Abstract— In this paper we present a version of the balancing technique for nonlinear positive/bounded real systems. We discuss an approach to study related nonlinear systems that allow us to apply the theory of balancing based upon the Hankel operator and the corresponding singular value analysis. We provide balanced realizations of these systems. If we truncate them accordingly, the reduced order models will preserve their original structure, i.e., they are positive/bounded real.

I. INTRODUCTION

Balancing with preservation of dissipativity properties for linear systems is an attractive tool for obtaining low order models in applications like electric circuits, mechanical systems or electromechanical systems, e.g. power systems (see [1], [2]). We refer to two particular and common cases: the passive systems case, i.e. the system is dissipative with respect to the supply rate

\[ s(u, y) = u^T y \]  

and to the bounded real (finite L2 gain) case, i.e. the system is dissipative with respect to

\[ s(u, y) = \frac{1}{2}(||u||_2 - ||y||_2) \]  

where \( u \) and \( y \) are the input and the output of the system, respectively. The notion of passivity is closely related to the notion of positive real systems, i.e. the supplied energy is always positive. In the linear case passivity is equivalent to positive realness. The positive and bounded real cases are important for stability analysis and control.

Basically, a system is called balanced when the required input energy to be supplied in the past to reach a state and the future available energy stored by the same state can be (directly) quantified, showing how much energy that state dissipates.

The linear case has been widely treated in literature, and we refer to e.g. [1], [3], for positive realness and to e.g. [1], [4], [5], [6] for the bounded real and other cases. For nonlinear systems, in [7] the positive real balanced case was treated, combining the nonlinear balancing method developed in [8] with the passivity theory ([9], [10], [11], [12]). The positive real singular value functions were defined, representing the energy measure of the states in a particular state space representation of the system. If the singular value function associated to a state is large, then that state is less dissipative and conversely, if the singular value is small then the state dissipates more energy. Truncating the more or the less dissipative states, a passive reduced order model is obtained.

Being dependent on the particular state space representation, these singular values are not invariants as in the linear case. To overcome this drawback we propose a different approach, based on the recent results from [13], [14]. Basically, we turn the problem of finding the positive/bounded real singular value functions into the Hankel singular value problem of an asymptotically stable extended realization. Then the axis singular value functions of this new system are directly related to the positive/bounded real singular value functions of the original system. They are a coordinate free energy measure of the states. We also provide positive/bounded real balanced realizations. In the end, truncation of the balanced realization yields a positive/bounded real reduced order model.

In Section II, a brief overview of the dissipativity theory is given with respect to a nonlinear system, as well as of the energy functions, the available storage and the required supply. We also present the positive/bounded real balancing case for linear systems. Section III deals with constructing the new state-space realization whose controllability and observability function are the storage functions defined for the original positive/bounded real system. Section IV deals with a factorization approach where a new pair of energy functions is derived. In Section V the relation between the axis singular value functions of the original system, with respect to the available storage and the required supply is provided. Also the balanced realizations are written and model reduction is performed yielding passive reduced order models. We end the paper with some conclusions.

II. PRELIMINARIES

We will treat the following class of nonlinear systems:

\[
\begin{align*}
\dot{x} &= f(x) + g(x)u \\
y &= h(x) + d(x)u
\end{align*}
\]  

where \( x \in \mathbb{R}^n \), is the state vector, \( u, y \in \mathbb{R}^m \), are the input vector and the output vector, respectively, of the dynamic system and the vectorfields \( f(x), g(x), h(x) \) are smooth.
A. Dissipativity and storage functions

The property of a system being dissipative with respect to a supply rate is described by the following:

**Definition 1.** [9], [10] A system (3) is called dissipative with respect to a supply rate \( s(u,y) \), if there exists a storage function \( S : \mathbb{R}^n \rightarrow \mathbb{R} \), such that

\[
S(x_0) + \int_{t_0}^{t_1} s(u,y)dt \geq S(x_1),
\]

where \( x_0 = x(t_0) \) and \( x_1 = x(t_1) \).

