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The Circle Criterion and Input-to-State Stability for Infinite-Dimensional Systems

B. Jayawardhana†, H. Logemann‡, and E.P. Ryan§

1 Introduction

In this paper, the focus is on absolute stability and input-to-state stability of the feedback interconnection of an infinite-dimensional linear system $\Sigma$ and a nonlinearity $\Phi : \text{dom}(\Phi) \subset L^2_{\text{loc}}(\mathbb{R}_+, Y) \to L^2_{\text{loc}}(\mathbb{R}_+, U)$, where $\text{dom}(\Phi)$ denotes the domain of $\Phi$ and $U$ and $Y$ (Hilbert spaces) denote the input and output spaces of $\Sigma$, respectively (see Figure 1, wherein $\nu$ is an essentially bounded input signal). The system $\Sigma$ is assumed to belong to the rather general class of well-posed systems (see, for example, [11, 13] and the references therein) and the nonlinearity is assumed to satisfy a (generalized) sector condition.

In the literature on the circle criterion for infinite-dimensional systems (see, for example, [3, 4, 5, 7, 9, 12], and the references therein), the emphasis is usually on $L^2$- or $L^1$-stability and global asymptotic or global exponential stability (or some variants thereof) of feedback systems of the type shown in Figure 1, with a static sector-bounded nonlinearity $\Phi$ in the feedback path. The new contribution of this paper as compared to the previous literature is twofold.

(i) In addition to static nonlinearities, we include a class of dynamic nonlinearities which may exhibit bias, but still satisfy a generalized pointwise sector condition.

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Figure 1. Feedback interconnection of linear system $\Sigma$ and nonlinearity $\Phi$

As specific subclasses, the class of nonlinearities encompasses both static nonlinearities with “negative resistance” and a wide range of hysteretic effects described by so-called Preisach operators.

(ii) The main results of the paper guarantee input-to-state-stability with “bias” (and “standard” input-to-state-stability if the nonlinearity is unbiased), thereby making contact with the important and rapidly developing input-to-state-stability theory in (finite-dimensional) nonlinear control.

As in the classical theory of absolute stability and circle criteria, the methodology involves a “symbiosis” of (generalized) sector data relating to the nonlinearity $\Phi$ and properties of the transfer function of the linear system $\Sigma$ to conclude stability properties of the feedback interconnection. We mention that the viewpoint of this paper is similar in spirit to that of [1]: however, the class of feedback systems considered here is very different to that in [1] as is the methodology adopted.

For sake of brevity, this paper does not contain any proofs: for these we refer to [6].

Notation and terminology. For $\alpha \in \mathbb{R}$, set $\mathbb{C}_\alpha := \{ s \in \mathbb{C} : \text{Re } s > \alpha \}$. If $S$ is a non-empty subset of $\mathbb{C}$, then a set $R \subset S$ is said to be discrete in $S$, if, for every $s \in S$, there exists a neighbourhood $N$ of $s$ such that $N \cap R$ is finite. For Hilbert spaces $U$ and $Y$, let $\mathcal{B}(U,Y)$ denote the space of all linear bounded operators mapping $U$ to $Y$. We write $\mathcal{B}(U)$ for $\mathcal{B}(U,U)$. For $T \in \mathcal{B}(U)$, we define

$$\text{Re } T := \frac{1}{2}(T + T^*) \in \mathcal{B}(U).$$

The space of all holomorphic and bounded functions $\mathbb{C}_\alpha \to \mathcal{B}(U,Y)$ is denoted by $H^\infty_\alpha(\mathcal{B}(U,Y))$. We write $H^\infty(\mathcal{B}(U,Y))$ for $H^\infty_0(\mathcal{B}(U,Y))$. Moreover, in the scalar case (that is $U = Y = \mathbb{C}$), we simply write $H^\infty_\alpha$, or, if $\alpha = 0$, $H^\infty$ for $H^\infty_0(\mathcal{B}(U,Y))$ and $H^\infty(\mathcal{B}(U,Y))$, respectively. For $\alpha \in \mathbb{R}$, we define the exponentially weighted $L^p$-space $L^p_\alpha(\mathbb{R}_+, X) := \{ f \in L^p_{\text{loc}}(\mathbb{R}_+, U) : f(\cdot) \exp(-\alpha \cdot) \in L^p(\mathbb{R}_+, U) \}$. The Laplace transform is denoted by $\mathcal{L}$.

