The Circle Criterion and Input-to-State Stability for Infinite-Dimensional Systems*

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1 Introduction

In this paper, the focus is on absolute stability and input-to-state stability of the feedback interconnection of an infinite-dimensional linear system $\Sigma$ and a nonlinearity $\Phi : \text{dom}(\Phi) \subset L^2_{\text{loc}}(\mathbb{R}^+, \mathbb{R}^+; \mathbb{R}^+, \mathbb{R}^+)$, where $\text{dom}(\Phi)$ denotes the domain of $\Phi$ and $U$ and $Y$ (Hilbert spaces) denote the input and output spaces of $\Sigma$, respectively (see Figure 1, wherein $v$ is an essentially bounded input signal). The system $\Sigma$ is assumed to belong to the rather general class of well-posed systems (see, for example, [11, 13] and the references therein) and the nonlinearity is assumed to satisfy a (generalized) sector condition.

In the literature on the circle criterion for infinite-dimensional systems (see, for example, [3, 4, 5, 7, 9, 12], and the references therein), the emphasis is usually on $L^2$- or $L^1$-stability and global asymptotic or global exponential stability (or some variants thereof) of feedback systems of the type shown in Figure 1, with a static sector-bounded nonlinearity $\Phi$ in the feedback path. The new contribution of this paper as compared to the previous literature is twofold.

(i) In addition to static nonlinearities, we include a class of dynamic nonlinearities which may exhibit bias, but still satisfy a generalized pointwise sector condition.

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As specific subclasses, the class of nonlinearities encompasses both static nonlinearities with “negative resistance” and a wide range of hysteretic effects described by so-called Preisach operators.

(ii) The main results of the paper guarantee input-to-state-stability with “bias” (and “standard” input-to-state-stability if the nonlinearity is unbiased), thereby making contact with the important and rapidly developing input-to-state-stability theory in (finite-dimensional) nonlinear control.

As in the classical theory of absolute stability and circle criteria, the methodology involves a “symbiosis” of (generalized) sector data relating to the nonlinearity \( \Phi \) and properties of the transfer function of the linear system \( \Sigma \) to conclude stability properties of the feedback interconnection. We mention that the viewpoint of this paper is similar in spirit to that of [1]: however, the class of feedback systems considered here is very different to that in [1] as is the methodology adopted.

For sake of brevity, this paper does not contain any proofs: for these we refer to [6].

**Notation and terminology.** For \( \alpha \in \mathbb{R} \), set \( \mathbb{C}_\alpha := \{ s \in \mathbb{C} : \text{Re } s > \alpha \} \). If \( S \) is a non-empty subset of \( \mathbb{C} \), then a set \( R \subset S \) is said to be discrete in \( S \), if, for every \( s \in S \), there exists a neighbourhood \( N \) of \( s \) such that \( N \cap R \) is finite.

For Hilbert spaces \( U \) and \( Y \), let \( \mathcal{B}(U, Y) \) denote the space of all linear bounded operators mapping \( U \) to \( Y \). We write \( \mathcal{B}(U) \) for \( \mathcal{B}(U, U) \). For \( T \in \mathcal{B}(U) \), we define

\[
\text{Re} T := \frac{1}{2}(T + T^*) \in \mathcal{B}(U).
\]

The space of all holomorphic and bounded functions \( \mathbb{C}_\alpha \to \mathcal{B}(U, Y) \) is denoted by \( H_\alpha^\infty(\mathcal{B}(U, Y)) \). We write \( H^\infty(\mathcal{B}(U, Y)) \) for \( H_0^\infty(\mathcal{B}(U, Y)) \). Moreover, in the scalar case (that is \( U = Y = \mathbb{C} \)), we simply write \( H_\alpha^\infty \), or, if \( \alpha = 0 \), \( H^\infty \) for \( H_0^\infty(\mathcal{B}(U, Y)) \) and \( H^\infty(\mathcal{B}(U, Y)) \), respectively. For \( \alpha \in \mathbb{R} \), we define the exponentially weighted \( L^p \)-space \( L_\alpha^p(\mathbb{R}_+, X) := \{ f \in L^p_{\text{loc}}(\mathbb{R}_+, U) : f(\cdot) \exp(-\alpha \cdot) \in L^p(\mathbb{R}_+, U) \} \). The Laplace transform is denoted by \( \mathcal{L} \).

