Full-order observer design for a class of port-Hamiltonian systems

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Abstract

We consider a special class of port-Hamiltonian systems for which we propose a design methodology for constructing globally exponentially stable full-order observers using a passivity based approach. The essential idea is to make the augmented system consisting of the plant and the observer dynamics strictly passive with respect to an invariant manifold defined on the extended state-space, on which the state estimation error is zero. We first introduce the concept of passivity of a system with respect to a manifold by defining a new input and output on the extended state-space and then perform a partial state feedback passivation which leads to the construction of the observer. We then illustrate this observer design procedure on two physical examples, the magnetic levitation system and the inverted pendulum on the cart system.

1. Introduction

The first attempts towards nonlinear observer design were to identify necessary and sufficient conditions on a nonlinear system for converting it into a simpler form like a linear or a bilinear system up to an output injection term, for which an observer can be easily constructed (Besancon, 1999; Hammouri & Gauthier, 1989; Krener & Respondek, 1985; Levine & Marino, 1986; Xia & Gao, 1989). See also Shim, Seo, and Teel (2003) and the references in there. Another class of nonlinear systems that was studied consisted of those in which the state-dependent nonlinearities satisfied certain conditions, like being globally Lipschitz as studied by Rajamani (1998), Thau (1973) and Tsinias (1989), being a monotonic function of a linear combination of the states as in Arcak and Kokotovic (2001), Fan and Arcak (2003) or have a bounded slope (Arcak & Kokotovic, 2001). The observer design for such systems was performed by employing quadratic Lyapunov functions. Quite recently, the observer design was studied as a problem of rendering a selected manifold in the extended state-space of the plant and observer as positively invariant and globally attractive (Astolfi, Karagiannis, & Ortega, 2008; Karagiannis & Astolfi, 2008).

Shim et al. (2003) proposes a different approach to observer design by invoking passivity based concepts. The underlying idea is to make the augmented system consisting of the plant and the observer dynamics strictly passive with respect to an invariant set in which the state estimation error is zero. In order to establish passivity, a new input and output is defined on the extended state-space and, under some assumptions on the plant and the observer, it is proved that passivation can be achieved. It has been shown that the proposed observer, on account of its passivity property, admits a redesign which makes it robust to measurement disturbances.

In this paper we consider port-Hamiltonian systems (van der Schaft, 1999, Chapter 4) and construct globally exponentially stable full-order observers for them by following a similar approach as stated in Shim et al. (2003). Our main contribution is to identify a special class of port-Hamiltonian systems that admit such a passivity based observer and give a constructive procedure for the observer design. We allow the observer gain matrices to depend on the observer states unlike in Shim et al. (2003) (where they are assumed constant) and thus also enlarge the admissible class of nonlinear systems. Interestingly, as a part of our full-order observer construction, we obtain a globally exponentially stable reduced-order observer whose formulation shares the same philosophy as the one stated in Astolfi et al. (2008) and Karagiannis and Astolfi (2008). We illustrate our observer design procedure on two well-known physical systems, the magnetic levitation system and the inverted pendulum on the cart, and perform simulations to show the efficacy of the proposed observer.

2. Passivity based observer design for port-Hamiltonian systems

2.1. Port-Hamiltonian systems

A port-Hamiltonian system model arises from network modeling of lumped-parameter physical systems. In the case of...
independent storage elements, they are represented as:

\[ \dot{x} = [J(x) - R(x)] \frac{\partial H}{\partial x}(x) + g(x)u, \]
\[ y = g^T(x) \frac{\partial H}{\partial x}(x), \]  

(1)

\[ y = g^T(x) \frac{\partial H}{\partial x}(x), \]