2 Well-posed linear systems with nonlinear feedback

There are a number of equivalent definitions of well-posed systems, see, for example, [11, 13] and the references therein. We will be brief in the following and refer the reader to the literature for more details. Throughout, we shall be considering a well-posed system $\Sigma$ with state-space $X$, input space $U$ and output space $Y$, generating operators $(A,B,C)$, input-output operator $G$ and transfer function $G$. Here $X$, $U$ and $Y$ are separable (complex) Hilbert spaces, $A$ is the generator of a strongly unbounded self-adjoint operator.
continuous semigroup $T = (T_t)_{t \geq 0}$ on $X$ and $B \in \mathcal{B}(U, X_{-1})$ and $C \in \mathcal{B}(X_1, Y)$, respectively, are admissible control and observations for $T$. The spaces $X_1$ and $X_{-1}$, respectively, are interpolation and extrapolation spaces associated with $X$: $X_1 = \text{dom}(A)$ (the domain of $A$), endowed with the graph norm of $A$, whilst $X_{-1}$ denotes the completion of $X$ with respect to the norm $\|x\|_{-1} = \|(\xi I - A)^{-1}x\|$, where $\xi \in \rho(A)$, the resolvent set of $A$ (different choices of $\xi$ lead to equivalent norms) and $\| \cdot \|$ denotes the norm on $X$. The control operator $B$ is said to be bounded if $\Psi$ is so as a map from the input space $U$ to the state space $X$, otherwise it is said to be unbounded; the observation operator $C$ is said to be bounded if it can be extended continuously to $X$, otherwise, $C$ is said to be unbounded.

The so-called $\Lambda$-extension $C_\Lambda$ of $C$ is defined by

$$C_\Lambda z = \lim_{s \to \infty, s \in \mathbb{R}} Cs(sI - A)^{-1}z,$$

with $\text{dom}(C_\Lambda)$ (the domain of $C_\Lambda$) consisting of all $z \in X$ for which the above limit exists. The transfer function $G$ has the property that $G \in H^\infty(\mathcal{B}(U, Y))$ for every $\omega > \omega(T)$, where $\omega(T)$ denotes the exponential growth constant of $T$. Moreover, the input-output operator $G : L^2_{\text{loc}}(\mathbb{R}_+, U) \to L^2_{\text{loc}}(\mathbb{R}_+, Y)$ is continuous and shift-invariant; for every $\omega > \omega(T)$, $G \in B(L^2(\mathbb{R}_+, U), L^2(\mathbb{R}_+, Y))$ and

$$(\mathcal{L}(Gu))(s) = G(s)\mathcal{L}(u)(s), \quad \forall s \in \mathbb{C}_\omega, \forall u \in L^2_{\text{loc}}(\mathbb{R}_+, U).$$

In the following, let $s_0 \in \mathbb{C}_\omega(T)$ be fixed, but arbitrary. For $x^0 \in X$ and $u \in L^2_{\text{loc}}(\mathbb{R}_+, U)$, let $x$ and $y$ denote the state and output functions of $\Sigma$, respectively, corresponding to the initial condition $x(0) = x^0 \in X$ and the input function $u$. Then $x(t) = T_1tx^0 + \int_0^t T_{t-}\tau Bu(\tau)d\tau$ for all $t \in \mathbb{R}_+$, $x(t) - (s_0I - A)^{-1}Bu(t) \in \text{dom}(C_\Lambda)$ for a.e. $t \in \mathbb{R}_+$ and

$$\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x^0, \quad \text{a.e. } t \in \mathbb{R}_+, \\
y(t) &= C_\Lambda (x(t) - (s_0I - A)^{-1}Bu(t)) + G(s_0)u(t), \quad \text{a.e. } t \geq 0.
\end{align*}$$

(1)

Of course, the differential equation in (1) has to be interpreted in $X_{-1}$. In the following, we identify $\Sigma$ and (1) and refer to (1) as a well-posed system.

We say that (1) is exponentially stable if $\omega(T) < 0$ and we say that (1) is input-output stable if $G \in H^\infty(\mathcal{B}(U, Y))$ or, equivalently, if $G \in B(L^2(\mathbb{R}_+, U), L^2(\mathbb{R}_+, Y))$. Furthermore, (1) is said to be observable, if for every $x^0$, there exists $u \in L^2(\mathbb{R}_+, U)$ such that the function $t \mapsto T_1tx^0 + \int_0^t T_{t-}\tau Bu(\tau)d\tau$ is in $L^2(\mathbb{R}_+, X)$. Writing $X_{-1} := (X_1)^{-1}$, we have that $X_{-1} = (X_1)^{-1}$ and $C^* \in \mathcal{B}(Y, X_{-1})$ is an admissible control operator for the adjoint semigroup $T^* = (T^*_t)_{t \geq 0}$. We say that (1) is estimatable if for every $x^0$, there exists $u^* \in L^2(\mathbb{R}_+, Y)$ such that the function $t \mapsto T^*_1x^0 + \int_0^t T^*_{t-}\tau C^*u^*(\tau)d\tau$ is in $L^2(\mathbb{R}_+, X)$.