**Remark 2.** Definition 1 can be also written in differential form as:

\[
\frac{dS(x)}{dt}(f(x) + g(x)u) \leq s(u,y),
\]

called the differential dissipation inequality.

We are interested in the study of two particular energy storage functions: the available stored energy at the ports at the state \( x_0 \) and the required supply at the ports.

**Definition 3.** [9], [10] The available storage function of a system (3) is the energy function:

\[
S_a(x(0)) = \sup_{u \in L_2(0,\infty)} - \int_0^\infty s(u(t),y(t)) dt, \; x(\infty) = 0.
\]

The required supply function of system (3) is the energy function:

\[
S_r(x(0)) = \inf_{u \in L_2(-\infty,0)} \int_{-\infty}^0 s(u(t),y(t)) dt, \; x(-\infty) = 0.
\]

\( S_a(x) \) represents the maximal amount of energy that can be extracted from the terminals of the system when starting at the initial state \( x_0 \). \( S_r(x) \) represents the minimal amount of energy required to be supplied to the system in order to reach \( x_0 \) from the equilibrium.

The property of the system being reachable from \( x_0 \) is a condition for the existence and nonnegativity of the energy functions defined in relations (6) and (7), see e.g. [15], [9].

**Lemma 4.** [15] Let system (3) be dissipative with respect to \( s(u,y) \) as in Definition 1. Moreover assume the system is asymptotically reachable. Then, the energy functions \( S_a \) and \( S_r \) as in Definition 6 exist and are nonnegative. Moreover, \( S_a \leq S_r \).

For the positive definiteness of the storage functions of a dissipative system, additional assumptions are required.

**Lemma 5.** [16] Assume system (3) is dissipative as in Definition 1 and it is reachable and zero-state observable. Moreover, assume there exists \( u \), such that \( s(u,y) \leq 0 \). Then any storage function \( S \) that fulfills (4) is positive definite.

Throughout the paper, we consider that the assumption on \( s(u,y) \) made in Lemma 5 holds.

B. Positive real balancing for linear systems

Consider a linear system: \( \dot{x} = Ax + Bu, \; y = Cx + Du \), where \( A, B, C, D \) are constant matrices of appropriate dimensions. We give a brief overview of the positive real balanced truncation technique that yields positive real reduced order models.

**Definition 6.** [3], [1] A linear, square system \( G(s) = C(sI - A)^{-1}B + D \) is called positive real if

\[
G^T(-j\omega) + G(j\omega) \geq 0, \; \forall \omega \in \mathbb{R}.
\]

If the inequalities are strict then the system is called strictly positive real.

The system is assumed to be reachable and observable (minimal) and strictly positive real. Then, strict positive realness, can be studied with the Kalman-Yakubovich-Popov lemma, see e.g. [1]. The energy functions are quadratic and related to a pair of matrices called the positive real Gramians of the system.

**Theorem 7.** [9] Assume that the linear system is strictly positive real. Then \( S_a(x) = \frac{1}{2}x^T K_{min}x \) and \( S_r(x) = \frac{1}{2}x^T K_{max}x \), where \( K_{min} > 0 \) and \( K_{max} > 0 \) are the minimal stabilizing, respectively maximal antistabilizing solution of the Positive Real Algebraic Riccati equation:

\[
KA + A^TK + (KB - C^T)R^{-1}(B^TK - C) = 0,
\]

where \( R = D + D^T \).

**Definition 8.** [1] A positive real linear system is called positive real balanced if \( K_{min} = K_{max}^1 = \text{diag}(\pi_1, \pi_2, \ldots, \pi_n) \), where \( 1 \geq \pi_1 > \pi_2 > \ldots > \pi_n \geq 0 \) are the positive real singular values.