(2)

where \( x \in \mathbb{R}^n \) are the energy variables, the smooth function \( H(x) : \mathbb{R}^n \to \mathbb{R} \) represents the total energy stored and \( u, y \in \mathbb{R}^m \), \( m \leq n \), are the port variables. The port variables \( u \) and \( y \) are conjugated variables, in the sense that their product defines the power flows exchanged between the system and its environment. Typical examples of such pairs would be currents and voltages in electrical circuits or forces and velocities in mechanical systems. The system’s interconnection structure is captured in the \( n \times n \) skew-symmetric matrix \( J(x) = -J(x) \) while \( R(x) = R^T(x) \geq 0 \) represents the dissipation structure. All the matrices \( J(x), R(x), \) \( g(x) \) have entries depending smoothly on \( x \). In some systems, the control \( u \) could also act through the interconnection structure, that is, the matrix \( J \) is skew-symmetric instead of the velocity. This might motivate to design observers for the system (3) composed of (3) and (5) strictly passive with respect to the manifold \( M = \{ (\hat{x}, \check{x}) : \hat{x} = x \} \), from which the input \( v_d \) to the new output \( v = \hat{x}_1 - \check{x}_1 \).

Definition 1. The dynamical system represented as

\[ \dot{\hat{x}} = [J(\hat{x}_1, u_1)] \frac{\partial H}{\partial \hat{x}}(\hat{x}) + g(y)u_2 + L(\hat{x})v, \]
\[ L(\hat{x}) = \begin{bmatrix} L_1(\hat{x}_1) \\ L_2(\hat{x}_1) \end{bmatrix}, \]

(5)

where \( \hat{x} = (\hat{x}_1, \check{x}_2), \hat{x}_1 \in \mathbb{R}^n, \check{x}_2 \in \mathbb{R}^{n-p}, v \in \mathbb{R}^p \), is a passivity based observer for the system (3) if there exist smooth matrices \( L_1 : \mathbb{R}^n \to \mathbb{R}^{p \times p}, L_2 : \mathbb{R}^{n-p} \to \mathbb{R}^{p \times p} \) and a continuous scalar function \( k : \mathbb{R}^n \times \mathbb{R}^{n-p} \to \mathbb{R} \) such that the feedback law

\[ v = L_1^{-1}(\hat{x}_1)X^{-1}\{k(\hat{x}, y_1, \check{x}_2)\} + \check{v}_d, \]

(6)

with \( X \in \mathbb{R}^{p \times n} = X^T > 0 \) makes the augmented system \(^1\) composed of (3) and (5) strictly passive with respect to the manifold \( M \). uniformly for all \( u_1 \in U \) and \( u_2 \in \mathbb{R}^m \) if there exists a storage function \( S(\check{x}, \hat{x}) > 0 \) for every \( x \neq \hat{x}, S(\check{x}, \hat{x}) = 0 \) on \( M \) and the time derivative of \( S \) along the system trajectory satisfies:

\[ \frac{\partial S}{\partial \hat{x}}(\hat{x}, \check{x}_2)F(\hat{x}_1, x_2, u_1) + \frac{\partial S}{\partial \check{x}_1}(\hat{x}, \check{x}_2)F(\hat{x}_1, \check{x}_2, u_1) + k(\hat{x}, \check{x}_2)\frac{\partial S}{\partial \check{x}_1}(\hat{x}, \check{x}_2) \]

(7)

\[ \frac{\partial S}{\partial \hat{x}}(\hat{x}, \check{x}_2)\frac{\partial S}{\partial \check{x}_1}(\hat{x}, \check{x}_2)L(\hat{x}_1)(XL(\hat{x}_1))^{-1}y_d \leq -\alpha(\|e\|), \]

where \( \alpha \) is a positive definite function, \( e = \hat{x} - \check{x} \)

\[ \frac{\partial S}{\partial \hat{x}}(\hat{x}, \check{x}_2)\frac{\partial S}{\partial \check{x}_1}(\hat{x}, \check{x}_2) = g(y), \]

(8)

\[ \frac{\partial S}{\partial \hat{x}}(\hat{x}, \check{x}_2)\frac{\partial S}{\partial \check{x}_1}(\hat{x}, \check{x}_2) \]

(9)

Definition 2. The system (3), (5) with the design input (6) is strictly passive with respect to the manifold \( M \) uniformly for all \( u_1 \in U \) and \( u_2 \in \mathbb{R}^m \) if there exists a storage function \( S(\check{x}, \hat{x}) > 0 \) for every \( x \neq \hat{x}, S(\check{x}, \hat{x}) = 0 \) on \( M \) and the time derivative of \( S \) along the system trajectory satisfies:

\[ \frac{\partial S}{\partial \hat{x}}(\hat{x}, \check{x}_2)F(\hat{x}_1, x_2, u_1) + \frac{\partial S}{\partial \check{x}_1}(\hat{x}, \check{x}_2)F(\hat{x}_1, \check{x}_2, u_1) + k(\hat{x}, \check{x}_2)\frac{\partial S}{\partial \check{x}_1}(\hat{x}, \check{x}_2) \]