In the following, we will consider the closed-loop system obtained by applying the nonlinear feedback

$$u = v - \Phi(y)$$

(2)

to the well-posed linear system (1), where $v \in L^\infty(\mathbb{R}_+, U)$ and the nonlinear operator $\Phi : \text{dom}(\Phi) \subset L^2_{\text{loc}}(\mathbb{R}_+, Y) \to L^2_{\text{loc}}(\mathbb{R}_+, U)$ is causal. To define the concept
of a (local) solution of the feedback system given by (1) and (2), we first need to show that $\Phi$ can be “localized” in the sense that it can be “extended” to spaces of functions with a finite time horizon. To this end, let $0 < \sigma \leq \infty$ be arbitrary and set

$$
\text{dom}_\sigma(\Phi) := \{ w \in L^2_{\text{loc}}([0, \sigma), Y) : \forall \tau \in (0, \sigma) \exists w_\tau \in \text{dom}(\Phi) \text{ s.t. } w = w_\tau \text{ on } [0, \tau] \}.
$$

Trivially, dom$_\infty(\Phi) = \text{dom}(\Phi)$. For $w \in \text{dom}_\sigma(\Phi)$ with $\sigma < \infty$, we define $\Phi(w)$ by

$$
(\Phi(w))(t) = (\Phi(w_\tau))(t), \quad 0 \leq t \leq \tau < \sigma,
$$

where $w_\tau \in \text{dom}(\Phi)$ such that $w = w_\tau$ on $[0, \tau]$. By causality of $\Phi$, this definition does not depend on the choice of $\tau$ and thus $\Phi(w)$ is a well-defined element in $L^2_{\text{loc}}([0, \sigma), U)$.

A solution on $[0, \sigma)$ (where $0 < \sigma \leq \infty$) of the feedback system given by (1) and (2) is a pair $(x, y) \in C([0, \sigma), X) \times \text{dom}_\sigma(\Phi)$ such that, with $u$ given by (2),

$$
x(t) = Ttx^0 + \int_0^t T_{t-\tau}Bu(\tau)d\tau, \quad \forall t \in [0, \sigma), \quad (3)
$$

$$
y(t) = C_\lambda (x(t) - (s_0I - A)^{-1}Bu(t)) + G(s_0)u(t), \quad \text{a.e. } t \in [0, \sigma). \quad (4)
$$

If $\sigma = \infty$, then we say that $(x, y)$ is a global solution. Let $S$ denote the set of all $(x^0, v) \in X \times L^\infty(\mathbb{R}_+, U)$ for which the feedback system given by (1) and (2) has at least one global solution. If $(x^0, v) \in S$, then the notation $(x(\cdot; x^0, v), y(\cdot; x^0, v))$ is used to denote any global solution corresponding to the initial condition $x^0$ and the closed-loop input $v$. Furthermore, a routine argument based on Zorn’s lemma shows that every solution $(x, y)$ can be extended to a maximal solution, that is, to a maximally defined solution which cannot be extended any further. The interval on which a maximal solution is defined is called the maximal interval of existence of the solution. We say that the feedback system given by (1) and (2) has the blow-up property if for every maximal solution $(x, y)$ defined on a finite maximal interval of existence $[0, \sigma)$, the $L^2$-norm of $y$ blows up, that is, $\|y\|_{L^2([0, \tau])} \to \infty$ as $\tau \uparrow \sigma$. In this paper, we are mainly concerned with stability properties of the feedback system given by (1) and (2): whilst of fundamental importance, the question of existence of solutions is not the main concern here; this question requires addressing on a less general basis, taking into account features of the particular system or subclass of systems under consideration (see [6] for further comments in this context).

3 The sector condition and input-to-state stability

First, we introduce a sector condition on the class of nonlinearities (in due course, this condition will be weakened to a generalized sector condition).

