Attractor switching in neuron networks
Chapter 2
Pulse coupled oscillator network with delay

Abstract

This chapter formulates a mathematical description for a system of pulse coupled oscillators with delay. The purpose is to introduce notations and to discuss some general results concerning a pulse-coupled system. In particular, a symbolic representation of the dynamics, called event representation, is described which is used in later chapters to prove several results.

In a system of pulse coupled oscillators, an individual oscillator is described by a phase variable that is a function of time. The phase variable varies between 0 and 1 and lives in $\mathbb{T}^1$ (or $\mathbb{S}^1$). When the phase of an oscillator reaches the threshold 1, the oscillator fires and its phase is reset to zero. Consequently, a pulse is sent to all the other oscillators which is received after a time delay $\tau$. The interactions between the oscillators occur through a linear monotonic function called pulse response function.

The goal of this chapter is to present a mathematical setting of the system and is organized as follows. We begin with the definition of pulse response function, the dynamics of the system, its state space and evolution operator in section 2.1.1. In section 2.1.2 a specific pulse response function called Mirollo Strogatz function is discussed. Then in section 2.1.3 we introduce a metric on the state space and study the continuity of the evolution operator with respect to the defined metric. Finally, in section 2.2, two other representations of the dynamics called past-firings representation and event representation are described. Certain general properties of the metric are discussed in the appendix of this chapter. The representations as well as the properties of the metric are used in later chapters to prove several results.

2.1 Setting of the problem

The system studied in this chapter is a delay system (Diekmann et al. 1995). The state space of such systems is an appropriate space $\mathcal{P}_{\tau}^n$ of functions (see definition 1) defined on the interval $(-\tau, 0]$, where $\tau > 0$ is the delay of the system, and taking values in an $n$-dimensional manifold $N$. The state space thus is infinite dimensional. In our case, points
in \( N \) represent the phases of the \( n \) coupled oscillators, which implies that \( N = \mathbb{T}^n \), the \( n \)-dimensional torus.

For a given \( \phi \in \mathcal{P}_\tau^n \) and for each \( t \in (-\tau, 0] \), \( \phi(t) \in N \) represents the phases of the oscillators at time \( t \). Using the dynamics of the system, \( \phi \) can be extended to a unique function \( \phi^+ : (-\tau, +\infty) \to N \), such that \( \phi^+(t) = \phi(t) \) for \( t \in (-\tau, 0] \) and \( \phi^+(t) \in N \) represents the phases of the oscillators at any time \( t \geq -\tau \). Then the evolution operator \( \Phi^t : \mathcal{P}_\tau^n \to \mathcal{P}_\tau^n \) is defined by \( \Phi^t(\phi)(s) = \phi^+(t + s) \) for any \( t \geq 0 \) and \( s \in (-\tau, 0] \).

In other words, the evolution operator maps the initial state \( \phi = \phi_0 \) to the state \( \phi^t \) of the system at time \( t \). The latter is the restriction of \( \phi^+ \) in \( (t - \tau, t] \) shifted back to the interval \( (-\tau, 0] \).

### 2.1.1 Definition of the dynamics

We now specialize the above notions of the theory of delay equations to the current setting. In this section we follow closely (Ashwin and Timme 2005a).

**Definition 1 (State space, cf. (Ashwin and Timme 2005a)).** The state space \( \mathcal{P}_\tau^n \) of the system of \( n \) pulse coupled oscillators with delay \( \tau > 0 \) is the space of phase history functions \( \phi : (-\tau, 0] \to \mathbb{T}^n : s \mapsto \phi(s) = (\phi_1(s), \ldots, \phi_n(s)) \), that satisfy the following conditions:

1. Each \( \phi_i \) is upper-semicontinuous, i.e., \( \phi_i(s^+) := \lim_{t \to s^+} \phi_i(t) = \phi_i(s) \) and \( \phi_i(s^-) := \lim_{t \to s^-} \phi_i(t) \leq \phi_i(s) \) for all \( s \in (-\tau, 0] \).
2. Each \( \phi_i \) is only discontinuous at a finite (or empty) set \( S_i = \{s_{i,1}, \ldots, s_{i,k_i}\} \subset (-\tau, 0] \) with \( k_i \in \mathbb{N} \) and \( s_{i,1} > s_{i,2} > \cdots > s_{i,k_i} \).
3. \( d\phi_i(s)/ds = 1 \) for \( s \not\in S_i \).

The coupling between the \( n \) oscillators is defined using the pulse response function.

**Definition 2 (Pulse response function, cf. (Ashwin and Timme 2005a)).** A pulse response function is a map

\[
V : \mathbb{T} \times \mathbb{R}_+ \to \mathbb{R} : (\theta, \varepsilon) \mapsto V(\theta, \varepsilon), \tag{2.1}
\]

that satisfies the following conditions:

1. \( V \) is smooth on \( (\mathbb{T} \setminus \{0\}) \times \mathbb{R}_+ \).
2. \( \partial V(\theta, \varepsilon)/\partial \theta > 0 \) on \( (\mathbb{T} \setminus \{0\}) \times (\mathbb{R}_+ \setminus \{0\}) \).
Setting of the problem

1. \( \theta, \varepsilon \in T \times \mathbb{R}_+ \)

2. \( \partial V(\theta, \varepsilon) / \partial \varepsilon > 0 \) on \( T \times \mathbb{R}_+ \).

3. \( V(\theta, 0) = 0 \) for all \( \theta \in T \).

4. \( 0 < V(0, \varepsilon) < 1 \) for all \( \varepsilon \in (0, 1) \).