Let \( (A,B,C,D) \) be a positive real balanced system. Assume that \( \pi_k \gg \pi_{k+1} \) and accordingly partition the state-space realization into:

\[
A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C = [C_1 \; C_2],
\]

\[
A_{11} \in \mathbb{R}^{k \times k}, \quad B_1 \in \mathbb{R}^{k \times m}, \quad C_1 \in \mathbb{R}^{p \times k}.
\]

**Theorem 9.** [1], [3] Let the reduced order model \( G_r(s) = C_1(sI - A_1)^{-1}B_1 + D \) be obtained by truncation, i.e. set the states from \( k+1 \) to \( n \) to 0. Then \( G_r(s) \) is asymptotically stable, minimal and strictly positive real.

C. Bounded real balancing for linear systems

**Definition 10.** [3], [17], [5]. A minimal, square, asymptotically stable system with the transfer function \( G(s) = C(sI - A)^{-1}B + D \) is bounded real if it satisfies the following property:

\[
I - G^T(-j\omega)G(j\omega) \geq 0, \; \forall \omega \in \mathbb{R}.
\]

If the inequalities are strict, then the system is called strictly bounded real.
Equivalently, $(A, B, C, D)$ is bounded real, if there exists $K > 0$ that satisfies the bounded real Riccati equation (15):

$$AK + KA + (KB - C^TD)R_b^{-1}(B^TK - D^TC) + C^TC = 0,$$

with $R_b = I - D^TD$. This equation admits a minimal stabilizing solution $K_{\text{min}} > 0$ and a maximal antistabilizing solution $K_{\text{max}} > 0$ and $S_a(x) = \frac{1}{2}x^TK_{\text{min}}x$, $S_r(x) = \frac{1}{2}x^TK_{\text{max}}x$.

Lemma 11. [17] $K_{\text{min}}$ and $K_{\text{max}}^{-1}$ are the observability and controllability Gramians of the state-space realization $(A_c, B_c, C_c)$, where denoting $L_b = I - DD^T$, we have:

$$A_c = A - BR_b^{-1}D^TC, \quad B_c = [BR_b^{-1/2} \ K_{\text{max}}^{-1}C^T L_b^{-1/2}],$$

$$C_c = \begin{bmatrix} L_b^{-1/2}C \\ R_b^{-1/2}B^TK_{\text{min}} \end{bmatrix}.$$

Definition 12. [3] A bounded real system is called bounded real balanced if $K_{\text{min}} = K_{\text{max}}^{-1} = \text{diag}(\nu_1, \nu_2, ..., \nu_n)$, where $1 \geq \nu_1 > \nu_2 > ... > \nu_n > 0$ are called the bounded real singular values.

Let $(A, B, C, D)$ be a bounded real balanced system. Assume that $\nu_k \gg \nu_{k+1}$ and accordingly partition the state-space realization as:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C = [C_1 \ C_2].$$

Theorem 13. [1], [3] Let the reduced order model $G_r(s) = C_1(sI - A_{11})^{-1}B_1 + D$ be obtained by truncation, i.e. set the states from $k+1$ to $n$ to 0. Then $G_r(s)$ is asymptotically stable, minimal and strictly bounded real.

D. Controllability and observability functions

In this section we briefly describe the controllability and observability energy functions associated to a nonlinear system. Assume system (3) is asymptotically stable in a neighbourhood of 0.

The controllability and observability energy functions of (3) are described by

$$L_c(x_0) = \min_{u \in L_2(-\infty, 0), x(0)=x_0} \frac{1}{2} \int_{-\infty}^{0} \|u(t)\|^2 dt$$

(9)

$$L_o(x_0) = \frac{1}{2} \int_{0}^{\infty} \|y(t)\|^2 dt, \quad x(0) = x_0, \quad x(\infty) = 0. \quad (10)$$

Under the assumption of asymptotic reachability, the controllability function is the antistabilizing solution of Hamilton-Jacobi equation. Respectively, the observability function is the solution of a nonlinear Lyapunov equation, if the system is zero-state observable.