**Definition 1.** A nonlinearity $\Phi : \text{dom}(\Phi) \subset L^2_{\text{loc}}(\mathbb{R}_+, Y) \to L^2_{\text{loc}}(\mathbb{R}_+, U)$ satisfies a sector condition if there exist operators $K_1, K_2 \in \mathcal{B}(Y, U)$ such that

$$
\Re \langle (\Phi(w))(t) - K_1w(t), (\Phi(w))(t) - K_2w(t) \rangle \leq 0, \quad \forall w \in \text{dom}(\Phi), \text{ a.e. } t \in \mathbb{R}_+. \quad (5)
$$
Example 2 (Static nonlinearities). Let $\varphi : Y \to U$ be continuous and assume that there exist $K_1, K_2 \in B(Y, U)$ such that
\[
\text{Re}\langle \varphi(\xi) - K_1 \xi, \varphi(\xi) - K_2 \xi \rangle_U \leq 0 \quad \forall \xi \in Y.
\] (6)

With $\varphi$ we may associate the Nemyckii operator $\Phi : L^2_{\text{loc}}(\mathbb{R}_+, Y) \to L^2_{\text{loc}}(\mathbb{R}_+, U)$, defined by $\Phi(w) := \varphi \circ w$. This operator satisfies the sector condition (5). Such operators provide a simple prototype class for the general nonlinearities considered in this section: at the simplest illustrative level, static sector-bounded scalar nonlinearities $\varphi : \mathbb{R} \to \mathbb{R}$ of the type shown in Figure 2 (ubiquitous in the literature on the classical circle criterion) are subsumed by the formulation. This observation extends mutatis mutandis to encompass time-dependent static nonlinearities $\varphi : \mathbb{R}_+ \times Y \to U$.

Anticipating Sections 4 and 5 below, we will also consider static nonlinearities for which the inequality in (6) is assumed to hold only outside some bounded set $E \subset Y$ (see Figure 3). To accommodate these and more general nonlinearities, in Section 4 we will introduce a generalized sector condition and remark here that the generalized formulation encompasses a large class of hysteresis operators, including hysteresis of Preisach type.

Let $K_1, K_2 \in B(Y, U)$ and define
\[
K := \frac{1}{2}(K_1 + K_2), \quad \kappa := \|K_2 - K_1\|^2.
\] (7)

We assemble the following hypotheses on the transfer function $G$ of (1) which will be variously invoked in the theory presented below.

(H1) There exists $\alpha < 0$ and an open set $\Omega \subset \mathbb{C}_\alpha$ such that $\mathbb{C}_\alpha \setminus \Omega$ is discrete in
\(C_\alpha \) and \(G\) is holomorphic on \(\Omega\), the frequency-domain condition

\[
G^*(i\omega)\left[\frac{\kappa + \delta}{4} I - K^* K\right]G(i\omega) \leq I + 2 \Re(KG(i\omega)), \quad \text{a.e. } \omega \in \mathbb{R}. \tag{8}
\]

holds for some \(\delta > 0\) and \(G(I + KG)^{-1} \in H^\infty(B(U,Y))\).

\((H2)\) \(G \in H^\infty(B(U,Y))\) and there exist \(\delta > 0\) and \(\rho < 1\) such that (8) holds and

\[
G^*(i\omega)\left[\frac{\kappa + \delta}{4} I - K^* K\right]G(i\omega) \geq -\rho I, \quad \text{a.e. } \omega \in \mathbb{R}. \tag{9}
\]

\((H3)\) There exists an open set \(\Omega \subset C_0\) such that \(C_0 \setminus \Omega\) is discrete in \(C_0\) and \(G\) is holomorphic on \(\Omega\), \(I + KG(s)\) is invertible for all \(s \in \Omega\) and the frequency-domain condition

\[
G^*(s)\left[\frac{\kappa + \delta}{4} I - K^* K\right]G(s) \leq I + 2 \Re(KG(s)), \quad \forall s \in \Omega \tag{10}
\]

holds for some \(\delta > 0\).

\((H4)\) There exists an open set \(\Omega \subset C_0\) such that \(C_0 \setminus \Omega\) is discrete in \(C_0\) and \(G\) is holomorphic on \(\Omega\), \(KG(s)\) is compact for all \(s \in \Omega\) and the frequency-domain condition (10) holds for some \(\delta > 0\).

**Remark 3.** (a) In the case of scalar “sector data”, that is \(U = Y\) and there exist \(k_1, k_2 \in \mathbb{C}\) such that \(K_1 = k_1 I\) and \(K_2 = k_2 I\), the term

\[
\frac{\kappa + \delta}{4} I - K^* K
\]

appearing on the left-hand sides of (8)-(10) simplifies to \((\delta/4 - \Re(\tilde{k}_1 \tilde{k}_2)) I\).

(b) Assume that one of the operators \(K_1\) and \(K_2\) is the zero operator and that the other is a scalar multiple of an isometry. Then it is not difficult to show that (H2) is satisfied, provided that \(G \in H^\infty(B(U,Y))\) and the positive-real condition

\[
\varepsilon I \leq I + 2 \Re(KG(i\omega)), \quad \text{a.e. } \omega \in \mathbb{R}
\]

holds for some \(\varepsilon > 0\).