(7)

\[ \frac{\partial S}{\partial \hat{x}}(\hat{x}, \check{x}_2)\frac{\partial S}{\partial \check{x}_1}(\hat{x}, \check{x}_2) \]

(9)

If the augmented system is strictly passive with respect to \( M \) for some functions \( L_1, L_2 \) and \( k \), then upon letting \( v_d = 0 \) the manifold \( M \) becomes positively invariant \(^2\) and globally attractive.\(^3\) The solution of the observer design problem then follows by noting that the state estimation error is zero on \( M \). Further, the asymptotic estimate of \( x \) is then given by \( \hat{x} \).

\(^1\) In the sequel, we shall always use the term augmented system to refer to the system composed of (3) and (5).

\(^2\) The manifold \( M \) is positively invariant if \( (x(0), \hat{x}(0)) \in M \Rightarrow (x(t), \hat{x}(t)) \in M \) for all \( t \geq 0 \) for every initial condition \( (x(0), \hat{x}(0)) \).

\(^3\) The manifold \( M \) is globally attractive if, for every initial condition \( (x(0), \hat{x}(0)) \), the distance of the augmented state vector to the manifold globally asymptotically goes to zero, i.e., \( \lim_{t \to \infty} \text{dist}(x(t), \hat{x}(t), M) = 0 \).
2.4. Observer design

The notion of passivity is usually associated with respect to a point in the state-space rather than a manifold and accordingly, necessary and sufficient conditions for feedback passivation have been proposed. It has been established in Brynes, Isidori, and Willems (1991) that any affine control system can be rendered strictly passive by a smooth static state feedback if and only if the system has a vector relative degree \{1, \ldots, 1\} and is globally minimum phase. When some of the states are not measurable, additional sufficiency conditions have been proposed for partial state feedback passivation in Jiang and Hill (1998) and Shim et al. (2003). Our situation is similar to Jiang and Hill (1998) and Shim et al. (2003) as we need to achieve strict passivity of the augmented system with respect to \(M\) by using a feedback law which is independent of \(x_2\).

We now state two key assumptions on (3), (5) and use them to prove that:

1. There exist matrices \(L_1(\hat{x}_1)\) and \(L_2(\hat{x}_1)\) such that the augmented system satisfies the vector relative degree and global minimum phase conditions with respect to \(M\) which are needed for static state feedback passivation.

2. The augmented system satisfies an additional nonlinear growth inequality which is sufficient to make it strictly passive with respect to \(M\) by a partial state feedback law \(v = L_1^{-1}(\hat{x}_1)X^{-1}\{k(x, \dot{x}, u_1)y_2 + u_1\}\), which is independent of \(x_2\).

**Assumption 1.** There exists a smooth globally invertible matrix \(L_1(x_1) \in \mathbb{R}^{p \times p}\) and a smooth matrix \(L_2(x_1) \in \mathbb{R}^{(p-m) \times p}\) such that
\[
A^T(x_1, u_1) + A(x_1, u_1) > \epsilon I_{p \times p}, \quad \epsilon > 0
\]
holds for all \(x_1\), uniformly for all \(u_1 \in U\), where
\[
A(x_1, u_1) = \{L_2(x_1)L_1^{-1}(x_1)T(x_1, u_1) + R_2(x_1)\}.
\]

**Assumption 2.** There exists a smooth function \(\beta : \mathbb{R}^p \to \mathbb{R}^{p-m}\) such that
\[
L_2(x_1)L_1^{-1}(x_1) = \frac{\partial \beta}{\partial x_1}(x_1)
\]
holds for all \(x_1 \in \mathbb{R}^p\).

We next state the following theorem.

**Theorem 1.** Under **Assumption 1**, 1. The augmented system has a vector relative degree \{1, \ldots, 1\} with respect to the input \(v\) and the output \(y_2 = \hat{x}_1 - x_1\).

