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Control of one-dimensional guided formations using coarsely quantized information

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Abstract— **Motivated by applications of platoon formations, the paper studies the problem of guiding mobile agents in a one-dimensional formation to their desired relative positions. Only coarsely quantized information is used which is communicated from a guidance system that monitors in real time the agents' motions. The desired relative positions are defined by the given distance constraints between the agents under which the overall formation is rigid in shape and thus admits locally a unique realization. It is firstly shown that even when the guidance system can only transmit at most four bits of information to each agent, it is still possible to design control laws to guide the agents to their desired positions. We further delineate the thin set of initial conditions for which the proposed control law may fail using the example of a three-agent formation. Tools from non-smooth analysis are utilized for the convergence analysis.**

I. Introduction

In recent years, various ideas have been proposed to realize intelligent highway systems to reduce traffic congestions and improve safety levels. It is envisioned that navigation, communication and automatic driver assistance systems are critical components [1], [2], [3]. A great deal of monitoring and controlling capabilities have been implemented through roadside infrastructures, such as cameras, sensors, and control and communication stations. Such systems can work together to monitor in real time the situations on highways and at the same time guide vehicles to move in a coordinated fashion, e.g. to keep appropriate distances from the vehicles in front of and behind each individual vehicle. In intelligent highway systems, the guiding commands are expected to be simple and formatted as short digital messages to scale with the number of vehicles and also to avoid conflict with the automatic driver assistance systems installed within the vehicles. Similar guided formation control problems also arise when navigating mobile robots or docking autonomous vehicles [4].

Motivated by this problem of guiding platoons of vehicles on highways, we study in this paper the problem of controlling a one-dimensional multi-agent formation using only *coarsely* quantized information. The formation to be considered are rigid under inter-agent distance constraints and thus its shape is uniquely determined locally. Most of the existing work on controlling rigid formations of mobile agents, e.g. [5], [6], [7], assumes that there is no communication bandwidth constraints and thus real-valued control signals are utilized. The idea of quantized control through digital communication channels has been applied to consensus problems, e.g. [8], [9] and references therein, and more recently to formation control problems [10]. The uniform quantizer and logarithmic quantizer [11] are among the most popular choices for designing such controllers with quantized information. Moreover, the paper [12] has discussed Krasowskii solutions and hysteretic quantizers in connection with continuous-time average consensus algorithms under quantized measurements.

The problem studied in this paper distinguishes itself from the existing work in that it explores the limit of the least bandwidth for controlling a one-dimensional rigid formation by using a quantizer in its simplest form with only two quantization levels. As a result, for each agent in the rigid formation, at most four bits of bandwidth is needed for the communication with the navigation controller. The corresponding continuous-time model describing the behavior of the overall multi-agent formation is, however, non-smooth and thus an appropriate notion of solution [13] has to be defined first. We use both the Lyapunov approach and trajectory-based approach to prove convergence since the former provides a succinct view about the dynamic behavior while the latter leads to insight into the set of initial positions for which the proposed controller may fail.

The rest of the paper is organized as follows. We first formulate the one-dimensional guided formation control problem with coarsely quantized information in section II. Then in section III, we provide the convergence analysis results first using the Lyapunov method and then the trajectory-based method. Simulation results are presented in section IV to validate the theoretical analysis. We make concluding remarks in section V.

II. Problem formulation

The one-dimensional guided formation that we are interested in consists of n mobile agents. We consider the case when the formation is rigid [5]; to be more specific, if we align the given one-dimensional space with the x-axis in the plane and label the agents along the positive direction of the x-axis by $1, \ldots, n$, then the geometric shape of the formation is specified by the given pairwise distance constraints $|x_i - x_{i+1}| = d_i$, $i = 1, \ldots, n-1$, where $d_i > 0$ are desired distances. Although the guidance system can monitor the motion of the agents in real time, we require that it can only broadcast to the mobile agents quantized guidance information through digital channels. In fact, we explore the

