^{ab}} B^{ab} + \frac{1}{2} \frac{\partial \mathcal{L}}{\partial C_2^{ab}} C_2^{ab} = 0, \]
\[ \frac{1}{4} F_{ab} \tilde{F}^{ab} + \frac{\partial L}{\partial a} + \frac{1}{2} \frac{\partial L}{\partial C_{ab}^c} B^{ab} + \frac{1}{8} \epsilon_{abcd} B^{ab} B^{cd} = 0, \]

(4.3.33b)

\[ \frac{1}{4} G_{ab} \tilde{G}^{ab} + 2a \frac{\partial L}{\partial \phi} + \frac{\partial L}{\partial a} \left( e^{-2\phi} - a^2 \right) + \frac{1}{2} \frac{\partial L}{\partial B^{ab} C_{ab}^c} C^{ab} + \frac{1}{8} \epsilon_{abcd} C_{ab}^c C_{cd}^c = 0. \]

(4.3.33c)

The fact that the \( \mathcal{L}_{DBI} \) depends upon the dilaton field \( \phi \) in the form of \( e^{-2\phi} F \) implies

\[ \frac{\partial \mathcal{L}_{DBI}}{\partial \phi} = -\frac{1}{4} \frac{\partial \mathcal{L}_{DBI}}{\partial F_{ab}} F_{ab}, \]

(4.3.34)

which comes in handy for verifying 4.3.33a. Notice that 4.3.33b is almost trivially satisfied.

The most intricate equation is 4.3.33c. After somewhat tedious but straightforward calculations it can be reduced to the following equation

\[ \epsilon^{abcd} \left( \frac{\partial \mathcal{L}_{DBI}}{\partial F_{ab}} \frac{\partial \mathcal{L}_{DBI}}{\partial F_{ab}} + e^{-2\phi} F_{ab} F_{cd} \right) = 0. \]

(4.3.35)

Plugging the explicit definition 4.3.27 of DBI Lagrangian into 4.3.35, it is easily shown that 4.3.33c is satisfied. We have thereby demonstrated that the D3-brane action 4.3.26 in type IIB supergravity backgrounds indeed satisfies the selfduality condition 4.3.13. Therefore the D3-brane worldvolume theory is selfdual.

The generalization of this analysis to the super D3-brane is somehow feasible and has been worked out in [118, 120, 121]. It is worthwhile also to note that in [120, 121] there have been achieved more than that. They have basically succeeded in uplifting the selfduality transformations from a symmetry of the field equations to a symmetry of the action: there has been argued that an off-shell duality transformation might be realized through the gauge potential \( A_a \), and by imposing the Gaillard-Zumino condition, namely selfduality condition 4.3.13, one can easily obtain the invariance of the action up to surface terms. This method can be regarded as an alternative approach, in some specific cases, to the PST formalism formerly mentioned.

### 4.4 Gaillard-Zumino Model: Selfduality in Supergravity

One might wonder whether the analysis of selfduality that we have developed so far may be extended to higher dimensions. We should admit that working in four dimensions makes life easy, however, there is no obstruction to go higher in dimensions, specifically even dimensions \( d = 2p \), and hence higher in the field strength rank [106, 122, 123]. Such a generalization of the Gaillard-Zumino model comes in handy when one wants to turn on duality in supergravity.

We consider theories of \( n (p - 1) \)-th rank antisymmetric tensor fields \( A^i_{a_1 \cdots a_{p-1}}(x) \)
(i = 1, · · · , n) interacting with matter fields \( \Phi^\alpha(x) \). The field strengths and its Hodge duals are defined by

\[
F_{a_1 \cdots a_p} = p \partial_{[a_1} A_{a_2 \cdots a_{p-1}]}^i \tag{4.4.1}
\]

\[
\tilde{F}^{i a_1 \cdots a_p} = \frac{1}{p!} \epsilon^{a_1 \cdots a_p b_1 \cdots b_p} F_{b_1 \cdots b_p}^i. \tag{4.4.2}
\]

The Hodge duality operation has a peculiar property in \( d \) dimensions

\[
\tilde{\tilde{F}} = \epsilon F, \quad \epsilon = \begin{cases} +1 & \text{for } d = 4r + 2 \\ -1 & \text{for } d = 4r. \end{cases} \tag{4.4.3}
\]