## 2 Well-posed linear systems with nonlinear feedback

There are a number of equivalent definitions of well-posed systems, see, for example, [11, 13] and the references therein. We will be brief in the following and refer the reader to the literature for more details. Throughout, we shall be considering a well-posed system \( \Sigma \) with state-space \( X \), input space \( U \) and output space \( Y \), generating operators \( (A, B, C) \), input-output operator \( G \) and transfer function \( \mathbf{G} \). Here \( X, U \) and \( Y \) are separable (complex) Hilbert spaces, \( A \) is the generator of a strongly
continuous semigroup $T = (T_t)_{t \geq 0}$ on $X$ and $B \in \mathcal{B}(U, X_{-1})$ and $C \in \mathcal{B}(X_1, Y)$, respectively, are admissible control and observations for $T$. The spaces $X_1$ and $X_{-1}$, respectively, are interpolation and extrapolation spaces associated with $X$: $X_1 = \text{dom}(A)$ (the domain of $A$), endowed with the graph norm of $A$, whilst $X_{-1}$ denotes the completion of $X$ with respect to the norm $\|x\|_{-1} = \|(\xi I - A)^{-1}x\|$, where $\xi \in \varrho(A)$, the resolvent set of $A$ (different choices of $\xi$ lead to equivalent norms) and $\| \cdot \|$ denotes the norm on $X$. The control operator $B$ is said to be bounded if it is so as a map from the input space $U$ to the state space $X$, otherwise is said to be unbounded; the observation operator $C$ is said to be bounded if it can be extended continuously to $X$, otherwise, $C$ is said to be unbounded.

In the following, let $\Sigma$ and (1) and refer to (1) as a well-posed system.

The so-called $\Lambda$-extension $C_\Lambda$ of $C$ is defined by

$$C_\Lambda z = \lim_{s \to \infty, s \in \mathbb{R}} Cs(sI - A)^{-1}z,$$

with $\text{dom}(C_\Lambda)$ (the domain of $C_\Lambda$) consisting of all $z \in X$ for which the above limit exists. The transfer function $G$ has the property that $G \in H^\infty(B(U, Y))$ for every $\omega > \omega(T)$, where $\omega(T)$ denotes the exponential growth constant of $T$. Moreover, the input-output operator $G : L^2_{\text{loc}}(\mathbb{R}^+, U) \to L^2_{\text{loc}}(\mathbb{R}^+, Y)$ is continuous and shift-invariant; for every $\omega > \omega(T)$, $G \in \mathcal{B}(L^2_2(\mathbb{R}^+, U), L^2_2(\mathbb{R}^+, Y))$ and

$$(\mathcal{L}(G u))(s) = G(s)(\mathcal{L}(u))(s), \quad \forall s \in \mathbb{C}_\omega, \forall u \in L^2_2(\mathbb{R}^+, U).$$

In the following, let $s_0 \in \mathbb{C}_{\omega(T)}$ be fixed, but arbitrary. For $x^0 \in X$ and $u \in L^2_{\text{loc}}(\mathbb{R}^+, U)$, let $x$ and $y$ denote the state and output functions of $\Sigma$, respectively, corresponding to the initial condition $x(0) = x^0 \in X$ and the input function $u$. Then $x(t) = T_t x^0 + \int_0^t T_{t-\tau} Bu(\tau)d\tau$ for all $t \in \mathbb{R}^+$, $x(t) - (s_0I - A)^{-1}Bu(t) \in \text{dom}(C_\Lambda)$ for a.e. $t \in \mathbb{R}^+$ and

$$\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x^0, \quad \text{a.e. } t \in \mathbb{R}^+, \\
y(t) &= C_\Lambda \left(x(t) - (s_0I - A)^{-1}Bu(t)\right) + G(s_0)u(t), \quad \text{a.e. } t \geq 0.
\end{align*}$$

Of course, the differential equation in (1) has to be interpreted in $X_{-1}$. In the following, we identify $\Sigma$ and (1) and refer to (1) as a well-posed system.

We say that (1) is exponentially stable if $\omega(T) < 0$ and we say that (1) is input-output stable if $G \in H^\infty(B(U, Y))$ or, equivalently, if $G \in \mathcal{B}(L^2(\mathbb{R}^+, U), L^2(\mathbb{R}^+, Y))$. Furthermore, (1) is said to be optimizable, if for every $x^0$, there exists $u \in L^2(\mathbb{R}^+, U)$ such that the function $t \mapsto T_t x^0 + \int_0^t T_{t-\tau} Bu(\tau)d\tau$ is in $L^2(\mathbb{R}^+, X)$. Writing $X^*_1 := (X^*)_1$, we have that $X^*_1 = (X_1)^*$ and $C^* \in \mathcal{B}(Y, X^*_1)$ is an admissible control operator for the adjoint semigroup $T^* = (T^*_t)_{t \geq 0}$. We say that (1) is estimatable if for every $x^0$, there exists $u^* \in L^2(\mathbb{R}^+, Y)$ such that the function $t \mapsto T^*_t x^0 + \int_0^t T^*_{t-\tau} C^* u^*(\tau)d\tau$ is in $L^2(\mathbb{R}^+, X)$.