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limit for the bit constraint by utilizing the quantizer that only has two quantization levels and consequently its output only takes up one bit of bandwidth. The quantizer that is under consideration takes the form of the following sign function: For any $z \in \mathbb{R}$,

$$\mathrm{sgn}(z) = \begin{cases} +1 & z \geq 0 \\ -1 & z < 0 \,. \end{cases}$$

Each agent, modeled by a kinematic point, then moves according to the following rules utilizing the coarsely quantized information:

$$\begin{array}{rcl} \dot{x}_1 &=& -k_1 \mathrm{sgn}(x_1 - x_2)\mathrm{sgn}(|x_1 - x_2| - d_1) \\ \dot{x}_i &=& \mathrm{sgn}(x_{i-1} - x_i)\mathrm{sgn}(|x_{i-1} - x_i| - d_{i-1}) - \\ && k_i \mathrm{sgn}(x_i - x_{i+1})\mathrm{sgn}(|x_i - x_{i+1}| - d_i), \\ && \qquad\qquad i = 2, \ldots, n-1 \\ \dot{x}_n &=& \mathrm{sgn}(x_{n-1} - x_n)\mathrm{sgn}(|x_{n-1} - x_n| - d_{n-1}) \end{array} \quad (1)$$

where $x_i \in \mathbb{R}$ is the position of agent i in the one-dimensional space aligned with the x-axis, and $k_i > 0$ are gains to be designed. Note that since each agent is governed by at most two distance constraints, as is clear from (1), a bandwidth of four bits is sufficient for the communication between the guidance system and the agents $2, \ldots, n-1$ and the required bandwidths for the guidance signals for agents 1 and n are both 2 bits. Hence, in total only $4n - 2$ bits of bandwidth is used.

We want to point out here that in the application of the intelligent highway systems that we have in mind, the ordering of the mobile agents in the one-dimensional space is usually preserved. So the bit of information of $\mathrm{sgn}(x_i - x_{i+1})$, $1 \leq i < n$, might be redundant. In deed, we have fully used this fact in [14] where a slightly different control strategy is proposed and only 2 bits of information is used for each agent.

The main goal of this paper is to demonstrate under this extreme situation of using coarsely quantized information, the formation still exhibits satisfying convergence properties under the proposed maneuvering rules. Towards this end, we introduce the variables of relative positions among the agents

$$z_i \stackrel{\Delta}{=} x_i - x_{i+1} \,, \quad i = 1, 2, \ldots, n-1 \,. \quad (2)$$

Let us express the system in the z-coordinates to obtain

$$\begin{array}{rcl} \dot{z}_1 &=& -(k_1 + 1)\mathrm{sgn}(z_1)\mathrm{sgn}(|z_1| - d_1) \\ && +k_2 \mathrm{sgn}(z_2)\mathrm{sgn}(|z_2| - d_2) \\ \dot{z}_i &=& \mathrm{sgn}(z_{i-1})\mathrm{sgn}(|z_{i-1}| - d_{i-1}) \\ && -(k_i + 1)\mathrm{sgn}(z_i)\mathrm{sgn}(|z_i| - d_i) \\ && +k_{i+1}\mathrm{sgn}(z_{i+1})\mathrm{sgn}(|z_{i+1}| - d_{i+1}), \\ && \qquad\qquad i = 2, \ldots, n-2 \\ \dot{z}_{n-1} &=& \mathrm{sgn}(z_{n-2})\mathrm{sgn}(|z_{n-2}| - d_{n-2}) \\ && -(k_{n-1} + 1)\mathrm{sgn}(z_{n-1})\mathrm{sgn}(|z_{n-1}| - d_{n-1}) \,. \end{array} \quad (3)$$

To study the dynamics of the system above, we need to first specify what we mean by the solutions of the system. Since the vector field $f(z)$ on the right-hand side is discontinuous, we consider Krasowskii solutions, namely solutions to the differential inclusion $\dot{z} \in \mathcal{K}(f(z))$, where

$$\mathcal{K}(f(z)) = \bigcap_{\delta > 0} \overline{\mathrm{co}}\,(f(B(z, \delta))) \,,$$

$\overline{\mathrm{co}}$ denotes the involutive closure of a set, and $B(z, \delta)$ is the ball centered at z and of the radius δ. The need to consider these solutions becomes evident in the analysis in the next section. Since the right-hand side of (1) is also discontinuous, its solutions are to be intended in the Krasowskii sense as well. Then we can infer conclusions on the behavior of (1) provided that each solution x of (1) is such that z defined in (2) is a Krasowskii solution of (3). This is actually the case by [15], Theorem 1, point 5), and it is the condition under which we consider (3). It turns out that the z-system (3) is easier to work with for the convergence analysis that we present in detail in the next section.