Given a Lagrangian \( \mathcal{L}(F, \Phi, \partial \Phi) \) which governs the dynamics of interacting theories, the equations of motion and the Bianchi identity read

\[
\partial_{a_1} G^{a_1 \cdots a_p} = 0, \quad \partial_{a_1} \tilde{F}^{i a_1 \cdots a_p} = 0, \tag{4.4.4}
\]

where the dual tensors \( G^{a_1 \cdots a_p} \) generated by \( \mathcal{L} \) are defined as

\[
G^{i a_1 \cdots a_p} = -p! \frac{\partial \mathcal{L}}{\partial F_{a_1 \cdots a_p}}. \tag{4.4.5}
\]

The selfduality requirement imposed on higher dimensional theories - the simultaneous covariance of 4.4.5 and of matter fields equations of motion under 4.3.2- gives rise to the following constraints on the parameters

\[
A_{ij} = -D_{ji}, \quad B_{ij} = \epsilon B_{ji}, \quad C_{ij} = -\epsilon C_{ji}, \tag{4.4.6}
\]

which can be recast into

\[
S^T \eta + \eta S = 0, \quad \text{where} \quad S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad \eta = \begin{pmatrix} 0 & \epsilon \cdot 1_{n \times n} \\ 1_{n \times n} & 0 \end{pmatrix}. \tag{4.4.7}
\]

Here one should distinguish two cases:

- If the dimension is \( d = 4r \) (\( \epsilon = -1 \)), then \( \eta \) is an antisymmetric bilinear form and the above condition corresponds to a Sp(2n, \( \mathbb{R} \)) duality group or its subgroup.

- For \( d = 4r + 2 \) case corresponding to \( \epsilon = 1 \), \( \eta \) is symmetric bilinear form which by an appropriate change of basis can be brought into its diagonal form \( \text{diag}(1, -1) \).

Then the duality symmetry associated to this case is SO(\( n, n \)) or its subgroup.

As usual, we are not seeking the duality invariance of the Lagrangian, however the variation of the Lagrangian must take on a very definite form under the duality transformation

\[
\delta \mathcal{L} = \frac{1}{2p!} (B^{ij} \text{tr} F^i \tilde{F}^j - C^{ij} \text{tr} G^i \tilde{G}^j) = -\delta \left( \frac{1}{2p!} \text{tr} F^i \tilde{G}^j \right). \tag{4.4.8}
\]
The variation 4.4.8 of the Lagrangian $\mathcal{L}$ is rather indicative and suggestive. As a consequence one can obtain an explicit form of the Lagrangian $\mathcal{L}$ satisfying 4.4.8

$$\mathcal{L} = \frac{1}{2p!} \text{tr} F^i G_i + \frac{1}{2p!} \text{tr}(\tilde{K}^i G_i + \epsilon L^i \tilde{F}_i) + \mathcal{L}_{\text{inv}}(\Phi, \partial \Phi), \quad (4.4.9)$$

where $p$-th antisymmetric tensors $(L_i(\Phi, \partial \Phi), \tilde{K}^i(\Phi, \partial \Phi))$ transform the same way as $(G^i, \tilde{F}^i)$ see 4.3.14, and $\mathcal{L}_{\text{inv}}$ is a manifestly dual invariant (selfdual) sector\(^{10}\).

One still have the possibility to eliminate $G$ in favor of the other fields. Substituting 4.4.9 into 4.4.5 we obtain the following differential equation\(^{11}\)

$$(G - \epsilon \tilde{L})_i = (F - \tilde{K})^j \frac{\partial}{\partial F^i}(G - \epsilon \tilde{L})_j. \quad (4.4.10)$$

In order to solve this equation, one might introduce the so-called $j$-operation

$$\tilde{F} = jF, \quad \text{with } j^2 = \epsilon. \quad (4.4.11)$$

The solution of this equation has then been found to be

$$G_i = \epsilon \tilde{L}_i - M_{ij}(\Phi)(F - \tilde{K})^j, \quad (4.4.12)$$

where $M_{ij}(\Phi)$ is a posteriori an arbitrary $n \times n$ symmetric matrix function of the matter fields\(^{12}\).