5. \( H \), given by (2.4), satisfies

\[
H_m(\theta) = H_1 \circ H_{m-1}(\theta) = H_1 \circ \cdots \circ H_1(\theta). \tag{2.2}
\]

6. Notice that in the above definition \( \partial V / \partial \theta > 0 \), therefore \( V \) cannot be smooth everywhere on \( T \). This is reflected in condition (i) of the definition. The pulse response function depends on the parameter \( \varepsilon \geq 0 \), called coupling strength. As a shorthand notation we introduce

\[
V_m(\theta) = V(\theta, m \hat{\varepsilon}), \text{ for } m = 1, 2, 3, \ldots, \tag{2.3}
\]

where \( \hat{\varepsilon} = \varepsilon / (n - 1) \). Given a pulse response function \( V \) we also define

\[
H : T \times \mathbb{R}_+ \rightarrow \mathbb{R} : (\theta, \varepsilon) \mapsto H(\theta, \varepsilon) = \theta + V(\theta, \varepsilon), \tag{2.4}
\]

and

\[
H_m(\theta) = H(\theta, m \hat{\varepsilon}), \text{ for } m = 1, 2, 3, \ldots. \tag{2.5}
\]

Definition 3 (Dynamics, cf. (Ashwin and Timme 2005a)). A system of \( n \) pulse coupled oscillators with delay is a quadruple \( D = (n, V, \varepsilon, \tau) \), where \( V \) is as in definition 2, \( \varepsilon \geq 0 \) and \( \tau \geq 0 \). Given a system \( D \) and an initial state \( \phi \in P_\tau^n \), we extend \( \phi \) to a function \( \phi^+ : (-\tau, +\infty) \rightarrow T^n \) using the following rules:

1. \( \phi^+(t) = \phi(t) \) for \( t \in (-\tau, 0] \).

2. \( d\phi^+_i(t) / dt = 1 \) for \( t \geq 0 \), if \( \phi^+_j(t - \tau) \neq 0 \) (mod \( \mathbb{Z} \)) for all \( j \neq i \).

3. \( \phi^+_i(t) = \min\{1, H_m(\phi^+_i(t^-)) \} \) (mod \( \mathbb{Z} \)), if there are \( j_1, \ldots, j_m \neq i \) such that \( \phi^+_j(t - \tau) = 0 \) (mod \( \mathbb{Z} \)) for all \( k = 1, \ldots, m \).

The dynamics described in definition 3 can be interpreted in the following way. The phase \( \phi_i \) of each oscillator \( O_i \), \( i = 1, \ldots, n \), increases linearly. When the phase reaches the value \( 1 = 0 \) (mod \( \mathbb{Z} \)), then the oscillator \( O_i \) fires and all the other oscillators \( O_j, j \neq i \) receive a pulse after a time delay \( \tau \). In general, an oscillator \( O_j \) may receive \( m \) simultaneous pulses at time \( t \) if \( m \) oscillators \( O_{i_1}, \ldots, O_{i_m} \) have fired simultaneously at time \( t - \tau \). Then the phase of \( O_j \) is increased to \( H(u_j, m \hat{\varepsilon}) = H_m(u_j) \) where \( u_j = \phi^+_j(t^-) \), unless the pulse causes the oscillator to fire and then the phase becomes exactly 1.
The evolution operator $\Phi^t$ for $t \geq 0$ is then defined by
\[
\Phi^t : \mathcal{P}_\tau^n \rightarrow \mathcal{P}_\tau^n : \phi \mapsto \Phi^t(\phi) = \phi^+|_{[t-\tau,t]} \circ T_t, \tag{2.6}
\]
where $T_t$ is the shift $s \mapsto s + t$ and the positive semiorbit of $\phi \in \mathcal{P}_\tau^n$ is given by
\[
\mathcal{O}_+(\phi) = \{ \Phi^t(\phi) : t \geq 0 \}. \tag{2.7}
\]

**Proposition 1.** The evolution operator $\Phi^t$ is well defined.

**Proof.** From definition 3 it follows that the extended function $\phi^+$ can be determined for all $t \geq 0$ and all $\phi \in \mathcal{P}_\tau^n$, given $\mathcal{D}$ and $\phi \in \mathcal{P}_\tau^n$. The only question is whether $\phi^t \in \mathcal{P}_\tau^n$ for all $t \geq 0$. First we show that $\phi^t$, $t \geq 0$ is discontinuous at a finite set. Note that, by definition 1, each component $\phi_i$ of $\phi$ has only a finite number $k_i$ of discontinuities in $(-\tau,0]$. Therefore, $\phi_i(0) < \phi_i(-\tau) + k_i + \tau$, since the phase $\phi_i$ increases linearly (outside discontinuities) and each discontinuity induces an increase of $\phi_i$ that is less than 1. This implies that $\phi_i(s) = 0 \text{ (mod } \mathbb{Z})$ in a finite set $\{s_{i,1}, \ldots, s_{i,\ell_i}\} \subset (-\tau,0]$ with $\ell_i$ elements. Then, the number of discontinuities of $\phi^+_j$, in $(0,\tau]$ (and hence of $\phi^+_j$) also is finite for all $j = 1, \ldots, n$. This follows from the fact that the number of discontinuities of $\phi^+_j$ in $(0,\tau]$ is less than or equal to the number of firings of all the $\phi_i$ in $(-\tau,0]$ for $i = 1, \ldots, n$ and $i \neq j$, therefore it is less than or equal to $\sum_{i=1}^n \ell_i$. This shows that advancing time by $\tau$ the number of discontinuities remains finite. It follows by induction, that the number of discontinuities of $\phi^+_i$ in any interval $((m-1)\tau, m\tau]$, $m \in \mathbb{N}$ is finite for all $i = 1, \ldots, n$. Thus, the number of discontinuities of $\phi^+_i$ in any interval $[t-\tau,t]$, $t \geq 0$ (and hence of $\phi^+_i$) is finite for all $i = 1, \ldots, n$. The facts that $\phi^t$ is upper-semicontinuous and $d\phi^t/ds = 1$ (outside discontinuities) are a direct consequence of properties (iii) and (ii) respectively of definition 3. \hfill \Box

For a given system $\mathcal{D} = (n,V,\varepsilon,\tau)$, the accessible state space is $\mathcal{P}_\mathcal{D} = \Phi^\tau(\mathcal{P}_\tau^n)$. In other words, $\phi \in \mathcal{P}_\mathcal{D}$ if there is a state $\psi \in \mathcal{P}_\tau^n$ such that $\Phi^\tau(\psi) = \phi$, i.e., $\mathcal{P}_\mathcal{D}$ includes only those states that are dynamically accessible. From now on, we restrict our attention to $\mathcal{P}_\mathcal{D}$.