III. Convergence analysis

In this section, after identifying the equilibria of the system, we present two different approaches for convergence analysis. The first is based on a Lyapunov-like function and the second examines the vector field in the neighborhood of the system's trajectories.

A. Equilibria of the system

We start the analysis of system (3) by looking at the discontinuity points of the system. A discontinuity point is a point at which the vector field on the right-hand side of the equations above is discontinuous. Hence, the set $\mathcal{D}$ of all the discontinuity points is:

$$\mathcal{D} = \{z \in \mathbb{R}^{n-1} : \Pi_{i=1}^{n-1} z_i(|z_i| - d_i) = 0\} \,.$$

It is of interest to characterize the set of equilibria:

Proposition 1: Let $k_1 + 1 > k_2$, $k_i > k_{i+1}$ for $i = 2, \ldots, n-2$, and $k_{n-1} > 0$. The set of equilibria, i.e. the set of points for which $\mathbf{0} \in \mathcal{K}(f(z))$ with $f(z)$ being the vector field on the right-hand side of (3), is given by

$$\mathcal{E} = \{z \in \mathbb{R}^{n-1} : \sum_{i=1}^{n-1} |z_i||\,|z_i| - d_i| = 0\} \,.$$

The proof of this proposition relies on the following lemma.

Lemma 1: For $i \in \{2, \ldots, n-2\}$, if $|z_j|\,||z_j| - d_j| = 0$ for $j = 1, 2, \ldots, i-1$, and $\mathbf{0} \in \mathcal{K}(f(z))$, then $|z_i|\,||z_i| - d_i| = 0$.

Proof: Suppose by contradiction that $|z_i|\,||z_i| - d_i| \neq 0$. Observe that z belongs to a discontinuity surface where in particular $|z_{i-1}|\,||z_{i-1}| - d_{i-1}| = 0$. This implies that in a neighborhood of this point, the state space is partitioned into different regions where $f(z)$ is equal to constant vectors. In view of (3), the component i of these vectors is equal to one of the following values: $1-(k_i+1)+k_{i+1}$, $1-(k_i+1)-k_{i+1}$, $-1-(k_i+1)+k_{i+1}$, $-1-(k_i+1)-k_{i+1}$, if $\mathrm{sgn}(z_i)\mathrm{sgn}(|z_i| - d_i) = 1$, or $1+(k_i+1)+k_{i+1}$, $1+(k_i+1)-k_{i+1}$, $-1+(k_i+1)+k_{i+1}$, $-1+(k_i+1)-k_{i+1}$, if $\mathrm{sgn}(z_i)\mathrm{sgn}(|z_i| - d_i) = -1$. Any $v \in \mathcal{K}(f(z))$ is such that its component i belongs to (a subinterval of) the interval $[-1-(k_i+1)-k_{i+1}, 1-(k_i+1)+k_{i+1}]$ if $\text{sgn}(z_i)\text{sgn}(|z_i|-d_i)=1$ (respectively, to the interval $[-1+(k_i+1)-k_{i+1}, 1+(k_i+1)+k_{i+1}]$ if $\text{sgn}(z_i)\text{sgn}(|z_i|-d_i)=-1$). In both cases, if $k_i > k_{i+1}$, then the interval does not contain 0 and this is a contradiction. This ends the proof of the lemma. □

Proof of Proposition 1: First we show that if $\mathbf{0} \in \mathcal{K}(f(z))$, then $z \in \mathcal{E}$. As a first step, we observe that $\mathbf{0} \in \mathcal{K}(f(z))$ implies $|z_1|||z_1|-d_1|=0$. In fact, suppose by contradiction that the latter is not true. This implies that at the point z for which $\mathbf{0} \in \mathcal{K}(f(z))$, any $v \in \mathcal{K}(f(z))$ is such that the first component takes values in the interval $[-(k_1+1)-k_2, -(k_1+1)+k_2]$, or in the interval $[(k_1+1)-k_2, (k_1+1)+k_2]$. In both cases, if $k_1+1>k_2$, then 0 does not belong to the interval and this contradicts that $\mathbf{0} \in \mathcal{K}(f(z))$. Hence, $|z_1|||z_1|-d_1|=0$.