The determination of the matrix $M_{ij}$ in terms of $\Phi$ rests upon two physical observations:

- The transformation law of $M$ follows from the covariance of Eq.4.4.12, therefore the matrix $M$ must transform nonlinearly as

$$\delta M = -j B + AM - MD + \epsilon j MCM. \quad (4.4.13)$$

- The form of the kinetic energy term for the vector fields requires

$$M_{ij}(\Phi) = \delta_{ij} + N_{ij}(\Phi). \quad (4.4.14)$$

\(^{10}\)We have assumed that we are dealing with the maximal duality group, i.e., $\text{Sp}(2n, \mathbb{R})$ in $d = 4r$ or $\text{SO}(n, n)$ in $d = 4r + 2$; any dual invariant quantity associated to this group should be proportional to $\tilde{K}G + \epsilon L\tilde{F}$. However, had we considered models that admit subgroups of them as duality symmetries, there have been other invariants than $\tilde{K}G + \epsilon L\tilde{F}$.

\(^{11}\)For simplicity, we drop spacetime indices in equation 4.4.10.

\(^{12}\)Note that the matrix $M_{ij}$, restricted only to scalar fields, is in some sense the scalar matrix which will parameterize the target space of the 4D nonlinear sigma-model in chapter 5.
Thus, if we can find out functions $K^i(\Phi, \partial \Phi)$, $L^i(\Phi, \partial \Phi)$ and $M_{ij}(\Phi)$ with appropriate transformations properties, we have then an explicit form of the Lagrangian

$$
\mathcal{L} = -\frac{1}{2p!} \text{tr} F^i M_{ij} F^j + \epsilon \frac{1}{p!} \text{tr} F^i (\tilde{L}_i - M_{ij} \tilde{K}^j) + \epsilon \frac{1}{2p!} \text{tr} K^i (L_i - M_{ij} K^j) + \mathcal{L}_{\text{inv}}(\Phi, \partial \Phi).
$$

(4.4.15)

Note that the Lagrangians of supergravities, whose equations of motion are invariant under selfduality transformations, are often of this type.

4.4.1 Special Case: Compact Selfduality

We now consider the case where $M_{ij} = \delta_{ij}$, reproducing some of the results we have obtained in section 4.3. In this special case we will see that the duality symmetry group must be demoted to a compact group in the absence of the matter fields. From 4.4.13 we see that the parameters of the transformations have to obey

$$
A_{ij} = D_{ij}, \quad B = \epsilon C.
$$

(4.4.16)

Again, two cases must be distinguished. For $d = 4r$, the condition 4.4.16 implies

$$
S = \begin{pmatrix}
A & B \\
-B & A
\end{pmatrix}, \quad A_{ij} = -A_{ji}, \quad B_{ij} = B_{ji}.
$$

(4.4.17)

In an appropriate complex basis the transformation law becomes

$$
\delta \begin{pmatrix}
G + i\tilde{F} \\
G + i\tilde{F}
\end{pmatrix} = \begin{pmatrix}
\overline{U} & 0 \\
0 & U
\end{pmatrix} \begin{pmatrix}
G + i\tilde{F} \\
G + i\tilde{F}
\end{pmatrix}, \quad \text{with } U = A + iB.
$$

(4.4.18)

Since $U$ is anti-hermitian, i.e. $U = -U^\dagger$, the duality symmetry group is $U(n)$, the maximal compact subgroup of $\text{Sp}(2n, \mathbb{R})$. For $d = 4k + 2$ the constraints 4.4.16 becomes

$$
S = \begin{pmatrix}
A & B \\
B & A
\end{pmatrix}, \quad A_{ij} = -A_{ji}, \quad B_{ij} = -B_{ji}.
$$

(4.4.19)

and

$$
\delta \begin{pmatrix}
G + \tilde{F} \\
G + \tilde{F}
\end{pmatrix} = \begin{pmatrix}
U_+ & 0 \\
0 & U_-
\end{pmatrix} \begin{pmatrix}
G + \tilde{F} \\
G + \tilde{F}
\end{pmatrix}.
$$

(4.4.20)

with $U_+ = A + B$, and $U_- = A - B$ are real and antisymmetric matrices. The duality symmetry group is hence $\text{SO}(n) \times \text{SO}(n)$, the maximal compact subgroup of $\text{SO}(n, n)$.