### 2.1.2 The Mirollo-Strogatz model

A pulse response function $V$ that satisfies all the requirements of definition 2 is provided by the Mirollo-Strogatz model (Mirollo and Strogatz 1990) where the pulse response function is
\[
V_{\text{MS}}(\theta, \varepsilon) = f^{-1}(f(\theta) + \varepsilon) - \theta, \tag{2.8}
\]
and $f$ is a function which is concave down ($f'' < 0$) and monotonically increasing ($f' > 0$). Moreover, $f(0) = 0$ and $f(1) = 1$. A concrete example is given by
\[
f_b(\theta) = \frac{1}{b} \ln(1 + (e^b - 1)\theta). \tag{2.9}
\]
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We present a sketch of the function \( f_b \) for various values of \( b \) in Figure 2.1a. For any given positive value of \( \varepsilon \), the pulse response function \( V_{MS}(\theta, \varepsilon) \) for \( f = f_b \) as in (2.9) is affine:

\[
V_{MS}(\theta, \varepsilon) = m_\varepsilon + K_\varepsilon \theta,
\]

where \( m_\varepsilon = (e^{b\varepsilon} - 1)/(e^b - 1) \) and \( K_\varepsilon = e^{b\varepsilon} - 1 \). The graph of \( V_{MS} \) (2.10) is depicted in Figure 2.1b for different values of \( \varepsilon \).

In the numerical computations in this paper, we use the Mirollo-Strogatz model with \( f_b \) as in (2.9) with fixed \( b = 3 \). After fixing \( b \), the parameter space of the system is \( \{(\varepsilon, \tau) : \varepsilon > 0, \tau > 0 \} = \mathbb{R}_+^2 \), where we recall that \( \tau \) is the delay and \( \varepsilon \) is the coupling strength. The qualitative results of our analysis depend only on the properties of the pulse response function \( V \) given in definition 2 and not on the specific choice of the Mirollo-Strogatz model (2.8) nor on the choice \( f = f_b \) (2.9).

2.1.3 Metric

In this section, we introduce a metric \( d \) on \( P_D \) and thereafter study the continuity of the evolution operator \( \Phi^t \) with respect to \( d \). Recall that given a phase history function \( \phi \in P^n_\tau \), we can define the extended phase history function \( \phi^+ \).

We define a lift (Katok and Hasselblatt 1995) of an extended phase history function \( \phi^+ \) as any function \( L_\phi : \langle -\tau, +\infty \rangle \rightarrow \mathbb{R}^n \) such that:

1. \( L_\phi(s) \pmod{\mathbb{Z}} = \phi^+(s) \), and

2. For any \( s \in (-\tau, +\infty) \) and for \( i = 1, \ldots, n \),

\[
(L_\phi)_i(s) - (L_\phi)_i(s^-) = \phi^+_i(s) - \phi^+_i(s^-).
\]
It follows from these properties that if $L^{(1)}_\phi$ and $L^{(2)}_\phi$ are two lifts of the same extended phase history function $\phi^+$ then they differ by a constant integer vector, i.e., $L^{(1)}_\phi(s) - L^{(2)}_\phi(s) = k \in \mathbb{Z}^n$, for all $s \in (-\tau, \infty)$.

**Definition 4 (Metric on $\mathcal{P}_D$).** The metric $d: \mathcal{P}_D \times \mathcal{P}_D \to \mathbb{R}$ is given by

$$d(\phi, \psi) = \min_{k \in \mathbb{Z}^n} \sum_{i=1}^n \int_{-\tau}^\tau |(L_\phi)_i(s) - (L_\psi)_i(s) - k_i| ds,$$

where $L_\phi$ and $L_\psi$ are arbitrary lifts of $\phi$ and $\psi$ respectively.

**Remark 1.** Because of the delay $\tau$, the distance in $\mathbb{T}^n$ between the points $\phi(0)$ and $\psi(0)$ is not a suitable metric for this system. Instead, it is important to take into account the values of $\phi$ and $\psi$ at least in the interval $(-\tau, 0]$. Nevertheless, if the integral in (2.11) runs from $-\tau$ to 0 defining thus a metric $d'$, there are several states at which the evolution operator is discontinuous with respect to $d'$. For the chosen metric $d$ in (2.11) the only states for which the evolution operator is discontinuous are those that are related to the overfiring effect which is discussed later in this section.

**Discontinuity of the evolution operator**

In general, the evolution operator $\Phi^t: \mathcal{P}_D \to \mathcal{P}_D$ is not continuous for all $t \geq 0$. We demonstrate this by a simple example. Consider a system of $n = 3$ oscillators and the initial state $\phi$ given by

$$\begin{align*}
\phi_1(s) &= \phi_2(s) = 1 - \frac{1}{2} \tau + s \\
\phi_3(s) &= \vartheta - \frac{3}{2} \tau + s
\end{align*}$$

for $s \in (-\tau, 0]$, where $\vartheta \in (0, 1)$ is close enough to 1, so that $H_1(\vartheta) > 1$ which also implies that $H_2(\vartheta) > 1$. Following the rules of definition 3 we extend $\phi$ to a function $\phi^+$ defined on $(-\tau, 2\tau]$. The graphs of $\phi^+_1$, $\phi^+_2$ and $\phi^+_3$ are depicted in Figure 2.2 with the solid lines. Recall that, $\phi^+|_{(-\tau, 0]} = \phi$. The most important thing to notice is that the oscillator $O_3$ receives two simultaneous pulses at $t_f = 3\tau/2$ while $\phi^+_3(t_f) = \vartheta$. Therefore, $\phi^+_3(t_f) = \min\{1, H_2(\vartheta)\} \pmod{\mathbb{Z}} = 0$.