In the following, we will consider the closed-loop system obtained by applying the nonlinear feedback

$$u = v - \Phi(y)$$

(2)

to the well-posed linear system (1), where $v \in L^\infty(\mathbb{R}^+, U)$ and the nonlinear operator $\Phi : \text{dom}(\Phi) \subset L^2_{\text{loc}}(\mathbb{R}^+, Y) \to L^2_{\text{loc}}(\mathbb{R}^+, U)$ is causal. To define the concept
of a (local) solution of the feedback system given by (1) and (2), we first need to show that $\Phi$ can be “localized” in the sense that it can be “extended” to spaces of functions with a finite time horizon. To this end, let $0 < \sigma \leq \infty$ be arbitrary and set

$$\text{dom}_{\sigma} (\Phi) := \{ w \in L^2_{\text{loc}}([0, \sigma], Y) : \forall \tau \in (0, \sigma) \exists w_{\tau} \in \text{dom} (\Phi) \text{ s.t. } w = w_{\tau} \text{ on } [0, \tau] \}.$$ 

Trivially, $\text{dom}_{\infty} (\Phi) = \text{dom} (\Phi)$. For $w \in \text{dom}_{\sigma} (\Phi)$ with $\sigma < \infty$, we define $\Phi (w)$ by

$$\Phi (w) (t) = (\Phi (w_{\tau})) (t), \quad 0 \leq t \leq \tau < \sigma,$$

where $w_{\tau} \in \text{dom} (\Phi)$ such that $w = w_{\tau}$ on $[0, \tau]$. By causality of $\Phi$, this definition does not depend on the choice of $\tau$ and thus $\Phi (w)$ is a well-defined element in $L^2_{\text{loc}}([0, \sigma], U)$.

A solution on $[0, \sigma]$ (where $0 < \sigma \leq \infty$) of the feedback system given by (1) and (2) is a pair $(x, y) \in C([0, \sigma], X) \times \text{dom}_{\sigma} (\Phi)$ such that, with $u$ given by (2),

$$x(t) = T_t x^0 + \int_0^t T_{t-s} B u(s) \, ds, \quad \forall t \in [0, \sigma),$$

$$y(t) = C_x (x(t) - (s_0 I - A)^{-1} B u(t)) + G (s_0) u(t), \quad \text{a.e. } t \in [0, \sigma).$$

If $\sigma = \infty$, then we say that $(x, y)$ is a global solution. Let $S$ denote the set of all $(x^0, v) \in X \times L^\infty (\mathbb{R}_+, U)$ for which the feedback system given by (1) and (2) has at least one global solution. If $(x^0, v) \in S$, then the notation $(x(\cdot; x^0, v), y(\cdot; x^0, v))$ is used to denote any global solution corresponding to the initial condition $x^0$ and the closed-loop input $v$. Furthermore, a routine argument based on Zorn’s lemma shows that every solution $(x, y)$ can be extended to a maximal solution, that is, to a maximally defined solution which cannot be extended any further. The interval on which a maximal solution is defined is called the maximal interval of existence of the solution. We say that the feedback system given by (1) and (2) has the blow-up property if for every maximal solution $(x, y)$ defined on a finite maximal interval of existence $[0, \sigma)$, the $L^2$-norm of $y$ blows up, that is, $\| y \|_{L^2 (0, \tau)} \to \infty$ as $\tau \uparrow \sigma$. In this paper, we are mainly concerned with stability properties of the feedback system given by (1) and (2): whilst of fundamental importance, the question of existence of solutions is not the main concern here; this question requires addressing on a less general basis, taking into account relevant features of the particular system or subclass of systems under consideration (see [6] for further comments in this context).

### 3 The sector condition and input-to-state stability

First, we introduce a sector condition on the class of nonlinearities (in due course, this condition will be weakened to a generalized sector condition).