2. The zero dynamics of the augmented system with respect to the output \(y_2\) renders the manifold \(\mathcal{P} = \{(x_1, x_2, \hat{x}_2) : \hat{x}_2 = x_2\}\) as positively invariant and globally exponentially attractive.

**Proof.** We compute the derivative of \(y_2\) and see that the input \(v\) appears in it pre-multiplied by the matrix \(L_1\). From **Assumption 1**, since \(L_1\) is invertible for all \(x_1\), we conclude that the augmented system has a vector relative degree \{1, \ldots, 1\} with respect to the input \(v\) and the output \(y_2\).

We next see that the zero dynamics of the augmented system with respect to the output \(y_2\), defined uniformly for all \(u_1 \in U\), \(u_2 \in \mathbb{R}^m\), essentially consists of (3) and the equations
\[
\dot{x}_2 = \{L_2(x_1, u_1) - R_2(x_1)\}Q\hat{x}_2 - T(x_1, u_1)\frac{\partial K}{\partial x_1}(x_1) + g_2(y)u_2 + L_2(x_1)v,
\]
where we make use of (4). We now consider the manifold \(\mathcal{P}\) and denote its off-the-manifold coordinate as \(z = \hat{x}_2 - x_2\). Computing the derivative of \(z\) along (3), (13) and using (12) yields
\[
\dot{z} = (L_2(x_1, u_1) - A(x_1, u_1))Qz.
\]

We can clearly see from (14) that the manifold \(\mathcal{P}\) is positively invariant and further if we consider the Lyapunov function \(V = \frac{1}{2}z^TQz\), then **Assumption 1** verifies \(\dot{V} \leq -\frac{\sigma_{\text{min}}}{\sigma_{\text{max}}}V\) with \(\lambda_m\), \(\lambda_M\) denoting the minimum and maximum eigenvalue. Thus \(V\) exponentially decays to zero with convergence rate \(\frac{\sigma_{\text{min}}}{\sigma_{\text{max}}}\).

An interesting corollary that follows from **Theorem 1** is

**Corollary 1.** Under **Assumptions 1 and 2**, the dynamical system
\[
\dot{\eta} = -\frac{\partial \beta}{\partial x_1}(x_1) \left\{J_1(x_1, u_1) - R_1(x_1)\frac{\partial K}{\partial x_1}(x_1) + g_1(y)u_2\right\}
- T^T(x_1, u_1)\frac{\partial K}{\partial x_1}(x_1) + g_2(y)u_2
+ \left\{J_2(x_1, u_1) - R_2(x_1) - \frac{\partial \beta}{\partial x_1}(x_1)T(x_1, u_1)\right\}Q\{\eta + \beta(x_1)\}
\]
where \(\eta \in \mathbb{R}^{n-p}\), is a reduced-order observer for \(x_2\) and the dynamics of \(\{x, \eta\}\) renders the manifold \(\mathcal{N} = \{(x_1, x_2, \eta) : \eta = x_2 - \beta(x_1)\}\) positively invariant and globally exponentially attractive. The asymptotic estimate of \(x_2\) thus is \(\eta + \beta(x_1)\).

**Proof.** The off-the-manifold coordinate is \(z = \eta - x_2 + \beta(x_1)\), which upon differentiating along the system dynamics yields
\[
\dot{z} = (L_2(x_1, u_1) - R_2(x_1) - \frac{\partial \beta}{\partial x_1}(x_1)T(x_1, u_1))Qz.
\]

Using **Assumptions 1, 2** and employing the Lyapunov function \(V = \frac{1}{2}z^TQz\), we can prove that (15) is globally exponentially stable. The asymptotic estimate of \(x_2\) would then be \(\eta + \beta(x_1)\).

**Remark 1.** The notion of vector relative degree is usually defined with respect to the output and the total input of the system, which for our augmented system (3)–(5) would be \(u_1, u_2\) and \(v\). However, our idea is to design the input \(v\) by the feedback law (6) such that the augmented system becomes strictly passive with respect to the input \(v\) and the output \(y_2\), uniformly for all \(u_1 \in U \subset \mathbb{R}^m\) and \(u_2 \in \mathbb{R}^m\). In other words, we consider \(v\) as our design input and the other inputs \(u_1, u_2\) can be any functions of time or state or both, belonging to their respective domains. Hence, we use the concept of vector relative degree between \(y_2\) and \(v\), which is a small modification of the definition that is usually found in the literature (Shastry, 1999).