Then, consider an initial state $\psi$ given by $\psi_1(s) = \phi_1(s)$, $\psi_2(s) = \phi_2(s) - \epsilon$ and $\psi_3(s) = \phi_3(s)$ for $s \in (-\tau, 0]$, where $\epsilon > 0$ is small. The distance between $\phi$ and $\psi$ is

$$d(\phi, \psi) = 2\tau\epsilon = O(\epsilon).$$

The graphs of the components $\psi^+_1$, $\psi^+_2$ and $\psi^+_3$ of the extended phase history function $\psi^+$ are depicted in Figure 2.2 with the dashed lines. The main difference between $\phi$ and $\psi$
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Figure 2.2: An initial state $\phi \in \mathcal{P}_D$ for which the map $\Phi^\tau$ is discontinuous. The graphs of $\phi^+_j$, $j = 1, 2, 3$ are represented by the solid lines. The graphs of $\psi^+_j$, $j = 1, 2, 3$ where $\psi^+$ is the extended phase history function that corresponds to the initial state $\psi$ are represented by the dashed lines. Recall that $\phi$ (resp. $\psi$) is the restriction of $\phi^+$ (resp. $\psi^+$) to $(-\tau, 0]$ (represented by the gray region). The two states diverge abruptly after $t = 3\tau/2$. On the vertical axis, $a = \vartheta - \frac{5}{2}\tau$ and $b = 1 - \frac{3}{2}\tau$.

is that while in the former the oscillators $O_1$ and $O_2$ are synchronized, in the latter they are not. For this reason, the oscillator $O_3$ receives a single pulse at $t = t_f = 3\tau/2$ while $\psi^+_3(t_f) = \vartheta$. Then, $\psi^+_3(t_f) = \min\{1, H_1(\vartheta)\} \mod \mathbb{Z} = 0$, since we assumed before that $H_1(\vartheta) > 1$. $O_3$ receives a second pulse at $t = t_f + \epsilon$ while $\psi^+_3(t_f + \epsilon^-) = \epsilon$. Hence its phase becomes $\psi^+_3(t_f + \epsilon) = V_1(\epsilon) = V_1(0) + O(\epsilon)$. This would imply that for $s \in (3\tau/2 + \epsilon, 2\tau]$, we have that $\psi^+_3(s) - \phi^+_3(s) = V_1(0) + O(\epsilon)$ where $V_1(0) > 0$ does not depend on $\epsilon$. Then, it is easy to see that

$$d(\Phi^\tau(\psi), \Phi^\tau(\phi)) = \frac{1}{2}\tau V_1(0) + O(\epsilon).$$

This shows that the evolution operator $\Phi^\tau$ is discontinuous at $\phi$ with respect to the metric $d$.

We conjecture that the discontinuity of the evolution operator is independent of the choice of a ‘reasonable’ metric but depends only on the dynamics of the system, and in particular, on the fact that in the example above $O_3$ fires by receiving two simultaneous pulses but could have fired after receiving a single pulse. Also this discontinuity should not be confused with the fact that the phases of the oscillators are discontinuous functions of time. Motivated by the this discussion we introduce the following definition.
Definition 5. Given a system $\mathcal{D} = (n, V, \varepsilon, \tau)$ and $\theta \in \mathbb{T}$, we define $\nu(\theta)$ as the minimum positive integer for which $H_{\nu(\theta)}(\theta) := H(\theta, \nu(\theta)\frac{\varepsilon}{n-1}) \geq 1$.

In other words, $\nu(\theta)$ is the minimum number of pulses that will make an oscillator with phase $\theta$ to fire. Consider an oscillator whose phase at time $t$ is $\theta$ and fires after receiving $m$ pulses at $t$. We say that the oscillator overfires by $m - \nu(\theta)$ pulses at $t$ if $\nu(\theta) < m$, i.e., if the oscillator fires after receiving more simultaneous pulses than the strictly necessary number $\nu(\theta)$.

In the context of the example shown in Figure 2.2, we can say that the map $\Phi: \mathcal{P}_D \rightarrow \mathcal{P}_D$ is discontinuous at $\phi \in \mathcal{P}_D$ because the oscillator $O_3$ overfires by 1 pulse at $t = t_f = 3\tau/2$.

The existence of discontinuous evolution have also been observed in (Timme 2002, Timme et al. 2002c) where initial states that give such evolution are characterized as superunstable.

2.2 Other representations of the dynamics

In this section we present alternative representations of the dynamics. In this section we introduce, following (Ashwin and Timme 2005a), the past firings and the event representation.

2.2.1 The past firings representation

It follows from definition 3 that the evolution of an initial state $\phi \in \mathcal{P}_D$ only depends on the values $\phi_i(0)$ and the firing sets $\Sigma_i(\phi)$ that are defined as follows:

Definition 6. Given a phase history function $\phi \in \mathcal{P}_D$, the firing sets $\Sigma_i(\phi) \subset (-\tau, 0]$, $i = 1, \ldots, n$ are the sets of solutions of the equation $\phi_i(s) = 0$ for $s \in (-\tau, 0]$. The total firing set is given by

$$\Sigma(\phi) = \{(i, \sigma) : i = 1, \ldots, n, \sigma \in \Sigma_i(\phi), \}$$

Therefore, if we are interested only in the future evolution of the system we can consider the following equivalence relation in $\mathcal{P}_D$.

Definition 7. Two phase history functions $\phi_1, \phi_2$ in $\mathcal{P}_D$ are equivalent, denoted by $\phi_1 \sim \phi_2$, if $\phi_1(0) = \phi_2(0)$ and $\Sigma(\phi_1) = \Sigma(\phi_2)$. Let $\mathbb{P}_D = \mathcal{P}_D/\sim$ be the quotient set of equivalence classes and by $[\phi] \in \mathbb{P}_D$ denote the equivalence class of $\phi \in \mathcal{P}_D$.

Points $[\phi] \in \mathbb{P}_D$ are completely determined by the values of the phases $\phi_i(0)$ and the firing sets $\Sigma(\phi)$ (which may be empty). We denote the elements of $\Sigma_i(\phi)$ by $\sigma_{i,1} > \sigma_{i,2} >$
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\[ \cdots > \sigma_{i,k_i} \text{ where } k_i \text{ is the cardinality of } \Sigma_i(\phi). \] Note that by definition, \( \phi_i(0) \geq \sigma_{i,1} \), and \( \phi_i(0) = 0 \) if and only if \( \sigma_{i,1} = 0 \).