**Definition 1.** A nonlinearity $\Phi : \text{dom} (\Phi) \subset L^2_{\text{loc}} (\mathbb{R}_+, Y) \to L^2_{\text{loc}} (\mathbb{R}_+, U)$ satisfies a sector condition if there exist operators $K_1, K_2 \in \mathcal{B} (Y, U)$ such that

$$\text{Re} \langle (\Phi (w) (t)) - K_1 w (t), (\Phi (w) (t)) - K_2 w (t) \rangle \leq 0, \quad \forall w \in \text{dom} (\Phi), \text{ a.e. } t \in \mathbb{R}_+.$$
Example 2 (Static nonlinearities). Let $\varphi : Y \to U$ be continuous and assume that there exist $K_1, K_2 \in \mathcal{B}(Y, U)$ such that

$$\text{Re}(\varphi(\xi) - K_1 \xi, \varphi(\xi) - K_2 \xi) \leq 0 \quad \forall \, \xi \in Y.$$  \hfill (6)

With $\varphi$ we may associate the Nemyckii operator $\Phi : L^2_{\text{loc}}(\mathbb{R}_+, Y) \to L^2_{\text{loc}}(\mathbb{R}_+, U)$, defined by $\Phi(w) := \varphi \circ w$. This operator satisfies the sector condition (5). Such operators provide a simple prototype class for the general nonlinearities considered in this section: at the simplest illustrative level, static sector-bounded scalar nonlinearities $\varphi : \mathbb{R} \to \mathbb{R}$ of the type shown in Figure 2 (ubiquitous in the literature on the classical circle criterion) are subsumed by the formulation. This observation extends mutatis mutandis to encompass time-dependent static nonlinearities $\varphi : \mathbb{R}_+ \times Y \to U$. \hfill \Diamond

Anticipating Sections 4 and 5 below, we will also consider static nonlinearities for which the inequality in (6) is assumed to hold only outside some bounded set $E \subset Y$ (see Figure 3). To accommodate these and more general nonlinearities, in Section 4 we will introduce a generalized sector condition and remark here that the generalized formulation encompasses a large class of hysteresis operators, including hysteresis of Preisach type.

Let $K_1, K_2 \in \mathcal{B}(Y, U)$ and define

$$K := \frac{1}{2}(K_1 + K_2), \quad \kappa := \|K_2 - K_1\|^2.$$ \hfill (7)

We assemble the following hypotheses on the transfer function $G$ of (1) which will be variously invoked in the theory presented below.

(H1) There exists $\alpha < 0$ and an open set $\Omega \subset C_\alpha$ such that $C_\alpha \setminus \Omega$ is discrete in
\( C_\alpha \) and \( G \) is holomorphic on \( \Omega \), the frequency-domain condition
\[
G^*(i\omega)\left[\frac{\kappa + \delta}{4} I - K^*K\right] G(i\omega) \leq I + 2 \text{Re}(K G(i\omega)), \quad \text{a.e. } \omega \in \mathbb{R}.
\] (8)
holds for some \( \delta > 0 \) and \( G(I + KG)^{-1} \in H^\infty(B(U,Y)) \).

(H2) \( G \in H^\infty(B(U,Y)) \) and there exist \( \delta > 0 \) and \( \rho < 1 \) such that (8) holds and
\[
G^*(i\omega)\left[\frac{\kappa + \delta}{4} I - K^*K\right] G(i\omega) \geq -\rho I, \quad \text{a.e. } \omega \in \mathbb{R}.
\] (9)

(H3) There exists an open set \( \Omega \subset C_0 \) such that \( C_0 \setminus \Omega \) is discrete in \( C_0 \) and \( G \) is holomorphic on \( \Omega \), \( I + KG(s) \) is invertible for all \( s \in \Omega \) and the frequency-domain condition
\[
G^*(s)\left[\frac{\kappa + \delta}{4} I - K^*K\right] G(s) \leq I + 2 \text{Re}(K G(s)), \quad \forall s \in \Omega
\] (10)
holds for some \( \delta > 0 \).

(H4) There exists an open set \( \Omega \subset C_0 \) such that \( C_0 \setminus \Omega \) is discrete in \( C_0 \) and \( G \) is holomorphic on \( \Omega \), \( KG(s) \) is compact for all \( s \in \Omega \) and the frequency-domain condition (10) holds for some \( \delta > 0 \).

Remark 3. (a) In the case of scalar “sector data”, that is \( U = Y \) and there exist \( k_1, k_2 \in \mathbb{C} \) such that \( K_1 = k_1 I \) and \( K_2 = k_2 I \), the term
\[
\frac{\kappa + \delta}{4} I - K^*K
\]
appearing on the left-hand sides of (8)-(10) simplifies to \((\delta/4 - \text{Re}(k_1 k_2))I\).

(b) Assume that one of the operators \( K_1 \) and \( K_2 \) is the zero operator and that the other is a scalar multiple of an isometry. Then it is not difficult to show that (H2) is satisfied, provided that \( G \in H^\infty(B(U,Y)) \) and the positive-real condition
\[
\varepsilon I \leq I + 2 \text{Re}(K G(i\omega)), \quad \text{a.e. } \omega \in \mathbb{R}
\]
holds for some \( \varepsilon > 0 \).