**Remark 2.** The zero dynamics of the augmented system with respect to the output \(y_2\) given by (3), (12), (13) differs slightly from the usual understanding of zero dynamics in the sense that the inputs \(u_1, u_2\) still remain in our equations. Once again, as already stated in the previous remark, we consider \(v\) to be the design input and define the zero dynamics uniformly for all \(u_1 \in U \subset \mathbb{R}^m\) and \(u_2 \in \mathbb{R}^m\).

**Remark 3.** **Assumption 1** involves finding matrices \(L_1(x_1), L_2(x_1)\) such that the augmented system has a vector relative degree \{1, \ldots, 1\} and is globally minimum phase with respect to \(M\) while **Assumption 2** states that the quantity \(L_2(x_1)L_1^{-1}(x_1)\) has to be integrable and satisfy (11) for some function \(\beta(x_1)\). Designing such state dependent matrices that satisfy (10) and (11) would

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\footnote{For \(p = 1\) this assumption is always satisfied.}
involve solving a set of algebraic and partial differential equations respectively and is usually a difficult task. Shim et al. (2003) studies the observer design problem by restricting \( L_1, L_2 \) to be constant matrices in which case Assumption 2 is trivially satisfied with \( \beta(x_1) = L_2L_1^{-1}x_1 \) and hence narrows the applicable class of nonlinear systems. Indeed, as we show later in our examples, whenever \( T \) is a constant matrix, letting \( L_1, L_2 \) to be constant would suffice for the observer design, whereas in situations where \( T \) depends on \( x_1 \), it is natural to allow \( L_1, L_2 \) to depend on \( x_1 \) in order to satisfy Assumption 1.

Remark 4. A very recent article Karagiannis and Astolfi (2008) presents a full-order observer design strategy for nonlinear systems that are affine in the unmeasurable state. The observer design is completed by introducing a dynamic scaling parameter which simplifies an integrability condition similar to our Eq. (11) in Assumption 2. However, the constructive procedure outlined in the article for computing the function \( \beta \) seems to involve some lengthy computations.

We next state a theorem to prove that the system (3), (5) admits a partial state feedback \( v \) (independent of \( x_2 \)) that renders it strictly passive with respect to \( M \) and also leads to the construction of the full-order observer.

Theorem 2. Under Assumptions 1 and 2,

1. The system (3), (5) expressed in the coordinates \((x_1, x_2, \zeta_1, \zeta_2)\) where

\[
\zeta_1 = \dot{x}_1 - x_1,
\]

\[
\zeta_2 = \dot{x}_2 - x_2 - \{\beta(\dot{x}_1) - \beta(x_1)\}
\]

assumes its global normal form with respect to the input \( v \) and the output \( y_d \).

2. Under the additional assumption \( g_1 \equiv 0 \) in (3), there exist non-negative scalar functions \( f_1(\zeta_1, \dot{x}_1, x_1, \dot{x}_2, u_1), f_2(\zeta_1, \dot{x}_1, x_1, \dot{x}_2, u_1) \) such that the feedback law

\[
v = L_1^{-1}(\dot{x}_1)X^{-1}[-(\delta + f_1 + f_2^2)\kappa_1 + u_2],
\]

where \( X \) was introduced in (6), \( \delta \in \mathbb{R} > 0 \), makes the system strictly passive with respect to the manifold \( M \), uniformly for all \( u_1 \in U, u_2 \in \mathbb{R}^n \), from the input \( v \) to the output \( \zeta_1 \) with the storage function being given by

\[
\|x\|^2
\]

and use (28), (29), (30) to finally obtain

\[
\dot{W} \leq -\delta \|\zeta_1\|^2 + \frac{3}{4}c \|\kappa_2\|^2 - \frac{3}{2}e \|\kappa_2\|^2 - \frac{1}{2} \sqrt{1+W} - \frac{1}{2} \|f_2\|^2.
\]

Thus, the system is strictly passive with respect to the manifold \( M \), from input \( v \) to the output \( y_d(= \zeta_1) \) with the storage function being \( W(\zeta_1, \zeta_2) \). Further, upon letting \( v_d = 0 \) and performing some simple computations, we get that \( W \leq -\frac{e}{2}W \) where \( c = \max \left( \|\lambda_M(x)\|, 1 \right) \) and hence the Lyapunov function \( W(\zeta_1, \zeta_2) \) exponentially decays to zero with convergence rate \( \frac{e}{2} \).