It is possible to give an equivalent description of the dynamics described by definition 3, using only the variables \( \phi_i(0) \) and \( \sigma_{i,j} \). For such a definition see (Ashwin and Timme 2005a).

Notice also that,

**Proposition 2.** If \( \phi_1 \sim \phi_2 \) then

1. \( \Phi^t(\phi_1) \sim \Phi^t(\phi_2) \) for \( t \geq 0 \), and
2. \( \Phi^t(\phi_1) = \Phi^t(\phi_2) \) for \( t \geq \tau \).

**Poincaré map**

Given a network of \( n \) oscillators with dynamics defined by the pulse response function \( V \), with pulse strength \( \varepsilon \) and with delay \( \tau \), we can simplify the study of the system \( D = (n, V, \varepsilon, \tau) \) by considering intersections of the positive semi-orbits \( O_+(\phi) \) with the set

\[ P = \{ \phi \in P_D : \phi_n(0) = 0 \}. \] (2.12)

The set \( P \) is called a (Poincaré) *surface of section* (Broer and Takens 2008 (to appear), Strogatz 1994) and it inherits the metric \( d \), see (2.11).

The evolution operator \( \Phi \), see (2.6), defines a map \( R : P \rightarrow P \) in the following way. Consider any \( \phi \in P \), i.e., such that \( \phi_n(0) = 0 \). Since the phases of the oscillators are always increasing there is a minimum time \( t(\phi) \) such that the phase of \( O_n \) becomes 0 again, i.e., such that \( \Phi^{t(\phi)}(\phi)_n(0) = 0 \). We define

\[ R(\phi) = \Phi^{t(\phi)}(\phi). \] (2.13)

The map \( R \) is called *Poincaré (return) map*. Furthermore, we can define the quotient map

\[ R_\sim : P / \sim \rightarrow P / \sim \text{ by } [\phi] \mapsto [R(\phi)], \]

of the Poincaré (return) map \( R \), where \( \sim \) is the equivalence relation given by definition 7. By Proposition 2 the map \( R_\sim \) is well defined.

2.2.2 The event representation

The *event representation* is a symbolic description of the dynamics in which the state of the system is represented by a sequence of events consisting of firings and pulse receptions that would occur. Each event \( E \) in the sequence is characterized by a triplet \([K(E), O(E), T(E)]\) where \( K(E) \) denotes the type of the event \( F \) or \( mP \). The event \( F \) denotes a firing event and \( mP \) \((m \in \mathbb{N})\) stands for the simultaneous reception of \( m \) pulses. The oscillator associated
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with event $K(E)$ is denoted by $O(E) \in \{1, \ldots, n\}$. Finally, $T(E) \in [0, 1]$ denotes how much time is left for the event to occur. For example, the event denoted by $[F, 2, 0.4]$ signifies that the oscillator $O_2$ will fire after 0.4 time units (and this means that its current phase is $1 - 0.4 = 0.6$), while the event denoted by $[P, 1, 0.3]$ signifies that $O_1$ is set to receive a pulse after 0.3 time units. We use the shorthand notation $[F, (i_1, \ldots, i_k), t]$ and $[mP, (i_1, \ldots, i_k), t]$ to indicate that the oscillators $O_{i_1}, \ldots, O_{i_k}$ fire or receive $m$ pulses respectively after time $t$.

Given a particular initial state $φ \in P_D$, such that its equivalence class $[φ] \in P_D$ is characterized by the phases $φ_i(0)$ and firing times $σ_{i,j}$ for $i = 1, \ldots, n$ and $j = 1, \ldots, k_i$, consider the space $A$ of event sequences $\{E_1, E_2, \ldots, E_k\}$ of finite (but not fixed) length and the map

$$ E : P_D \rightarrow A : [φ] \rightarrow E([φ]), \quad (2.14) $$

which maps $[φ]$ to the event sequence $E([φ])$ constructed in the following way. First, consider the set $Y$ consisting of the following events:

1. $[F, i, 1 - φ_i(0)]$ for $i = 1, \ldots, n$, and
2. $[P, ℓ, τ + σ_{i,j}]$ for $ℓ = 1, \ldots, n$ with $ℓ \neq i$ and $j = 1, \ldots, k_i$.

Then, impose time-ordering on $Y$ (i.e., order the events so that events that occur earlier appear first) and in the case that there are $m > 1$ identical events $[P, i, t]$ collect them together to $[mP, i, t]$ to obtain $E([φ])$. It follows that $E$ is injective and hence the inverse map $E^{-1} : E(P_D) \subset A \rightarrow P_D$ is well defined.

Next, define the map

$$ Φ_A : E(P_D) \rightarrow E(P_D) \quad (2.15) $$

using the following algorithm:

1. For $Z \in E(P_D)$, consider the first event $E_1 \in Z$ and let $t = T(E_1)$. If $T(E_1) \neq 0$ then set $T(E)$ to $T(E) - t$ for all $E \in Z$.

2. Take the sequence $Z_0$ of events $E \in Z$ with $T(E) = 0$ and define $Z_+ = Z \setminus Z_0$. For each event $E \in Z_0$ do the following:

   a) If $K(E) = F$, then
      i. append to $Z_+$ the event $[F, O(E), 1]$;
      ii. append to $Z_+$ the events $[P, ℓ, τ]$ for all $ℓ \in \{1, \ldots, n\}$ with $ℓ \neq O(E)$.

   b) If $K(E) = mP$, then
      i. find the (unique) event $E' \in Z_+$ with $K(E') = F$ and $O(E') = O(E)$;
      ii. set $T(E')$ to max$\{T(E') - V(1 - T(E'), m\bar{ε}), 0\}$. 