Theorem 14. [8] Consider system (3) such that the system is asymptotically stable about 0. If the system is asymptotically reachable, then $L_c(x_0) > 0$ $L_c(0) = 0$ exists and satisfies the Hamilton-Jacobi equation:

$$\frac{\partial L_c(x)}{\partial x} f(x) + \frac{1}{2} \frac{\partial^2 L_c(x)}{\partial x^2} g(x)g^T(x) \frac{\partial L_c(x)}{\partial x} = 0 \quad (11)$$

such that $- (f(x) + g(x)g^T(x) \frac{\partial L_c(x)}{\partial x})$ is asymptotically stable. If the system is zero-state observable, then $L_o(x_0) > 0$, $L_o(0) = 0$ exists and satisfies the nonlinear Lyapunov equation:

$$\frac{\partial L_o(x)}{\partial x} f(x) + \frac{1}{2} h^T(x)h(x) = 0. \quad (12)$$

III. ENERGY FUNCTIONS OF STATE-SPACE REALIZATIONS

A. The passivity case

A system (3) is called passive if it is dissipative with respect to the supply rate (1). Assume for the rest of the sequel that matrix

$$r(x) = d(x) + d^T(x) \quad (13)$$

is positive definite.

The notion of passivity is related to the (linear) notion of positive realness, i.e. the energy supply is always positive.

Definition 15. [11] A system (3) is called positive real if, for all $u \in L_2(\mathbb{R}^p)$,

$$\int_{0}^{t} u(\tau)^T y(\tau) d\tau \geq 0, \quad (14)$$

whenever $x(0) = 0$.\[4312\]

Combined with Lemma 4, we obtain the link between passivity and positive realness:

Proposition 16. [11] A passive system (3) is positive real. Conversely, a positive real system (3), that is reachable, is passive.\[4312\]

Remark 17. If inequality (14) is strict, the system is strictly positive real. Equivalently, if a system is strictly passive, then the system is also strictly positive real.\[4312\]

For a positive real, reachable nonlinear system, the available storage (6) and the required supply (7) are the stabilizing and antistabilizing solutions, respectively, of a Hamilton-Jacobi equation, which is the nonlinear extension of the Positive Real Riccati Equation (8).

Theorem 18. [7] Let system (3) be positive real and reachable. Then the Hamilton-Jacobi equation:

$$\frac{\partial S}{\partial x} f(x) + \frac{1}{2} \left( \frac{\partial S}{\partial x} g(x) - h^T(x) \right) r^{-1} \left( \frac{\partial S}{\partial x} g(x) - h(x) \right)^T = 0 \quad (15)$$

has the smooth solution $S_a(x), S_a(0) = 0$, such that

$$f(x) + g(x)r^{-1}(x) \left( g^T(x) \frac{\partial S_a}{\partial x} - h(x) \right) = 0 \quad (16)$$
is asymptotically stable and the smooth solution \( S_r(x), S_r(0) = 0 \), such that
\[
- \left( f(x) + g(x)r^{-1}(x) \left( g^T(x) \frac{\partial S_r}{\partial x} - h(x) \right) \right) \tag{17}
\]
is asymptotically stable.

Moreover, the available storage of system (3) can be written as the observability function of an asymptotically stable, zero-state observable extended system.

**Theorem 19.** Assume that system (3) is reachable, zero-state observable and positive real and that the vector field \( f(x) - g(x)r^{-1}(x)h(x) \) is asymptotically stable. Then \( S_n(x) \) is the observability function for the following system:
\[
\begin{align*}
\dot{x} &= f(x) - g(x)r^{-1}(x)h(x) \\
y_1 &= -r^{-\frac{1}{2}}(x)g^T(x)\frac{\partial S_n(x)}{\partial x} \\
y_2 &= r^{-\frac{1}{2}}(x)h(x)
\end{align*}
\tag{18}
\]

**Proof:** Since the system (3) is assumed passive, reachable and zero state observable, then \( S_n(x) > 0 \) exists and satisfies equation (15). Using the asymptotic stability of \( f - gr^{-1}h \), we find that \( S_n \) satisfies equation (12) for (18). Finally applying Theorem 14, the result is proven. \( \square \)

Following the reasoning in the proof of Theorem 19 we find the required supply (7) to be the controllability function of an extended system as in the following:

**Theorem 20.** Assume system (3) is asymptotically reachable and positive real. Then \( S_r(x) \) is the controllability function for the following system:
\[
\begin{align*}
\dot{x} &= f(x) - g(x)r^{-1}(x)h(x) + g(x)r^{-\frac{1}{2}}u_1 + K(x)r^{-\frac{1}{2}}u_2, \tag{19}
\end{align*}
\]
where \( K(x) \) satisfies \( \frac{\partial S_r(x)}{\partial x}K(x) = h^T(x) \).