This and Lemma 1 show that $|z_i|\,||z_i|-d_i|=0$ for $i=1,2,\ldots,n-2$. To prove that also $|z_{n-1}|\,||z_{n-1}|-d_{n-1}|=0$, consider the last equation of (3), and again suppose by contradiction that $|z_{n-1}|\,||z_{n-1}|-d_{n-1}| \neq 0$. Then the last component of $v \in \mathcal{K}(f(z))$ belongs to a subinterval of $[-1-(k_{n-1}+1), 1-(k_{n-1}+1)]$ or to a subinterval of $[-1+(k_{n-1}+1), 1+(k_{n-1}+1)]$. If $k_{n-1}>0$, then neither of these intervals contain 0 and this is again a contradiction. This concludes the first part of the proof, namely that $\mathbf{0} \in \mathcal{K}(f(z))$ implies $z \in \mathcal{E}$.

Now we let $z \in \mathcal{E}$ and prove that $\mathbf{0} \in \mathcal{K}(f(z))$. By definition, if $z \in \mathcal{E}$, then z lies at the intersection of $n-1$ planes, which partition $\mathbb{R}^n$ into $\nu \doteq 2^{n-1}$ regions, on each one of which $f(z)$ is equal to a different constant vector. Any $v \in \mathcal{K}(f(z))$ is the convex combination of these ν vectors, which we call $v^{(1)}, \ldots, v^{(\nu)}$. We construct $v \in \mathcal{K}(f(z))$ such that $v=\mathbf{0}$. We observe first that, the component 1 of the vectors $v^{(i)}$'s can take on four possible values, namely $(k_1+1)+k_2$, $(k_1+1)-k_2$, $-(k_1+1)+k_2$, $-(k_1+1)-k_2$, and that there are exactly $\frac{\nu}{4}$ (we are assuming that $n \geq 3$, as the case $n=2$ is simpler and we omit the details) vectors among $v^{(1)}, \ldots, v^{(\nu)}$ whose first component is equal to $(k_1+1)+k_2$, $\frac{\nu}{4}$ whose first component is equal to $(k_1+1)-k_2$ and so on. As a consequence, if $\lambda_i = \frac{1}{\nu}$ for all $i=1,2,\ldots,\nu$, then $\sum_{j=1}^{\nu} \lambda_j v_1^{(j)} = 0$.
Similarly, the component i, with $i=2,\ldots,n-2$, can take on eight possible values $(1+(k_i+1)+k_{i+1}$, $1+(k_i+1)-k_{i+1}, \ldots, -1-(k_i+1)-k_{i+1}$ – see the expression of $\dot{z}_i$ in (3)) and as before, the set $v^{(1)}, \ldots, v^{(\nu)}$ can be partitioned into $\frac{\nu}{8}$ sets, and each vector in a set has the component i equal to one and only one of the eight possible values. Moreover, these values are such that $\sum_{j=1}^{\nu} \lambda_j v_i^{(j)} = 0$.
Finally, if $i=n-1$, the set $v^{(1)}, \ldots, v^{(\nu)}$ can be partitioned into four sets, and each vector in a set has the last component equal to one and only one of the four possible values $1+(k_{n-1}+1)$, $1-(k_{n-1}+1)$, $-1+(k_{n-1}+1)$, $-1-(k_{n-1}+1)$. Hence, $\sum_{j=1}^{\nu} \lambda_j v_{n-1}^{(j)} = 0$. Let now $v \in \mathcal{K}(f(z))$ be such that $v = \sum_{i=1}^{\nu} \lambda_i v^{(i)}$, with $\lambda_i = \frac{1}{\nu}$ for all i. Since $\sum_{j=1}^{\nu} \lambda_j v_i^{(j)} = 0$ for all $i=1,2,\ldots,n-1$, then $v=\mathbf{0}$ and this proves that for all $z \in \mathcal{E}$, we have $\mathbf{0} \in \mathcal{K}(f(z))$. This completes the proof. □

Next, we show that the equilibrium set $\mathcal{E}$ is attractive.

B. Lyapunov function based analysis

Now we are in a position to present the main convergence result.

Theorem 1: If

$$k_1 \geq k_2\ ,\ k_i \geq k_{i+1}+1\ ,\ i=2,\ldots,n-2\ ,\ k_{n-1} \geq 1\ , \quad (4)$$

then all the Krasowskii solutions to (3) converge to (a subset of) the equilibria set $\mathcal{E}$.