We are now in the position to state the main result of this section.

Theorem 4. Assume that (1) is optimizable and estimatable and that there exist operators \( K_1, K_2 \in B(Y,U) \) such that \( \Phi \) satisfies the sector condition (5). Let \( K \in B(Y,U) \) and \( \kappa \geq 0 \) be given by (7). If at least one of hypotheses (H1)–(H4) holds, then there exist positive constants \( \Gamma \) and \( \gamma \), such that, for each \( (x^0,v) \in S \),
\[
\|x(t; x^0, v)\| \leq \Gamma (\exp(-\gamma t)\|x^0\| + \|v\|_{L^\infty}), \quad \forall t \in \mathbb{R}_+.
\] (11)

For the above theorem to be non-vacuous, \( S \) should be non-empty: thus, there is a tacit assumption of global existence of solutions. However, if the feedback
system given by (1) and (2) has the blow-up property, then it can be shown that the assumptions of Theorem 4 imply that every (local) solution can be extended to a global solution. Furthermore, we emphasize that (11) implies in particular that the feedback system is input-to-state stable in the sense of Sontag (see [10] for a recent survey of the theory of input-to-state stability).

Theorem 4 can be considered as a generalization and refinement of the circle criterion (see, for example, [4, 12]): in particular, it shows that, under the standard assumptions of the circle criterion (see also Corollaries 5 and 6 below), input-to-state stability is guaranteed. The proof of Theorem 4 (see [6]) is based on a well-known exponential weighting technique which has been used to prove stability results of input-output type (see [4, Section V.3] and the references therein). The application of this technique in an input-to-state stability context seems to be new (even in the finite-dimensional case). In particular, whilst the standard text-book version of the circle criterion for finite-dimensional state-space systems is usually proved using Lyapunov techniques combined with the positive-real lemma (see, for example, [12, p. 227]), the approach based on the exponential weighting technique provides a more elementary alternative.

The following corollary considers the case of scalar “sector data”.

**Corollary 5.** Assume that (1) is optimizable and estimatable, \( U = Y \) and that there exists an open set \( \Omega \subset \mathbb{C}_0 \) such that \( \mathbb{C}_0 \setminus \Omega \) is discrete in \( \mathbb{C}_0 \) and \( G \) is holomorphic on \( \Omega \). Furthermore, assume that there exist \( k_1, k_2 \in \mathbb{C} \) and \( \varepsilon > 0 \) such that \( \Phi \) satisfies (5) with \( K_1 = k_1 I \) and \( K_2 = k_2 I \), \( I + k_1 G(s) \) is invertible for every \( s \in \Omega \) and
\[
\Re \left[ \left( I + k_2 G(s) \right) \left( I + k_1 G(s) \right)^{-1} \right] \geq \varepsilon I, \quad \forall s \in \Omega. \tag{12}
\]
Then there exist positive constants \( \Gamma \) and \( \gamma \), such that, for each \((x^0, v) \in \mathcal{S}, (11)\) holds.

For non-zero real numbers \( k_1 \) and \( k_2 \), we define
\[
\Delta(k_1, k_2) := \text{open disk in } \mathbb{C} \text{ with centre in } \mathbb{R} \text{ and } -\frac{1}{k_1} \text{ and } -\frac{1}{k_2} \text{ in its boundary.}
\]

The next corollary focuses on the single-input-single-output case. In particular, the classical circle criterion is recovered.

**Corollary 6.** Assume that (1) is optimizable and estimatable, \( U = Y = \mathbb{R} \) and there exist real numbers \( k_1 < k_2 \) such that
\[
((\Phi(w))(t) - k_1 w(t))((\Phi(w))(t) - k_2 w(t)) \leq 0, \quad \forall w \in \text{dom}(\Phi), \ a.e. \ t \in \mathbb{R}_+. \tag{13}
\]
Then there exist positive constants \( \Gamma \) and \( \gamma \), such that, for each \((x^0, v) \in \mathcal{S}, (11)\) holds, provided that one of the following conditions is satisfied:

(C1) \( 0 < k_1 < k_2 \), \( G/(1 + [(k_1 + k_2)/2]*G) \in H^\infty \), \( G(i\omega) \) is bounded away from \( \Delta(k_1, k_2) \) for all \( \omega \in \mathbb{R} \) for which \( i\omega \) is not a pole of \( G \);

(C2) \( 0 = k_1 < k_2 \), \( G \in H^\infty \) and there exists \( \delta > 0 \) such that \( 1 + k_2 \Re G(i\omega) \geq \delta \) for all \( \omega \in \mathbb{R} \).
Figure 3. Static nonlinearity \( \varphi \) satisfying a generalized sector condition

\((C3)\) \( k_1 < 0 < k_2, \mathbf{G} \in H^\infty, \mathbf{G}(i\omega) \in \Delta(k_1, k_2) \) for all \( \omega \in \mathbb{R} \) and \( \mathbf{G}(i\omega) \) is bounded away from \( \partial \Delta(k_1, k_2) \) for all \( \omega \in \mathbb{R} \).