We are now in the position to state the main result of this section.

**Theorem 4.** Assume that (1) is optimizable and estimatable and that there exist operators \(K_1, K_2 \in B(Y,U)\) such that \(\Phi\) satisfies the sector condition (5). Let \(K \in B(Y,U)\) and \(\kappa \geq 0\) be given by (7). If at least one of hypotheses (H1)-(H4) holds, then there exist positive constants \(\Gamma\) and \(\gamma\), such that, for each \((x^0, v) \in S\),

\[
\|x(t; x^0, v)\| \leq \Gamma \left(\exp(-\gamma t)\|x^0\| + \|v\|_{L^\infty}\right), \quad \forall t \in \mathbb{R}_+.
\tag{11}
\]

For the above theorem to be non-vacuous, \(S\) should be non-empty: thus, there is a tacit assumption of global existence of solutions. However, if the feedback
system given by (1) and (2) has the blow-up property, then it can be shown that the assumptions of Theorem 4 imply that every (local) solution can be extended to a global solution. Furthermore, we emphasize that (11) implies in particular that the feedback system is input-to-state stable in the sense of Sontag (see [10] for a recent survey of the theory of input-to-state stability).

Theorem 4 can be considered as a generalization and refinement of the circle criterion (see, for example, [4, 12]): in particular, it shows that, under the standard assumptions of the circle criterion (see also Corollaries 5 and 6 below), input-to-state stability is guaranteed. The proof of Theorem 4 (see [6]) is based on a well-known exponential weighting technique which has been used to prove stability results of input-output type (see [4, Section V.3] and the references therein). The application of this technique in an input-to-state stability context seems to be new (even in the finite-dimensional case). In particular, whilst the standard text-book version of the circle criterion for finite-dimensional state-space systems is usually proved using Lyapunov techniques combined with the positive-real lemma (see, for example, [12, p. 227]), the approach based on the exponential weighting technique provides a more elementary alternative.

The following corollary considers the case of scalar “sector data”.

Corollary 5. Assume that (1) is optimizable and estimatable, $U = Y$ and that there exists an open set $\Omega \subset \mathbb{C}_0$ such that $\mathbb{C}_0 \setminus \Omega$ is discrete in $\mathbb{C}_0$ and $G$ is holomorphic on $\Omega$. Furthermore, assume that there exist $k_1, k_2 \in \mathbb{C}$ and $\varepsilon > 0$ such that $\Phi$ satisfies (5) with $K_1 = k_1 I$ and $K_2 = k_2 I$, $I + k_1 G(s)$ is invertible for every $s \in \Omega$ and

$$\text{Re}[(I + k_2 G(s))(I + k_1 G(s))^{-1}] \geq \varepsilon I, \quad \forall s \in \Omega. \quad (12)$$

Then there exist positive constants $\Gamma$ and $\gamma$, such that, for each $(x^0, v) \in S$, (11) holds.

For non-zero real numbers $k_1$ and $k_2$, we define

$$\Delta(k_1, k_2) := \text{open disk in } \mathbb{C} \text{ with centre in } \mathbb{R} \text{ and } -\frac{1}{k_1} \text{ and } -\frac{1}{k_2} \text{ in its boundary.}$$

The next corollary focuses on the single-input-single-output case. In particular, the classical circle criterion is recovered.

Corollary 6. Assume that (1) is optimizable and estimatable, $U = Y = \mathbb{R}$ and there exist real numbers $k_1 < k_2$ such that

$$((\Phi(w))(t) - k_1 w(t))(\Phi(w))(t) - k_2 w(t)) \leq 0, \quad \forall w \in \text{dom}(\Phi), \text{ a.e. } t \in \mathbb{R}_+. \quad (13)$$

Then there exist positive constants $\Gamma$ and $\gamma$, such that, for each $(x^0, v) \in S$, (11) holds, provided that one of the following conditions is satisfied:

1. $0 < k_1 < k_2$, $G/(1 + [(k_1 + k_2)/2]G) \in H^\infty$, $G(\imath \omega)$ is bounded away from $\Delta(k_1, k_2)$ for all $\omega \in \mathbb{R}$ for which $\imath \omega$ is not a pole of $G$;

2. $0 = k_1 < k_2$, $G \in H^\infty$ and there exists $\delta > 0$ such that $1 + k_2 \text{Re } G(\imath \omega) \geq \delta$ for all $\omega \in \mathbb{R}$;
Figure 3. Static nonlinearity $\varphi$ satisfying a generalized sector condition

(C3) $k_1 < 0 < k_2$, $G \in H^{\infty}$, $G(i\omega) \in \Delta(k_1, k_2)$ for all $\omega \in \mathbb{R}$ and $G(i\omega)$ is bounded away from $\partial\Delta(k_1, k_2)$ for all $\omega \in \mathbb{R}$.