For a non-compact duality group, one may construct a $M_{ij}(\Phi)$, with $\Phi$ confined to scalar fields, by resorting to the well-known description of nonlinear sigma models for scalars valued in the coset spaces $G/H$ of a group by a subgroup. Then $G$ is a non-compact semisimple duality group but the subgroup $H$ is its maximal compact subgroup. This nonlinear realization of $G$ and others will be discussed in chapter 5.
4.5 Charges and Selfdual Invariant Quantities

4.5.1 Charges of Duality Symmetry

The duality symmetries that have been examined in this chapter are global symmetries, i.e. $A$, $B$ and $C$ are constant parameters. Although we are dealing with 4-dimensional models whose actions are not manifestly invariant under selfduality 4.3.2, one can still partially perform the usual Noether’s argument and find the conservation laws. It is known from Noether’s procedures that the conserved currents should be constructed in terms of the basic fields of the theory. But we know from section 4.2 that the action of the duality transformation on the gauge field is through the field strength $F$, therefore the realization of duality symmetry on the basic field $A$ is nonlocal as $F = dA$. The way out of this problem, namely locality violation, is to construct the current so that the basic fields $A$ show up on equal footing with their duals $\hat{A}$ in the final expression.

By contrast, the covariance of matter fields equations of motion under selfduality is locally realized and via the basic fields $\Phi$’s. Therefore all one requires so as to apply consistently the Noether’s theorem is to extract $\delta \Phi$ out of $\delta L$. Then we get

$$
\delta \Phi L(F^i, \Phi^\alpha, \partial \phi^\alpha) = -\frac{1}{4} \left( B^{ij} F_{ab} \hat{F}^{j \ d} + C^{ij} G_{ab} \hat{G}^{j \ ab} + 2D^{ij} G_{ab} F^{j \ ab} \right). \tag{4.5.1}
$$

Invoking field equations 4.2.2, we are able to re-express 4.5.1 as a divergence of a quantity $J^a$

$$
\delta \Phi L = -\partial_a J^a, \tag{4.5.2}
$$

where

$$
J^a_1 = \frac{1}{2} \left( B^{ij} F_i \hat{A}^j + C^{ij} G_i \hat{A}^j + D^{ij} G_i \hat{A}^j - A^{ij} F_i \hat{A}^j \right) \tag{4.5.3}
$$

with $G^i = \partial_a \hat{A}^i_0 - \partial_b \hat{A}^i_0$ follows from the equations of motion $\partial_a G^{ab} = 0$.

Now the standard argument due to Emmy Noether, applying for $\delta \Phi L$, comes into play

$$
\delta \Phi L = \partial_a \left( \frac{\partial L}{\partial (\partial_a \Phi^\alpha)} \delta \Phi^\alpha \right) = \partial_a J^a_2, \tag{4.5.4}
$$

where we have made use of the equations of motion of the matter fields. Then one can easily infer that $J^a_2 = \zeta^\alpha \frac{\partial L}{\partial (\partial_a \Phi^\alpha)}$, with $\zeta^\alpha$ are the infinitesimal parameters defined in 4.3.2.

Equation 4.5.2 together with 4.5.4, using all equations of motion, lead to the result that selfduality symmetries imply the existence of conserved currents

$$
J^a = J^a_1 + J^a_2, \quad \text{with} \quad \partial_a J^a = 0. \tag{4.5.5}
$$
4.5 Charges and Selfdual Invariant Quantities

Notice that the current 4.5.5 is not invariant under the gauge transformation associated with the gauge fields

\[ \mathcal{A}_a^i \to \mathcal{A}_a^i + \partial_a \kappa^i, \quad \tilde{\mathcal{A}}_a^i \to \tilde{\mathcal{A}}_a^i + \partial_a \pi^i, \quad (4.5.6) \]

it still changes by the divergence of antisymmetric tensor

\[ J^a \to J^a + \frac{1}{2} \partial_b (B^{ij} F^{i a b} \kappa^j + C^{ij} G^{i a b} \pi^j + D^{ij} G^{i a b} \kappa^j - A^{ij} F^{i a b} \pi^j). \quad (4.5.7) \]