Observe that, in this single-input-single-output setting, the sector condition (13) can be expressed in the equivalent form:

\[
k_1 w^2(t) \leq (\Phi(w))(t) w(t) \leq k_2 w^2(t), \quad \forall w \in \text{dom}(\Phi), \text{ a.e. } t \in \mathbb{R}_+.
\] (14)

In many situations, the input-output stability condition \( \mathbf{G} = 1 + [(k_1 + k_2)/2] \mathbf{G} \in H^\infty \) (imposed in (C1)) is satisfied, provided that the number of anticlockwise encirclements of \( \Delta(k_1, k_2) \) by the Nyquist diagram of \( \mathbf{G} \) is equal to the number of poles of \( \mathbf{G} \) in \( \mathbb{C}_0 \), see, for example, [4, 12].

4 Generalized sector condition and input-to-state stability with bias

Next, we seek to relax the condition (5) to a generalized sector condition. Loosely speaking, we wish to impose the (pointwise) inequality in (5) only when \( t \in \mathbb{R}_+ \) and \( w \in \text{dom}(\Phi) \) are such that \( w(t) \in Y \setminus E \), where \( E \) (the exceptional set) is some bounded subset of \( Y \). A prototype to bear in mind is the case wherein \( \Phi \) is the \( \text{Nemyckii} \) operator, given by \( \Phi(w) := \varphi \circ w \), associated with a static nonlinearity \( \varphi : \mathbb{R} \to \mathbb{R} \), of the form shown in Figure 3 (a nonlinearity with negative resistance), satisfying a sector condition outside the interval \( E = [-1, 1] \). Extrapolating this prototype to our abstract setting requires care. The issue is to circumvent the technical difficulty engendered by the fact that the general operator \( \Phi \) has domain \( \text{dom}(\Phi) \subset L^2_{\text{loc}}(\mathbb{R}_+, Y) \) and so \( \Phi \) acts on equivalence classes of functions \( \mathbb{R}_+ \to Y \). Let \( w \in L^2_{\text{loc}}(\mathbb{R}_+, Y) \) and \( Z \subset Y \) be arbitrary. Let \( w_r : \mathbb{R}_+ \to Y \) be any representative of \( w \) and denote the preimage of \( Z \) under \( w_r \) by \( w_r^{-1}(Z) := \{ t \in \mathbb{R}_+ : w_r(t) \in Z \} \). Let \( \mathbb{I}_{w_r^{-1}(Z)} \) be the indicator or characteristic function of the set \( w_r^{-1}(Z) \) and define \( \chi_Z(w) \in L^2_{\text{loc}}(\mathbb{R}_+, Y) \) to be the equivalence class of this function,
that is, 
\[ \chi_Z(w) := \left[ I_{w^{-1}(Z)} \right] \].

Every choice of representative \( w_r \) of \( w \) yields the same equivalence class \( \left[ I_{w^{-1}(Z)} \right] \) and so \( \chi_Z(w) \) is a well-defined element of \( L^2_{\text{loc}}(\mathbb{R}^+, Y) \) for all \( w \in L^2_{\text{loc}}(\mathbb{R}^+, Y) \). We are now in a position to define the requisite generalized sector condition.

**Definition 7.** A nonlinearity \( \Phi : \text{dom}(\Phi) \subset L^2_{\text{loc}}(\mathbb{R}^+, Y) \to L^2_{\text{loc}}(\mathbb{R}^+, U) \) satisfies a generalized sector condition if there exist operators \( K_1, K_2 \in \mathcal{B}(Y, U) \), a bounded set \( E \subset Y \) and a constant \( b \geq 0 \) such that, for all \( w \in \text{dom}(\Phi) \) and a.e. \( t \in \mathbb{R}^+ \),
\[ \text{Re} \left( (\Phi(w))(t) - K_1 w(t), (\Phi(w))(t) - K_2 w(t) \right) (\chi_Y \setminus E)(w))(t) \leq 0 \tag{15} \]
and
\[ \| (\Phi(w))(t) \| \| (\chi_E)(w))(t) \| \leq b. \tag{16} \]

The following result generalizes Theorem 4.