Observe that, in this single-input-single-output setting, the sector condition (13) can be expressed in the equivalent form:

$$k_1 w^2(t) \leq (\Phi(w))(t)w(t) \leq k_2 w^2(t), \quad \forall w \in \text{dom}(\Phi), \text{ a.e. } t \in \mathbb{R}_+.$$  \hspace{1cm} (14)

In many situations, the input-output stability condition $G/(1 + [(k_1 + k_2)/2]G) \in H^{\infty}$ (imposed in (C1)) is satisfied, provided that the number of anticlockwise encirclements of $(k_1, k_2)$ by the Nyquist diagram of $G$ is equal to the number of poles of $G$ in $\mathbb{C}_0$, see, for example, [4, 12].

4 Generalized sector condition and input-to-state stability with bias

Next, we seek to relax the condition (5) to a generalized sector condition. Loosely speaking, we wish to impose the (pointwise) inequality in (5) only when $t \in \mathbb{R}_+$ and $w \in \text{dom}(\Phi)$ are such that $w(t) \in Y \setminus E$, where $E$ (the exceptional set) is some bounded subset of $Y$. A prototype to bear in mind is the case wherein $\Phi$ is the Nemyckii operator, given by $\Phi(w) := \varphi \circ w$, associated with a static nonlinearity $\varphi : \mathbb{R} \to \mathbb{R}$, of the form shown in Figure 3 (a nonlinearity with negative resistance), satisfying a sector condition outside the interval $E = [-1, 1]$. Extrapolating this prototype to our abstract setting requires care. The issue is to circumvent the technical difficulty engendered by the fact that the general operator $\Phi$ has domain $\text{dom}(\Phi) \subset L^2_{\text{loc}}(\mathbb{R}_+, Y)$ and so $\Phi$ acts on equivalence classes of functions $\mathbb{R}_+ \to Y$. Let $w \in L^2_{\text{loc}}(\mathbb{R}_+, Y)$ and $Z \subset Y$ be arbitrary. Let $w_r : \mathbb{R}_+ \to Y$ be any representative of $w$ and denote the preimage of $Z$ under $w_r$ by $w_r^{-1}(Z) := \{t \in \mathbb{R}_+ : w_r(t) \in Z\}$. Let $\mathbb{I}_{w_r^{-1}(Z)}$ be the indicator or characteristic function of the set $w_r^{-1}(Z)$ and define $\chi_Z(w) \in L^2_{\text{loc}}(\mathbb{R}_+, Y)$ to be the equivalence class of this function,
that is,
\[ \chi_Z(w) := [I_{w_r^{-1}(Z)}]. \]
Every choice of representative \( w_r \) of \( w \) yields the same equivalence class \( [I_{w_r^{-1}(Z)}] \) and so \( \chi_Z(w) \) is a well-defined element of \( L^2_{\text{loc}}(\mathbb{R}_+, Y) \) for all \( w \in L^2_{\text{loc}}(\mathbb{R}_+, Y) \). We are now in a position to define the requisite generalized sector condition.

**Definition 7.** A nonlinearity \( \Phi: \text{dom}(\Phi) \subset L^2_{\text{loc}}(\mathbb{R}_+, Y) \to L^2_{\text{loc}}(\mathbb{R}_+, U) \) satisfies a **generalized sector condition** if there exist operators \( K_1, K_2 \in B(Y, U) \), a bounded set \( E \subset Y \) and a constant \( b \geq 0 \) such that, for all \( w \in \text{dom}(\Phi) \) and a.e. \( t \in \mathbb{R}_+ \),
\[
\text{Re} \left( (\Phi(w))(t) - K_1w(t), (\Phi(w))(t) - K_2w(t) \right)(\chi_{Y \setminus E}(w))(t) \leq 0
\]
and
\[
\| (\Phi(w))(t) \| (\chi_{E}(w))(t) \leq b.
\]

The following result generalizes Theorem 4.