**Proof:** Summarizing, if the positive real system (3) asymptotically reachable and zero-state observable then the system
\[
\begin{align*}
\dot{x} &= f(x) - g(x)r^{-1}(x)h(x) - g(x)r^{-\frac{1}{2}}u_1 + K(x)r^{-\frac{1}{2}}u_2 \\
y_1 &= -r^{-\frac{1}{2}}(x)g^T(x)\frac{\partial S_n(x)}{\partial x} \\
y_2 &= r^{-\frac{1}{2}}(x)h(x)
\end{align*}
\tag{20}
\]
is asymptotically reachable and zero-state observable with the controllability function \( S_r(x) > 0 \) and the observability function \( S_n(x) > 0 \).

**B. The bounded real case**

In this case, the system is considered dissipative with respect to (2). Define
\[
r(x) = I - d(x)^Td(x). \tag{21}
\]
and assume \( r(x) > 0 \). In this case, the available storage and the required supply are the stabilizing and antistabilizing solutions of the Hamilton-Jacobi equation (consistent with [18] when \( d(x) = 0 \)):
\[
\frac{\partial S}{\partial x}f(x) + \frac{1}{2} \left( \frac{\partial S}{\partial x}g(x) - h^T(x)ds \right) + \frac{1}{2}h^T(x)h(x) = 0. \tag{22}
\]

Following the lines in Section III-A we find that the available storage and the required supply are the controllability and observability function of an extended system:

**Theorem 21.** Assume that system (3) is asymptotically reachable, zero-state observable and bounded real and moreover assume that \( l(x) = I - d(x)^Td(x) > 0 \). Then the available storage \( S_n \) and the required supply \( S_r \) are the observability and controllability functions, respectively, of the following extended system:
\[
\begin{align*}
\dot{x} &= f(x) - g(x)r^{-1}(x)h(x) - g(x)r^{-1/2}u_1 + K(x)r^{-1/2}u_2 \\
y_1 &= -r^{-\frac{1}{2}}(x)g^T(x)\frac{\partial S_n(x)}{\partial x} \\
y_2 &= l^{-\frac{1}{2}}(x)h(x),
\end{align*}
\tag{23}
\]
where \( K(x) \) satisfies \( \frac{\partial S_r(x)}{\partial x}K(x) = h^T(x) \).

**IV. A FACTORIZATION APPROACH**

In this section we give a different interpretation of the available storage and the required supply, in terms of a different pair of energy functions for a factorization system. This is a general extension of the normalized coprime factorization idea \( (J = I) \) presented in [19].

We will consider here the positive real case, the bounded real one being treated in a similar fashion. We assume that \( r(x) > 0 \). If a storage function \( S(x) \) satisfying (1) or (5) is the solution of the Hamilton-Jacobi equation (15) then it also immediately satisfies the following equation, equivalently:
\[
\frac{\partial S}{\partial x} \left( f(x) + g(x)r^{-1}(x) \left( g^T(x)\frac{\partial S}{\partial x} - h(x) \right) \right) - \frac{1}{2} \frac{\partial S}{\partial x}g(x)r^{-1}(x)g^T(x)\frac{\partial S}{\partial x} + \frac{1}{2}h^T(x)r^{-1}(x)h(x) = 0 \tag{24}
\]

We have the following result, relating the minimal and the maximal solutions of (15) or (24) to the controllability, and observability of a closed loop system.