Remark 5. If we let \( e = (e_1, e_2) \) denote the state estimation error (as also introduced in Definition 2), then the storage function \( W \) when expressed in the coordinates \((x, e)\) takes the form \( W(x, e) = \frac{e_1^2}{2}Xe_1 + \frac{1}{2}e_2(\beta(x_1 + e_1) + \beta(x_1))\). We thus obtain a Lyapunov function that depends both on the state and error coordinates, similar to the quadratic error Lyapunov functions.

Remark 6. The inequalities (28)–(30) are nonlinear growth condition on \( W \) which require the growth rate to be linearly bounded in \( \zeta_2 \). If \( R_2(x_2) > 0 \) then Assumption 1 holds with \( L_2 = 0 \). In this situation if we allow the matrices \( J_1, R_1 \) to also depend on \( x_2 \), then by performing some simple computations we can show that \( W \) can be linearly bounded in \( \zeta_2 \) provided the quantity...
\(f_1(x_1, x_2, u_1) - R_1(x_1, x_2)\) is globally Lipschitz\(^7\) in \(x_2\), uniformly for all \(u_1 \in U\).

**Remark 7.** The assumption \(g_1 \equiv 0\) ensures that (8) is satisfied, that is, the input \(u_2\) is decoupled from the dynamics of \((\zeta_1, \zeta_2)\) and hence the observer design is independent of \(u_2\). This would be the case in mechanical systems where the input is the external force applied and it appears in the dynamics of the (unmeasured) generalized momenta. When \(\beta\) is a linear function of its argument (as considered in Shim et al. (2003)), the assumption \(g_1 \equiv 0\) can be relaxed.

**Remark 8.** If \(u_1\) is a continuous time varying external signal taking values in a compact set \(U \subset \mathbb{R}^m\) and further has a bounded derivative, then the matrices \(L_1\) and \(L_2\) used in the observer dynamics can be allowed to depend smoothly on \(u_1\). This is also natural from the view point of having to satisfy Assumption 1 because the matrix \(T\) depends on \(u_1\). Shim et al. (2003) considers all plant inputs to be external time varying signals in their observer design.

In the next section, we illustrate our proposed observer design by considering some physical examples which come under the class of (3).

### 3. Physical examples

#### 3.1. Magnetic levitation system

We consider the magnetic levitation system as described in Astolfi et al. (2008), consisting of an iron ball in a vertical magnetic field created by a single electromagnet, described by the port-Hamiltonian model

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{pmatrix} =
\begin{bmatrix}
-R_2 & 0 & 0 \\
0 & 0 & 1 \\
-1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\frac{\partial H}{\partial x_1} \\
\frac{\partial H}{\partial x_2} \\
\frac{\partial H}{\partial x_3}
\end{bmatrix} + \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix} u
\]  \quad (31)

where \(x_1\) corresponds to the flux and \(x_2, x_3\) are the position and momentum of the ball respectively. The system’s energy is given as \(H(x_1, x_2, x_3) = \frac{1}{2} x_1^2 + mgx_2 + \frac{1}{2} K(1 - x_2)\) with \(m\) being the mass of the ball and \(K\) is some positive constant that depends on the number of coil turns. In (31), the ball position \(x_2\) is scaled such that when \(x_2 = 1\), the ball touches the electromagnet. To avoid this singularity, we assume that \(x_2(t) < 1\), that is, the ball remains strictly below the magnet.