**Corollary 8.** Assume that (1) is optimizable and estimatable and that there exist operators \( K_1, K_2 \in B(Y, U) \), \( b \geq 0 \) and a bounded set \( E \subset Y \) such that \( \Phi \) satisfies (15) and (16) for all \( w \in \text{dom}(\Phi) \) and a.e. \( t \in \mathbb{R}_+ \). Let \( K \in B(Y, U) \) and \( \kappa \geq 0 \) be given by (7). If at least one of hypotheses (H1)-(H4) holds, then there exist positive constants \( \Gamma \) and \( \gamma \) such that, for each \( (x^0, v) \in S \),
\[
\| x(t; x^0, v) \| \leq \Gamma \left( \exp(-\gamma t) \| x^0 \| + \| v \|_{L^\infty} + \beta \right), \quad \forall t \in \mathbb{R}_+,
\]
where
\[
\beta := \sup \left\{ \| (\Phi(w) - Kw)(\chi_E(w)) \|_{L^\infty} : w \in \text{dom}(\Phi) \right\} \leq b + \sup_{\xi \in E} \| K \xi \|.
\]

In particular, (17) provides an input-to-state stability estimate with bias \( \beta \) (input-to-state stability with bias \( \beta \)). Under the additional assumption that the feedback system given by (1) and (2) has the blow-up property, it can be shown that the hypotheses of Corollary 8 imply that every maximal solution is global, so that every (local) solution can be extended to a global solution (to which then the stability conclusions of Corollary 8 apply).

The following results are generalizations of Corollaries 5 and 6.

**Corollary 9.** Assume that (1) is optimizable and estimatable, \( U = Y \) and that there exists an open set \( \Omega \subset \mathbb{C}_0 \) such that \( \mathbb{C}_0 \setminus \Omega \) is discrete in \( \mathbb{C}_0 \) and \( G \) is holomorphic on \( \Omega \). Furthermore, assume that there exist \( k_1, k_2 \in \mathbb{C} \), a bounded set \( E \subset Y \) and constants \( b \geq 0 \) and \( \varepsilon > 0 \) such that, for all \( w \in \text{dom}(\Phi) \) and a.e. \( t \in \mathbb{R}_+ \), \( \Phi \) satisfies (15) and (16) (with \( K_1 = k_1I \) and \( K_2 = k_2I \)), \( I + k_1G(s) \) is invertible for every \( s \in \Omega \) and the positive-real condition
\[
\text{Re} \left[ (I + k_2G(s))(I + k_1G(s))^{-1} \right] \geq \varepsilon I, \quad \forall s \in \Omega
\]
holds. Then there exist constants $\Gamma > 0$ and $\gamma > 0$ such that, for each $(x^0,v) \in \mathcal{S}$, (17) holds, where $\beta \geq 0$ is given by (18).

Corollary 10. Assume that (1) is optimizable and estimatable, $U = Y = \mathbb{R}$ and there exist real numbers $k_1 < k_2$, a bounded set $E \subset \mathbb{R}$ and $b \geq 0$ such that

$$
((\Phi(w))(t) - k_1 w(t))((\Phi(w))(t) - k_2 w(t))\chi_{Y \setminus E}(w)(t) \leq 0, \; \forall w \in \text{dom}(\Phi), \; \text{a.e.} \; t \in \mathbb{R}^+ \quad \text{and}
$$

$$
|((\Phi(w))(t)|\chi_{E}(w))(t) \leq b, \; \forall w \in \text{dom}(\Phi), \; \text{a.e.} \; t \in \mathbb{R}^+.
$$

If at least one of the conditions (C1)–(C3) of Corollary 6 is satisfied, then there exist $\Gamma > 0$ and $\gamma > 0$ such that, for each $(x^0,v) \in \mathcal{S}$, (17) holds, where

$$
\beta := \sup \left\{ \|(\Phi(w) - (k_1 + k_2)w/2)\chi_{E}(w)\|_{L^\infty} : w \in \text{dom}(\Phi) \right\} \leq b + |k_1 + k_2| \sup_{\xi \in E} |\xi|/2, \; \forall w \in \text{dom}(\Phi).
$$

(19)

5 Hysteretic feedback systems

Consider again the feedback interconnection of Figure 1, but now in a single-input ($U = \mathbb{R}$), single-output ($Y = \mathbb{R}$) setting and with a hysteresis operator $\Phi$ in the feedback path. An operator $\Phi : C(\mathbb{R}^+) \to C(\mathbb{R}^+)$ is a hysteresis operator if it is causal and rate independent. Here rate independence means that $\Phi(w \circ \zeta) = (\Phi(w)) \circ \zeta$ for every $w \in C(\mathbb{R}^+)$ and every time transformation $\zeta$, where $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be a time transformation if it is continuous, non-decreasing and surjective.