**Theorem 22.** Let system (3) be asymptotically reachable, zero-state observable and positive real such that \( S_n(x) < S_r(x) \). Then \( S_r(x) - S_n(x) \) is the controllability function of the following closed loop system:
\[
\dot{x} = f(x) + g(x)r^{-1} \left( g^T(x)\frac{\partial S_n(x)}{\partial x} - h(x) \right) + g(x)r^{-1/2}v. \tag{25}
\]

For the following line of thinking we need a slightly more general observability function definition.

**Definition 23.** Given a system (3) we can define the following observability function \( F_{\alpha}^M(x) = \).
that (16) and (17) hold and moreover, 0 < S_a < S_r.

If we assume that the system fulfills the Assumption that $f(x) - g(x)r^{-1}(x)h(x)$ is asymptotically stable, in Section III, then the available storage and the required supply of 3 are the observability and controllability functions of the asymptotically stable system (20). Hence, the Hankel singular value problem of (20) is equivalent to the singular value problem with respect to the available storage and the required supply. Hence, the positive real singular value functions of (3) are the Hankel singular value functions of the extended system (20).

If the assumption in Section III is not satisfied, then we turn to the results in Section IV. If Assumption 25 holds, then the system (28) is asymptotically stable with the energy functions $\bar{L}_c(x) > 0$, $\bar{L}_o(0) = 0$ and $\bar{L}_d'(x) > 0$, $\bar{L}_d'(0) = 0$. Then for (28) the nonlinear Hankel singular value problem developed in [13] is treated. For this system the axis singular value functions can be defined as:

$$\rho_i^2(s) = \frac{\bar{L}_d'(\xi_i(s))}{\bar{L}_c(\xi_i(s))}$$

and they satisfy the following theorem:

**Theorem 26.** [13] Suppose that the linearization of (28) has non-zero distinct Hankel singular values. Then, there exists a neighbourhood $U \subset \mathbb{R}$ of 0 and $\rho_i(s) > 0$, $i = 1, ... n$ such that: $\min\{\rho_i(s), \rho_i(-s)\} \geq \max\{\rho_{i+1}(s), \rho_{i+1}(-s)\}$ holds for all $s \in U$, $i = 1, ..., n - 1$. Moreover, there exist $\xi_i(s)$, satisfying the following:

$$L_c(\xi_i(s)) = \frac{s^2}{2}, L_o(\xi_i(s)) = \frac{\rho_i(s)s^2}{2},$$

$$\frac{\partial L_o}{\partial x}(\xi_i(s)) = \lambda_i(s)\frac{\partial L_c}{\partial x}(\xi_i(s)),$$

$$\lambda_i(s) = \rho_i^2(s) + \frac{s}{2}d\rho_i^2(s).$$

Even more, if $U = \mathbb{R}$, the Hankel norm of the system is $\sup_{s \in U} \rho_i(s)$.

For the original system (3), we define positive real the axis singular values, in the coordinates $x = \xi(s)$ with respect to the balancing of $S_a$ and $S_r$ as:

$$\pi_i^2(\xi_i(s)) = \frac{S_a(\xi_i(s))}{S_r(\xi_i(s))}.$$  

They express the gain between the effort port energy to reach the state $x_i = \xi_i(s)$ to 0 and the maximum stored energy available at the ports in the future, at the same state. The $\pi_i$'s of (3) are straightforwardly related to the $\rho_i$'s of (28):

**Theorem 27.** Assume that system (28) satisfies the assumptions from the preamble of Theorem 26 so that $\rho_i(s)$ exist. Then if $\pi_i(s)$ are the axis singular values from balancing $S_a$ and $S_r$, then:

$$\pi_i(s) = \frac{\rho_i(s)}{\sqrt{1 + \rho_i^2(s)}}$$

This equation constitutes the relation between the axis positive real singular value functions of (3) and the axis singular value functions of system (28). It can be easily checked that $\pi$ is a monotonously increasing function of $\pi$ and moreover $\rho_i(s) \leq 1$ and $\pi_i(s) \leq 1$. Now we are ready to write input-normal output diagonal and balanced realizations for a strictly positive/bounded real system, with the balanced energy functions $S_{a,r}$.
Theorem 28. Assume that system (3) satisfies Assumption 25. Then there exists a coordinate transformation \( x = \Phi(z) \) such that:
\[
S_r(\Phi(z)) = \frac{1}{2} z^T z \quad \text{and} \quad S_a(\Phi(z)) = \frac{1}{2} \sum_{i=1}^{n} z_i^2 \pi_i^a(z_i),
\]
(33)
where \( \pi_i(z_i) = \pi_i(\Phi(z_i)) \).
□

Using the same line of thinking we write a balanced realization.