We assume the flux and position to be measurable while the momentum cannot be measured. Thus, (31) fits in the framework of (3). We let \((\dot{x}_1, \dot{x}_2, \dot{x}_3)\) be the state estimates and define their dynamics as in (5). If \((e_1, e_2, e_3) = (\dot{x}_1, \dot{x}_2, \dot{x}_3) - (x_1, x_2, x_3)\) denotes the error, then upon choosing \(L_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, L_2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}\) in our observer construction, we obtain the zero dynamics of \((\dot{x}, \dot{\chi})\) with respect to the outputs \((e_1, e_2)\) as \(e_3 = -\frac{\zeta_3}{m}\). Then, computing the time derivative of the Lyapunov function \(V(e_1, e_2, e_3) = \frac{1}{2} e_1^2 + \frac{1}{2} e_2^2\) along the zero dynamics yields \(\dot{V} = -\frac{1}{2} e_3^2\) and hence \(\epsilon = 1/m^2\). We introduce the change of coordinates \(\zeta = (\zeta_1, \zeta_2, \zeta_3) = (e_1, e_2, e_3 - e_2)\) to obtain the dynamics in the global normal form. We choose the total storage function as \(W(\zeta) = \frac{1}{2} \zeta_1^2 + \frac{1}{2} \zeta_2^2 + \frac{1}{2} \zeta_3^2\) and compute the inequalities (28), (29), (30) to get \(f_1 = -\frac{1}{m} z_1 + \frac{1}{2} \zeta_2(1 - |z_2| + |x_1|)\) and \(f_2 = \frac{1}{4} \zeta_1 |x_1| + \zeta_1\). We accordingly choose the observer feedback \(v\) as

\[
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix} =
\begin{bmatrix}
\delta + \frac{1}{m^2} + \frac{R_2}{k} [1 - |\hat{x}_2| + |x_1|] \\
\frac{1}{4k^2} |x_1 + \hat{x}_2|^2 \left( \frac{1}{\zeta_1} + \frac{v_1}{v_2} \right)
\end{bmatrix},
\]  \quad (32)

and verify that \(\dot{W} < e_1 v_1 + e_2 v_2\) and hence the system is strictly passive with respect to the input \((v_1, v_2)\) and the output \((e_1, e_2)\).

#### 3.2. Inverted pendulum on cart system

We consider the inverted pendulum on a cart example (Acosta, Ortega, Astolfi, & Mahindrakar, 2005; Teel, 1996; Venkatraman, Ortega, Sarras, & van der Schaft, 2008) which is a two degree of freedom mechanical system. It can be modeled in the port-Hamiltonian form

\[
\begin{pmatrix}
\dot{\theta} \\
\dot{\beta}
\end{pmatrix} =
\begin{pmatrix}
0 & T(q) \\
-T^T(q) & 0
\end{pmatrix}
\begin{pmatrix}
\frac{\partial V}{\partial q} \\
\frac{\partial V}{\partial p}
\end{pmatrix} +
\begin{pmatrix}
0 \\
G(q)
\end{pmatrix} u, \quad (33)
\]

where \(q = (q_1, q_2)\) with \(q_1\) being the angle made by the pendulum with the vertical axis, \(q_2\) being the horizontal position of the cart and \(\beta = (\beta_1, \beta_2)\) are the pseudo momenta. We obtain \(\dot{p}\) by the change of coordinates \(\hat{p} = T^T(p)\), where \(p = M(q)q\) are the actual momenta with \(M(q)\) being the inertia matrix and \(TT^T = M^{-1}\). Refer to the paper by Venkatraman et al. (2008) for more details. The matrices \(T(q), G(q)\) and the potential energy function \(V(q)\) are given as

\[
T(q) = \begin{bmatrix}
\sqrt{m_3 - b^2 \cos^2 q_1} & 0 \\
-b \cos q_1 & \sqrt{m_3}
\end{bmatrix}, \quad G(q) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad (34)
\]

and \(V(q) = a \cos q_1\). We assume that only \(q\) is measurable and see that (33) fits in the framework of (3). We next compute that \(P(q)T(q)^2 + P(q)T(q) > \epsilon I\), \(\epsilon = 2 \min[1, \frac{1}{\sqrt{m_3}}]\) where \(P(q) = \nabla V(q)\) given as

\[
P(q) = \begin{pmatrix}
& 0 \\
& 1
\end{pmatrix}, \quad \beta(q) = \begin{pmatrix}
q_1 \\
-\frac{m_3}{b} q_2
\end{pmatrix}. \quad (35)
\]