**Corollary 8.** Assume that (1) is optimizable and estimatable and that there exist operators \( K_1, K_2 \in \mathcal{B}(Y, U) \), \( b \geq 0 \) and a bounded set \( E \subset Y \) such that \( \Phi \) satisfies (15) and (16) for all \( w \in \text{dom}(\Phi) \) and a.e. \( t \in \mathbb{R}^+ \). Let \( K \in \mathcal{B}(Y, U) \) and \( \kappa \geq 0 \) be given by (7). If at least one of hypotheses (H1)-(H4) holds, then there exist positive constants \( \Gamma \) and \( \gamma \) such that, for each \( \{x^0, v\} \in \mathcal{S} \),
\[ \| x(t; x^0, v) \| \leq \Gamma \left( \exp(-\gamma t) \| x^0 \| + \| v \|_{L^\infty} + \beta \right), \quad \forall t \in \mathbb{R}^+, \tag{17} \]
where
\[ \beta := \sup \left\{ \| \Phi(w) - Kw \|_{L^\infty} : w \in \text{dom}(\Phi) \right\} \leq b + \sup_{\xi \in E} \| K \xi \|, \tag{18} \]

In particular, (17) provides an input-to-state stability estimate with bias \( \beta \) (input-to-state stability with bias \( \beta \)). Under the additional assumption that the feedback system given by (1) and (2) has the blow-up property, it can be shown that the hypotheses of Corollary 8 imply that every maximal solution is global, so that every (local) solution can be extended to a global solution (to which then the stability conclusions of Corollary 8 apply).

The following results are generalizations of Corollaries 5 and 6.

**Corollary 9.** Assume that (1) is optimizable and estimatable, \( U = Y \) and that there exists an open set \( \Omega \subset \mathbb{C}_0 \) such that \( \mathbb{C}_0 \setminus \Omega \) is discrete in \( \mathbb{C}_0 \) and \( G \) is holomorphic on \( \Omega \). Furthermore, assume that there exist \( k_1, k_2 \in \mathbb{C} \), a bounded set \( E \subset Y \) and constants \( b \geq 0 \) and \( \varepsilon > 0 \) such that, for all \( w \in \text{dom}(\Phi) \) and a.e. \( t \in \mathbb{R}^+ \), \( \Phi \) satisfies (15) and (16) (with \( K_1 = k_1 I \) and \( K_2 = k_2 I \)), \( I + k_1 G(s) \) is invertible for every \( s \in \Omega \), and the positive-real condition
\[ \text{Re} \left( (I + k_2 G(s))(I + k_1 G(s))^{-1} \right) \geq \varepsilon I, \quad \forall s \in \Omega \]
holds. Then there exist constants $\Gamma > 0$ and $\gamma > 0$ such that, for each $(x^0, v) \in S$, (17) holds, where $\beta \geq 0$ is given by (18).

**Corollary 10.** Assume that (1) is optimizable and estimatable, $U = Y = \mathbb{R}$ and there exist real numbers $k_1 < k_2$, a bounded set $E \subset \mathbb{R}$ and $b \geq 0$ such that

$$((\Phi(w))(t) - k_1 w(t))((\Phi(w))(t) - k_2 w(t)) (\chi_{Y \setminus E}(w))(t) \leq 0, \; \forall w \in \text{dom}(\Phi), \; \text{a.e.} \; t \in \mathbb{R}_+$$

and

$$|(\Phi(w))(t)|(\chi_{E}(w))(t) \leq b, \; \forall w \in \text{dom}(\Phi), \; \text{a.e.} \; t \in \mathbb{R}_+.$$  

If at least one of the conditions (C1)--(C3) of Corollary 6 is satisfied, then there exist $\Gamma > 0$ and $\gamma > 0$ such that, for each $(x^0, v) \in S$, (17) holds, where

$$\beta := \sup \left\{ \|(\Phi(w) - (k_1 + k_2)w/2)\chi_{E}(w)\|_{L^\infty}: w \in \text{dom}(\Phi) \right\} \leq b + |k_1 + k_2| \sup_{\xi \in E} |\xi|/2,$$

(19)

## 5 Hysteretic feedback systems

Consider again the feedback interconnection of Figure 1, but now in a single-input ($U = \mathbb{R}$), single-output ($Y = \mathbb{R}$) setting and with a hysteresis operator $\Phi$ in the feedback path. An operator $\Phi : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ is a hysteresis operator if it is causal and rate independent. Here rate independence means that $\Phi(w \circ \zeta) = (\Phi w) \circ \zeta$ for every $w \in C(\mathbb{R}_+)$ and every time transformation $\zeta$, where $\zeta : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a time transformation if it is continuous, non-decreasing and surjective.