3. Impose time-ordering on $Z_+$ and collect together identical pulse events.

4. Set $\Phi_A(Z) = Z_+$.

It follows from the definition of $\Phi_A$ that:

**Proposition 3.**

1. The map $\Phi_A : \mathcal{E}(\mathbb{P}_D) \to \mathcal{E}(\mathbb{P}_D)$ is well defined.

2. $[\Phi^t(\phi)] = \mathcal{E}^{-1}(\Phi_A(Z))$ where $Z = \mathcal{E}([\phi])$ and $t$ is determined at the first step of the algorithm.

3. Consider an initial state $\phi \in \mathbb{P}_D$ and the corresponding event sequence $\mathcal{E}([\phi])$. If we apply $\Phi_A$, $m$ times to $\mathcal{E}([\phi])$ and the time that elapses at the $j$th ($j = 1, \ldots, m$) application is $t_j$ with $t = \sum_j t_j$, then there exists a unique reconstruction of the extended phase history function $\phi^+$ on the interval $[0, t]$.

The last part of Proposition 3 implies that if $t \geq \tau$ then it is possible to obtain from the sequence $\{Z, \Phi_A(Z), \Phi_A^2(Z), \ldots, \Phi_A^m(Z)\}$, where $Z = \mathcal{E}([\phi])$, not only the equivalence class $[\Phi^t(\phi)]$ but also the phase history function $\Phi^t(\phi) = \phi^+|_{(\tau,t]} \circ T_t$ for any time $t \in [\tau, t]$.

With this mathematical setting, we prove in chapters 3 and 4 that unstable attractors and heteroclinic cycles exist in an open region $(\epsilon, \tau)$ space for a global network of pulse coupled oscillators.


Appendices

2. Pulse coupled oscillator network with delay

2.A Properties of nearby phase history functions

Consider a phase history function \( \phi \in \mathcal{P}_D \). Then we show that the characteristics of \( \phi \) determine to a large extent the characteristics of nearby phase history functions \( \psi \in \mathcal{P}_D \). In particular, we have the following three Propositions. Notice that in the following we make no distinction between a phase history function \( \phi : (-\tau, 0] \rightarrow \mathbb{T}^n \) and the corresponding extending phase history function \( \phi^+ : (-\tau, \infty) \rightarrow \mathbb{T}^n \) and we denote both by \( \phi \).

**Proposition 4.** Assume that \( \phi_i \) has no discontinuities in an interval \( (s_1, s_2) \subset [-\tau, \tau] \) and that \( \phi_i(s) \neq 0 \) (mod \( \mathbb{Z} \)) for all \( s \in (s_1, s_2) \). Define \( E = \frac{1}{M}(s_2 - s_1) \) where \( M = \min\{V_1(0), 1 - \phi_i(s_2)\} \). Then, if \( \psi \in \mathcal{P}_D \) satisfies \( d(\phi, \psi) = \epsilon < E \) we find that \( |\phi_i(s) - \psi_i(s)| < \epsilon_2 = 2\epsilon/(s_2 - s_1) \) for all \( s \in (s_1 + \epsilon_1, s_2 - \epsilon_1) \), where \( \epsilon_1 = 2\epsilon/M \). In particular, \( \psi_i \) has no discontinuities in \( (s_1 + \epsilon_1, s_2 - \epsilon_1) \).

**Proof.** Assume, for simplicity, that \( s_1 = 0, s_2 = S < \tau \) and that \( \phi_i(s) = u + s \) with \( u > 0 \) and \( u + S < 1 \). Then \( M = \min\{V_1(0), 1 - (u + S)\} \).

Suppose that \( \psi_i(s) \) has one or more discontinuities in \((\epsilon_1, S - \epsilon_1)\) and that one of these discontinuities (caused by \( m \geq 1 \) simultaneous pulses) is at \( p \).

If \( \psi_i(p^+) \geq \phi_i(p) \), then

\[
d(\phi, \psi) \geq \int_p^S |\phi_i(s) - \psi_i(s)|ds \geq (\psi_i(p^+) - \phi_i(p))(S - p) \geq (\psi_i(p^+) - \phi_i(p))\epsilon_1
\]

The second inequality follows from the fact that \( \phi_i \) increases linearly, while \( \psi_i \) increases at least linearly. The third inequality follows from \( p < S - \epsilon_1 \). Similarly, if \( \psi_i(p^-) \leq \phi_i(p) \), then

\[
d(\phi, \psi) \geq \int_0^p |\phi_i(s) - \psi_i(s)|ds \geq (\phi_i(p) - \psi_i(p^-))p \geq (\phi_i(p) - \psi_i(p^-))\epsilon_1
\]

Again, the second inequality follows from the fact that \( \phi_i \) increases linearly, while \( \psi_i \) increases at least linearly. The third inequality follows from \( p > \epsilon_1 \). Then, we can distinguish three cases.

If both \( \psi_i(p^-) \) and \( \psi_i(p^+) \) are greater than \( \phi_i(p) \), then \( \psi_i(p^+) = \min\{1, \psi_i(p^-) + V_m(\psi_i(p^-))\} \). If \( \psi_i(p^+) = 1 \) then \( \psi_i(p^+) - \phi_i(p) = 1 - \phi_i(p) > 1 - (u + S) \geq M \). This means that

\[
d(\phi, \psi) \geq M\epsilon_1 = 2\epsilon
\]

which is a contradiction. If \( \psi_i(p^+) = \psi_i(p^-) + V_m(\psi_i(p^-)) \), then \( \psi_i(p^+) - \phi_i(p) = V_m(\psi_i(p^-)) + \psi_i(p^-) - \phi_i(p) \geq V_1(0) \geq M \) and we get again a contradiction.
2.A. Properties of nearby phase history functions

In the second case we assume that both $\psi_i(p^-)$ and $\psi_i(p^+)$ are smaller than $\phi_i(p)$. Then $\phi_i(p) > \psi_i(p^+) = \psi_i(p^-) + V_m(\psi_i(p^-)) > \psi_i(p^-) + V_1(0)$ so we get that $\phi_i(p) - \psi_i(p^-) \geq V_1(0) \geq M$ and

$$d(\phi, \psi) \geq M\epsilon_1 = 2\epsilon.$$

In the third case we assume that $\psi_i(p^-) < \phi_i(p)$ and $\psi_i(p^+) > \phi_i(p)$. This implies that $\phi_i(p) - \psi_i(p^-) = \kappa V_m(\psi_i(p^-)) > \kappa V_1(0) \geq \kappa M$ for some $\kappa \in (0, 1)$ and that $\psi_i(p^+) - \phi_i(p) = \min\{1 - \phi_i(p), (1 - \kappa)V_m(\psi_i(p^-))\} \geq (1 - \kappa)M$. Therefore

$$d(\phi, \psi) \geq \kappa M\epsilon_1 + (1 - \kappa)M\epsilon_1 = 2\epsilon.$$

In all cases we have reached a contradiction and this implies that $\psi_i$ can not have any discontinuities in $(\epsilon_1, S - \epsilon_1)$.