Theorem 29. There exists a coordinate transformation \( x = \Phi(z) \) such that:
\[
S_a(\Phi(z)) = \frac{1}{2} \sum_{i=1}^{n} z_i^2 \pi_i(z_i) \quad \text{and} \quad S_r(\Phi(z)) = \frac{1}{2} \sum_{i=1}^{n} z_i^2 \pi_i(z_i),
\]
(34)
where \( \pi_i(z_i) = \pi_i(\Phi(z_i)) \).
□

For model reduction we assume that the system is in the form (34). Moreover assume there exists a \( k = 1, \ldots, n \) such that the singular value functions \( \pi(s) \) or \( \rho(s) \) satisfy the following relation:
\[
\max_{\pm s} \pi_k(s) \gg \max_{\pm s} \pi_{k+1}(s),
\]
equivalent to \( \max_{\pm s} \rho_k(s) \gg \max_{\pm s} \rho_{k+1}(s) \). Then the states \( z^1 = [z_1 \ldots z_k]^T \) require less energy at the ports to be reached and there is more available stored energy at the ports that the states \( z^2 = [z_{k+1} \ldots z_n]^T \). Splitting system (3) accordingly, we get:
\[
f(z) = \begin{bmatrix} f^1(z^1, z^2) \\ f^2(z^1, z^2) \end{bmatrix}, \quad g(z) = \begin{bmatrix} g^1(z^1, z^2) \\ g^2(z^1, z^2) \end{bmatrix}, \quad h(z) = h(z_1, z_2).
\]

If we truncate the states \( z^2 \), that is we set \( z^2 = 0 \) i.e. set \( z^2 = 0 \), we obtain two subsystems:
\[
\Sigma^1 : \quad z^1 = f^1(z^1, 0) + g^1(z^1, 0)u, \quad y^1 = h(z^1, 0) + d(z_1, 0)u,
\]
\[
\Sigma^2 : \quad z^2 = f^2(0, z^2) + g(0, z^2)u, \quad y^2 = h(0, z^2) + d(0, z^2)u.
\]
(35)
which have the following properties:

Theorem 30. The subsystems \( \Sigma^{1,2} \) are balanced in the sense of Theorem 29 with the following properties:
\[
S_a^1(z^1) = S_a(z^1, 0), \quad S_r^1(z^1) = S_r(z^1, 0) \quad \text{and} \quad S_a^2(z^2) = S_a(0, z^2), \quad S_r^2(z^2) = S_r(0, z^2).
\]
(36)
The singular value functions of subsystem \( \Sigma^1 \) are \( \pi_i(z_i, 0) \), \( i = 1, k \) and the singular value functions of subsystem \( \Sigma^2 \) are \( \pi_j(0, z_j) \), \( j = k+1, n \). Moreover \( \Sigma^{1,2} \) are positive/bounded real.
□

Remark 31. A physical example of applying the passivity preserving balanced truncation is found in [21] consisting of a nonlinear electrical circuit which is passive.
□

VI. CONCLUSIONS

In this paper we applied the balancing theory based on the nonlinear Hankel norm approach, to the case of positive/bounded real systems. The starting idea is turning the available storage and the required supply into the controllability and observability functions of new state-space realizations. The Hankel singular value functions of these systems are related to the original singular value functions defined with respect to the available storage and the required supply. Using this relation balanced realization are provided. Truncating the more or less dissipative states, lower order approximations are obtained. These approximations preserve the original passivity/L-2 gain property.

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REFERENCES