For simplicity of presentation, henceforth we restrict attention to the class of Preisach hysteresis operators which model complex hysteresis effects: for example, nested loops in input-output characteristics. A basic building block for the Preisach operator is the hysteresis operator $P_{\xi}$, the so-called backlash operator with width $\sigma \geq 0$ and “initial condition” $\xi \in \mathbb{R}$. A discussion of the backlash operator (also called play operator) can be found in a number of references, see for example, [2] and [8].

Let $\xi : \mathbb{R}_+ \to \mathbb{R}$ be a compactly supported and globally Lipschitz function with Lipschitz constant 1. Let $\mu$ be a regular signed Borel measure on $\mathbb{R}_+$. Denoting Lebesgue measure on $\mathbb{R}$ by $\mu_L$, let $f : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ be a locally $(\mu_L \otimes \mu)$-integrable function and let $f_0 \in \mathbb{R}$. The operator $P_{\xi} : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ defined by

$$(P_{\xi}(w))(t) = \int_0^\infty \int_0^t (B_{\sigma, \xi}(w))(s)(t) f(s, \sigma) \mu_L(ds) \mu(d\sigma) + f_0 \; \forall w \in C(\mathbb{R}_+), \; \forall t \in \mathbb{R}_+,$$

(20)

is called a Preisach operator, cf. [2, p. 55]. It is well-known that $P_{\xi}$ is a hysteresis operator (this follows from the fact that $B_{\sigma, \xi}$ is a hysteresis operator for every $\sigma \geq 0$).

Setting $f(\cdot, \cdot) = 1$ and $f_0 = 0$ in (20), we obtain the Prandtl operator $P_{\xi} : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ defined by

$$(P_{\xi}(w))(t) = \int_0^\infty (B_{\sigma, \xi}(w))(t) \mu(d\sigma) \; \forall w \in C(\mathbb{R}_+), \; \forall t \in \mathbb{R}_+.$$  

(21)
For $\xi(\cdot) = 0$ and $\mu$ given by $\mu(S) = \int_S \mathbb{I}_{[0,5]}(\sigma) \, d\sigma$ (where $\mathbb{I}_{[0,5]}$ denotes the indicator function of the interval $[0,5]$), the Prandtl operator is illustrated in Figure 4.

![Figure 4. Example of Prandtl hysteresis](image)

The next proposition identifies (rather “mild”) conditions under which the Preisach operator (20) satisfies a generalized sector bound and hence fits into the theory developed in Section 4. For simplicity, we assume that the measure $\mu$ and the function $f$ are non-negative (an important case in applications), although the proposition can be extended to signed measures $\mu$ and sign-indefinite functions $f$.

**Proposition 11.** Let $\mathcal{P}_\xi$ be the Preisach operator defined in (20). Assume that the measure $\mu$ is non-negative, $a_1 := \mu(\mathbb{R}_+) < \infty$ and $a_2 := \int_0^\infty \sigma \mu(d\sigma) < \infty$. Furthermore, assume that

$$b_1 := \text{ess inf}_{(s,\sigma) \in \mathbb{R} \times \mathbb{R}_+} f(s,\sigma) \geq 0, \quad b_2 := \text{ess sup}_{(s,\sigma) \in \mathbb{R} \times \mathbb{R}_+} f(s,\sigma) < \infty$$

and set

$$a_P := a_1 b_1, \quad b_P := a_1 b_2, \quad c_P := a_2 b_2 + |f_0|. \tag{22}$$

Then, for all $w \in C(\mathbb{R}_+)$ and all $t \in \mathbb{R}_+$,

$$w(t) \geq 0 \implies a_P w(t) - c_P \leq (\mathcal{P}_\xi(w))(t) \leq b_P w(t) + c_P, \tag{23}$$

$$w(t) \leq 0 \implies b_P w(t) - c_P \leq (\mathcal{P}_\xi(w))(t) \leq a_P w(t) + c_P, \tag{24}$$

and, furthermore, for every $\eta > 0$,

$$|w(t)| \geq c_P / \eta \implies (a_P - \eta) w^2(t) \leq (\mathcal{P}_\xi(w))(t) y(t) \leq (b_P + \eta) w^2(t). \tag{25}$$

In particular, for every $\eta > 0$, the generalized sector conditions (15) and (16) hold with $U = \mathbb{R} = Y$, $E = [-c_P / \eta, c_P / \eta]$, $K_1 = (a_P - \eta) I$, $K_2 = (b_P + \eta) I$, and $b = (b_P / \eta + 1)c_P$. 
Bibliography