This implies also that $\psi_i(s) = u' + s$ for $s \in (\epsilon_1, S - \epsilon_1)$ and in particular that $\phi_i(s) - \psi_i(s) = u - u'$ is constant in this interval. Then, we obtain

$$d(\phi, \psi) \geq \int_{\epsilon_1}^{S - \epsilon_1} |\phi_i(s) - \psi_i(s)| ds = |u - u'|(S - 2\epsilon_1).$$

Hence, $|u - u'| \leq \epsilon/(S - 2\epsilon_1) \leq 2\epsilon/S$. This concludes the proof of the first statement.$\square$

Proposition 5. Assume that $\phi_i$ has a discontinuity at $p \in (-\tau, \tau)$, such that $\phi_i(p^+) = H_m(\phi_i(p^-))$ (i.e., the oscillator $O_i$ receives $m$ simultaneous pulses). Also assume that $\phi_i$ has no other discontinuities in the open interval $(p - \delta, p + \delta)$ and that $\phi_i(s) \notin \mathbb{Z}$ for all $s \in (p - \delta, p + \delta)$. Then, there is $E' > 0$ such that if $\psi \in \mathcal{P}_D$ satisfies $d(\phi, \psi) = \epsilon < E'$ we find that $\psi_i$ receives $m$ pulses in the interval $(p - \epsilon_1, p + \epsilon_1)$, where $\epsilon_1 = 2\epsilon/M$ and $M = \min\{\phi_i(p + \delta), V_1(0)\}$.

Proof. Since $\phi_i$ has no discontinuities in $(p - \delta, p)$ and $(p, p + \delta)$ we can apply the previous result in each one of these intervals. Define $M = \min\{1 - \phi_i(p + \delta), V_1(0)\}$. Then for any $\psi$ with $d(\phi, \psi) = \epsilon < \frac{1}{2}M\delta$ we conclude that $\psi_i$ has no discontinuities in the intervals $W_1 = (p - \delta + \epsilon_1, p - \epsilon_1)$ and $W_2 = (p + \epsilon_1, p + \delta - \epsilon_1)$, where $\epsilon_1 = 2\epsilon/M$ and $|\phi_i(s) - \psi_i(s)| < \epsilon_2 = 2\epsilon/\delta$ in the same intervals. Hence,

$$|\psi_i(p - \epsilon_1) - \phi_i(p^-) + \epsilon_1| < \epsilon_2,$$

and

$$|\psi_i(p + \epsilon_1) - \phi_i(p^+) - \epsilon_1| < \epsilon_2.$$

Combining the two inequalities we obtain,

$$2(\epsilon_1 - \epsilon_2) < \psi_i(p + \epsilon_1) - \psi_i(p - \epsilon_1) - V_m(\phi_i(p^-)) < 2(\epsilon_1 + \epsilon_2).$$

If $\psi_i$ has discontinuities in $(p - \epsilon_1, p + \epsilon_1)$ that correspond to reception of $\kappa$ pulses, then

$$\psi_i(p - \epsilon_1) + V_k(\psi_i(p - \epsilon_1)) + 2\epsilon_1 \leq \psi_i(p + \epsilon_1)$$

$$\leq \psi_i(p - \epsilon_1) + 2\epsilon_1 + V_k(\psi_i(p - \epsilon_1) + 2\epsilon_1),$$
Assume that
\[ V_\kappa(\psi_i(p - \epsilon_1)) + 2\epsilon_1 \leq \psi_i(p + \epsilon_1) - \psi_i(p - \epsilon_1) \leq 2\epsilon_1 + V_\kappa(\psi_i(p - \epsilon_1)) + V_\kappa'(\psi_i(p - \epsilon_1))2\epsilon_1 + O(\epsilon^2). \]

From \(|\psi_i(p - \epsilon_1) - \phi_i(p^-) + \epsilon_1| < \epsilon_2\), we obtain the estimate:
\[ V_\kappa(\phi_i(p^-) - (\epsilon_1 + \epsilon_2)) < V_\kappa(\psi_i(p - \epsilon_1)) < V_\kappa(\phi_i(p^-) + (\epsilon_2 - \epsilon_1)), \]
or
\[ V_\kappa(\phi_i(p^-)) - V_\kappa'(\phi_i(p^-))(\epsilon_1 + \epsilon_2) + O(\epsilon^2) < V_\kappa(\psi_i(p - \epsilon_1)) < V_\kappa(\phi_i(p^-)) + V_\kappa'(\phi_i(p^-))(\epsilon_2 - \epsilon_1) + O(\epsilon^2), \]

This implies that
\[ V_\kappa(\phi_i(p^-)) - V_\kappa'(\phi_i(p^-))(\epsilon_1 + \epsilon_2) + 2\epsilon_1 + O(\epsilon^2) \leq \psi_i(p + \epsilon_1) - \psi_i(p - \epsilon_1) \leq 2\epsilon_1 + V_\kappa(\phi_i(p^-)) + V_\kappa'(\phi_i(p^-))(\epsilon_2 - \epsilon_1) + V_\kappa'(\psi_i(p - \epsilon_1))2\epsilon_1 + O(\epsilon^2), \]

or,
\[ -V_\kappa'(\phi_i(p^-))(\epsilon_1 + \epsilon_2) + 2\epsilon_1 + O(\epsilon^2) \leq \psi_i(p + \epsilon_1) - \psi_i(p - \epsilon_1) - V_\kappa(\phi_i(p^-)) \leq 2\epsilon_1 + V_\kappa'(\phi_i(p^-))(\epsilon_1 + \epsilon_2) + O(\epsilon^2). \]