For simplicity of presentation, henceforth we restrict attention to the class of Preisach hysteretic operators which model complex hysteretic effects: for example, nested loops in input-output characteristics. A basic building block for the Preisach operator is the hysteresis operator $B_{\sigma, \xi}$, the so-called backlash operator with width $\sigma \geq 0$ and “initial condition” $\xi \in \mathbb{R}$. A discussion of the backlash operator (also called play operator) can be found in a number of references, see for example, [2] and [8].

Let $\xi : \mathbb{R}^+ \to \mathbb{R}$ be a compactly supported and globally Lipschitz function with Lipschitz constant 1. Let $\mu$ be a regular signed Borel measure on $\mathbb{R}^+$. Denoting Lebesgue measure on $\mathbb{R}$ by $\mu_L$, let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a locally $(\mu_L \otimes \mu)$-integrable function and let $f_0 \in \mathbb{R}$. The operator $P_{\xi} : C(\mathbb{R}^+) \to C(\mathbb{R}^+)$ defined by

$$
(P_{\xi}(w))(t) = \int_0^\infty \int_0^t (B_{\sigma, \xi}(w))(t) f(s, \sigma) \mu_L(ds) \mu(d\sigma) + f_0 \quad \forall w \in C(\mathbb{R}^+), \; \forall t \in \mathbb{R}^+,
$$

(20)
is called a Preisach operator, cf. [2, p. 55]. It is well-known that $P_{\xi}$ is a hysteresis operator (this follows from the fact that $B_{\sigma, \xi}(\sigma)$ is a hysteresis operator for every $\sigma \geq 0$).

Setting $f(\cdot, \cdot) = 1$ and $f_0 = 0$ in (20), we obtain the Prandtl operator $P_{\xi} : C(\mathbb{R}^+) \to C(\mathbb{R}^+)$ defined by

$$
P_{\xi}(w)(t) = \int_0^\infty (B_{\sigma, \xi}(w))(t) \mu(d\sigma) \quad \forall w \in C(\mathbb{R}^+), \; \forall t \in \mathbb{R}^+. 
$$

(21)
For $\xi(\cdot) = 0$ and $\mu$ given by $\mu(S) = \int_S 1_{[0,5]}(\sigma) d\sigma$ (where $1_{[0,5]}$ denotes the indicator function of the interval $[0,5]$), the Prandtl operator is illustrated in Figure 4.

![Figure 4. Example of Prandtl hysteresis](image)

The next proposition identifies (rather “mild”) conditions under which the Preisach operator (20) satisfies a generalized sector bound and hence fits into the theory developed in Section 4. For simplicity, we assume that the measure $\mu$ and the function $f$ are non-negative (an important case in applications), although the proposition can be extended to signed measures $\mu$ and sign-indefinite functions $f$.

**Proposition 11.** Let $\mathcal{P}_\xi$ be the Preisach operator defined in (20). Assume that the measure $\mu$ is non-negative, $a_1 := \mu(\mathbb{R}_+) \in (0, \infty)$ and $a_2 := \int_{\mathbb{R}^+} \sigma d\mu(\sigma) < \infty$. Furthermore, assume that

$$b_1 := \inf_{(s, \sigma) \in \mathbb{R} \times \mathbb{R}^+} f(s, \sigma) \geq 0, \quad b_2 := \sup_{(s, \sigma) \in \mathbb{R} \times \mathbb{R}^+} f(s, \sigma) < \infty$$

and set

$$a_p := a_1 b_1, \quad b_p := a_1 b_2, \quad c_p := a_2 b_2 + |f_0|. \tag{22}$$

Then, for all $w \in C(\mathbb{R}_+)$ and all $t \in \mathbb{R}_+$,

$$w(t) \geq 0 \implies a_p w(t) - c_p \leq (\mathcal{P}_\xi(w))(t) \leq b_p w(t) + c_p, \tag{23}$$

$$w(t) \leq 0 \implies b_p w(t) - c_p \leq (\mathcal{P}_\xi(w))(t) \leq a_p w(t) + c_p, \tag{24}$$

and, furthermore, for every $\eta > 0$,

$$|w(t)| \geq c_p/\eta \implies (a_p - \eta) w^2(t) \leq (\mathcal{P}_\xi(w))(t) y(t) \leq (b_p + \eta) w^2(t). \tag{25}$$

In particular, for every $\eta > 0$, the generalized sector conditions (15) and (16) hold with $U = \mathbb{R} = Y$, $E = [-c_p/\eta, c_p/\eta]$, $K_1 = (a_p - \eta)I$, $K_2 = (b_p + \eta)I$, and $b = (b_p/\eta + 1)c_p$. 

Bibliography