Since, \(P(q) = L_2(q)L_2^{-1}(q)\), we choose \(L_2 = P, L_1 = L_2 x_2z_2\). For constructing the full-order observer we introduce the coordinates \(\zeta_1 = \hat{q} - q, \zeta_2 = \hat{\beta} - \beta - (\hat{\beta} - \beta)\), where \(\hat{p} = T^T(p)\) and \([\hat{q}, \hat{\beta}]^T\) is the estimate of \([q, \beta]^T\). Next, using standard results of functional analysis we get:

\[
\|\beta(\hat{q}) - \beta(q)\| \leq \left\{ \sup_q \|\nabla V(q)\| \right\} \|\zeta_1\|.
\]

\[
\|T(q) \frac{\partial V}{\partial q} - T(q) \frac{\partial V}{\partial q}\| \leq \left\{ \sup_q \nabla \left\{ T(q) \frac{\partial V}{\partial q} \right\} \right\} \|\zeta_1\|.
\]

\[
\|\hat{q}(\hat{\beta}) - q(\beta)\| \leq \left\{ \sup_q \frac{\partial T(q)}{\partial q} \right\} \|\zeta_1\|.
\]

\[
\|P(\hat{q})T(q) - P(q)(q)\| \leq \left\{ \sup_q \frac{\partial T(q)}{\partial q} \right\} \|\zeta_1\|.
\]

If \(\lambda\) denotes the eigenvalue, then we obtain the following bounds:

\[
\|\lambda (P^T(p))\| \leq 1 + \frac{b^2}{2m_3} + \frac{b}{2m_3} \sqrt{4 + \frac{b^2}{m_3}} = M_1,
\]

\[
\lambda \left( \nabla \left\{ T(q) \frac{\partial V}{\partial q} \right\} \right) \leq a \sqrt{m_3} \sqrt{m_3 - b^2} = M_2.
\]
Theorem 2
Venkatraman (28)
We refer to the paper by the methodology as the inverted pendulum on cart example. The matrices $L_1$ and $L_2$ once again depend on the generalized position coordinate $q$.

Remark 9. We refer to the paper by Venkatraman et al. (2008) for two other physical examples, namely a 3-link underactuated planar manipulator and a planar redundant manipulator with one elastic degree of freedom, both of which can be rendered linear in the unmeasured coordinates and hence follow the same observer design methodology as the inverted pendulum on cart example. The matrices $L_1$ and $L_2$ once again depend on the generalized position coordinate $q$.

4. Conclusion
We have proposed a passivity based full-order observer design framework for a class of port-Hamiltonian systems which leads to the construction of a globally exponentially stable observer. The idea is to render the augmented system (composed of the plant and the observer dynamics) strictly passive with respect to an invariant manifold defined on the extended state-space on which the state estimation error is zero. We also obtained as a part of the full-order observer construction, a globally exponentially stable reduced-order observer.

The observer construction is done in two steps:

1. Compute the observer gain matrices $L_1(\hat{x}_1)$ and $L_2(\hat{x}_1)$ by solving a set of algebraic and partial differential equations such that the augmented system has a vector relative degree $\{1, \ldots, 1\}$ and is globally minimum phase with respect to the manifold $\mathcal{M}$.

2. Compute the partial state feedback law, $u(y, x, u_1)$ by following the procedure given in the proof of Theorem 2, in order to render the augmented system strictly passive with respect to the manifold $\mathcal{M}$.

We finally demonstrated the observer design on two well-known physical examples, the magnetic levitation system and the inverted pendulum on cart.

Under some additional assumptions, we proved the separation principle for the proposed full-order observer when employed in closed-loop with a passivity based control (PBC) state feedback law, by using concepts from nonlinear cascaded systems theory. Refer to the internal report by Venkatraman and van der Schaft (2009) for more details.

References


Table 1
Simulation parameters for the inverted pendulum example.

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<th>$q_1(0)$</th>
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</table>

Fig. 1. Open-Loop trajectories for the Inverted Pendulum on Cart and the Observer with $u = 0$.


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