Combining inequalities we obtain
\[ -V_\kappa'(\phi_i(p^-))(\epsilon_1 + \epsilon_2) - 2\epsilon_2 + O(\epsilon^2) \leq V_\kappa(\phi_i(p^-)) - V_m(\phi_i(p^-)) \leq V_\kappa'(\phi_i(p^-))(\epsilon_2 + \epsilon_1) + 2\epsilon_2 + O(\epsilon^2). \]

If \(\kappa \neq m\) then the difference \(|V_\kappa(\phi_i(p^-)) - V_m(\phi_i(p^-))| > |\kappa - m|V_i(0)\) is bounded away from zero. This implies there is some positive \(E' < \frac{1}{\delta}M\delta\) such that the inequality 2.16 does not hold for any \(\kappa \neq m\) and \(\epsilon < E'\). Therefore, if \(\epsilon < E'\) we conclude that \(\kappa = m\). This concludes the proof of this part.

**Proposition 6.** Assume that \(\phi_i\) has a discontinuity at \(p \in (-\tau, \tau)\), such that \(\phi_i(p^+) = 1\) (i.e., the oscillator \(O_i\) receives \(m \geq \nu(\phi_i(p^+))\) simultaneous pulses and fires). Also assume that \(\phi_i\) has no other discontinuities in the open interval \((p - \delta, p + \delta)\) and that \(\phi_i(s) \notin \mathbb{Z}\) for all \(s \in (p - \delta, p + \delta)\). Then, there is \(E' > 0\) such that if \(\psi \in \mathcal{P}_D\) satisfies \(d(\phi, \psi) = \epsilon < E'\) we find that \(\psi_i\) receives at least \(\nu(\phi_i(p^-))\) pulses in the interval \((p - \epsilon_1, p + \epsilon_1)\), where \(\epsilon_1 = 2\epsilon/M\) and \(M = \min\{\phi_i(p + \delta), V_i(0)\}\). If \(\psi_i\) receives \(m'\) pulses with \(m' > \nu(\phi_i(p^-))\) then the last \(m' - \nu(\phi_i(p^-)) + 1\) pulses are simultaneous.
Proof. Since $\phi_i$ has no discontinuities in $(p - \delta, p)$ and $(p, p + \delta)$ we can apply the previous result in each one of these intervals. Define $M = \min\{1 - \phi_i(p + \delta), V_i(0)\}$. Then for any $\psi$ with $d(\phi, \psi) = \epsilon < \frac{1}{2} M \delta$ we conclude that $\psi_i$ has no discontinuities in the intervals $W_1 = (p - \delta + \epsilon_1, p - \epsilon_1)$ and $W_2 = (p + \epsilon_1, p + \delta - \epsilon_1)$, where $\epsilon_1 = 2\epsilon / M$ and $|\phi_i(s) - \psi_i(s)| < \epsilon_2 = 2\epsilon / \delta$ in the same intervals. Hence,

$$|\psi_i(p - \epsilon_1) - \phi_i(p^-) + \epsilon_1| < \epsilon_2; \text{ and}$$

$$|\psi_i(p + \epsilon_1) - 1 - \epsilon_1| < \epsilon_2.$$

Combining the two inequalities we obtain,

$$2(\epsilon_1 - \epsilon_2) < \psi_i(p + \epsilon_1) - \psi_i(p - \epsilon_1) - (1 - \phi_i(p^-)) < 2(\epsilon_1 + \epsilon_2).$$

If $\psi_i$ has discontinuities in $(p - \epsilon_1, p + \epsilon_1)$ that correspond to reception of $\kappa < \nu(\phi_i(p))$ pulses, then

$$\psi_i(p + \epsilon_1) \leq \psi_i(p - \epsilon_1) + 2\epsilon_1 + V'_\kappa(\psi_i(p - \epsilon_1) + 2\epsilon_1),$$

or

$$\psi_i(p + \epsilon_1) \leq \psi_i(p - \epsilon_1) + 2\epsilon_1 + V'_\kappa(\psi_i(p - \epsilon_1) + V'_\kappa(\psi_i(p - \epsilon_1))2\epsilon_1 + O(\epsilon^2).$$

As in the previous Proposition, since $|\psi_i(p - \epsilon_1) - \phi_i(p^-) + \epsilon_1| < \epsilon_2$, we obtain the estimate:

$$V'_\kappa(\phi_i(p^-)) - V'_\kappa(\phi_i(p^-))(\epsilon_1 + \epsilon_2) + O(\epsilon^2) < V'_\kappa(\psi_i(p - \epsilon_1))$$

$$< V'_\kappa(\phi_i(p^-)) + V'_\kappa(\phi_i(p^-))(\epsilon_2 - \epsilon_1) + O(\epsilon^2),$$

Therefore

$$\psi_i(p + \epsilon_1) \leq H'_\kappa(\phi_i(p^-)) + \epsilon_2 + \epsilon_1 + V'_\kappa(\phi_i(p^-))(\epsilon_2 - \epsilon_1) + V'_\kappa(\psi_i(p - \epsilon_1))2\epsilon_1 + O(\epsilon^2).$$

Since $\kappa < \nu(\phi_i(p))$ we deduce that by taking $\epsilon$ small enough we could have $\psi_i(p + \epsilon_1) < 1$ which is a contradiction. This implies that $\kappa \geq \nu(\phi_i(p^-))$. Moreover, if $\kappa > \nu(\phi_i(p^-))$ then the last $\kappa - \nu(\phi_i(p^-)) + 1$ pulses must be simultaneous, otherwise $\psi_i(p + \epsilon_1) > H_1(0)$ which gives a contradiction since $\psi_i(p + \epsilon_1) = O(\epsilon)$.  

\[\Box\]