Therefore the corresponding charge \( Q = \int J^0 d^3 x \) is gauge invariant and turns out actually to be the generator\(^{13}\) of the selfduality symmetry. It is worth pointing out that the charges of global symmetries in general and duality symmetries in particular will play an essential role in classifying solutions of (super)gravity theories in chapter 6.

4.5.2 Selfdual Invariant Quantities

Although the Lagrangian is not invariant under the duality transformation, a suitably defined derivative of the Lagrangian with respect to an invariant parameter is invariant. Assume that \( L \) depends upon an invariant parameter \( \omega \). If \( \zeta^\alpha (\Phi) \) is independent of \( \omega \), we differentiate 4.3.11 with respect to \( \omega \), then obtain

\[ \frac{\partial}{\partial \omega} \delta L = \frac{1}{2} \frac{\partial G_{i a b}^i}{\partial \omega} C^{ij} \tilde{G}^{j a b}. \quad (4.5.8) \]

On the other hand, the derivative of

\[ \delta L = \delta_\Phi L + \frac{1}{2} \delta F_{a b} \frac{\partial L}{\partial F_{a b}} \quad (4.5.9) \]

in terms of \( \omega \) yields

\[ \frac{\partial}{\partial \omega} \delta L = \delta \left( \frac{\partial L}{\partial \omega} \right) + \frac{1}{2} G_{a b}^i C^{ij} \frac{\partial G_{i a b}^i}{\partial \omega}. \quad (4.5.10) \]

Comparing 4.5.8 and 4.5.10, we find

\[ \delta \frac{\partial L}{\partial \omega} = 0. \quad (4.5.11) \]

The result 4.5.11 provides a way of checking that a theory admits selfduality or of constructing the Lagrangian for such a theory, by switching on couplings in an invariant way. The case when \( \zeta^\alpha \) depend on \( \omega \) is a little more delicate (see [106]).

As an example, if \( \omega \) represents an external gravitational field, 4.5.11 implies that the energy-momentum tensor, which is the variational derivative of the Lagrangian w.r.t to the gravitational field, is invariant under selfduality transformations.

\(^{13}\)By using Coulomb-like gauge and developing the appropriate canonical formalism.
4.6 Summary

The main purpose behind this chapter was first to present E-M duality symmetries as a physical requirement for constraining the higher derivative structure for a given nonlinear electromagnetic theory, in particular the Born-Infeld theory. It’s well-known that already the Born-Infeld theory is not the only selfdual extension of the Maxwell theory. As shown in [108] the ambiguity is labelled by a real function of one real argument. In this chapter we have moreover demonstrated that electromagnetic self-duality can only constrain but not determine the higher derivative corrections terms. This has been established for derivative corrections to the terms related to the 4-point function.

The rest of this chapter has been devoted to show how a general theory invariant under dual rotations may be constructed. We tried to clarify the structure of the theories admitting both compact and non-compact duality. A non-compact duality invariance is possible only when there are matter fields such as scalar fields. It has been moreover exhibited that the scalar fields must transform nonlinearly under the action of the duality group $G$. We derived the transformation property of the Lagrangian which is required for the equations of motion to be duality invariant, and show that this property implies the existence of conserved currents and the invariance of the energy-momentum tensor. We further exploited this property for explicit construction of the selfdual supergravity Lagrangians, which we illustrated by specializing to the compact case. We found that these theories still have a considerable degree of arbitrariness, namely in the choice of the matter Lagrangian $L_{\text{inv}}(\Phi, \partial \Phi)$. In supergravity theories this quantity, and in fact the field content itself, are fixed by supersymmetry. It appears that duality invariance of supergravity theories is implied by supersymmetry.

Motivated and inspired by our study of selfdual theories we will try in the next chapter to formulate Kaluza-Klein theories resp. extended supergravities as a nonlinear realization of the duality group $G$. 