Proof: Let

$$V(z) = \frac{1}{4}\sum_{i=1}^{n-1}(z_i^2-d_i^2)^2$$

be a smooth non-negative function. We want to study the expression taken by $\nabla V(z) f(x)$, where $f(z)$ is the vector field on the right-hand side of (3). We obtain:

$$\begin{array}{l} \nabla_{z_i} V(z)\dot{z}_i \\ = \left\{ \begin{array}{l} z_1(z_1^2-d_1^2)[-(k_1+1)\text{sgn}(z_1)\text{sgn}(|z_1|-d_1) \\ +k_2\text{sgn}(z_2)\text{sgn}(|z_2|-d_2)] \\ \qquad i=1 \\ \\ z_i(z_i^2-d_i^2)[-(k_i+1)\text{sgn}(z_i)\text{sgn}(|z_i|-d_i) \\ +\text{sgn}(z_{i-1})\text{sgn}(|z_{i-1}|-d_{i-1}) \\ +k_{i+1}\text{sgn}(z_{i+1})\text{sgn}(|z_{i+1}|-d_{i+1})] \\ \qquad i=2,\ldots,n-2 \\ \\ z_{n-1}(z_{n-1}^2-d_{n-1}^2)[\text{sgn}(z_{n-2})\text{sgn}(|z_{n-2}|-d_{n-2}) \\ -(k_{n-1}+1)\text{sgn}(z_{n-1})\text{sgn}(|z_{n-1}|-d_{n-1})] \\ \qquad i=n-1 \end{array} \right. \end{array}$$

If $z \notin \mathcal{D}$, i.e. if z is not a point of discontinuity for $f(z)$, then:

$$\begin{array}{l} \nabla_{z_i} V(z)\dot{z}_i \\ \leq \left\{ \begin{array}{ll} -(k_1+1-k_2)|z_1|\,|z_1^2-d_1^2| & i=1 \\ -(k_i-k_{i+1})|z_i|\,|z_i^2-d_i^2| & i=2,\ldots,n-2 \\ -k_{n-1}|z_{n-1}|\,|z_{n-1}^2-d_{n-1}^2| & i=n-1 \end{array} \right. \end{array}$$

where we have exploited the fact that $\text{sgn}(z_i^2-d_i^2) = \text{sgn}(|z_i|-d_i)$. Hence, if (4) holds, then

$$\nabla V(z) f(z) \leq -\sum_{i=1}^{n-1} |z_i|\,|z_i^2-d_i^2| < 0\ .$$

If $z \in \mathcal{D}$, we look at the set

$$\dot{\overline{V}}(z) = \{a \in \mathbb{R} : \exists v \in \mathcal{K}(f(z)) \text{ s.t. } a = \nabla V(z) \cdot v\}\ .$$

We distinguish two cases, namely (i) $z \in \mathcal{E} \subseteq \mathcal{D}$ and (ii) $z \in \mathcal{D} \setminus \mathcal{E}$. In case (i), $\nabla V(z) = \mathbf{0}^T$, and therefore, $\dot{\overline{V}}(z) = \{0\}$. In case (ii), there must exist at least one agent such that $|z_i|\,|z_i^2-d_i^2|=0$ and at least one agent such that $|z_j|\,|z_j^2-d_j^2| \neq 0$. Let $\mathcal{I}_1(z)$ (respectively, $\mathcal{I}_2(z)$) be the set of indices corresponding to agents for which $|z_i|\,|z_i^2-d_i^2|=0$

($|z_j|\,|z_j^2-d_j^2| \neq 0$). Clearly, $\mathcal{I}_1(z)\cup\mathcal{I}_2(z) = \{1,2,\ldots,n-1\}$. Since $\nabla_{z_i} V(z) = z_i(z_i^2-d_i^2) = 0$ if $i \in \mathcal{I}_1(z)$, then

$$\begin{aligned}\nabla V(z)\cdot v &= \sum_{i=1}^{n-1} z_i(z_i^2-d_i^2)v_i \\ &= \sum_{i\in\mathcal{I}_2(z)} z_i(z_i^2-d_i^2)v_i \ .\end{aligned}$$

Let $i \in \mathcal{I}_2(z)$ and $v \in \mathcal{K}(f(z))$. In view of (1), for $i = 1,2,\ldots,n-1$, it holds:

$$\begin{aligned}v_i \in \{\mu \in \mathbb{R} : \mu = (2\lambda_1 - 1)\tilde{k}_{i+1} \\ -(k_i+1)\mathrm{sgn}(z_i)\mathrm{sgn}(|z_i|-d_i),\ \lambda_i \in [0,1]\}\ ,\end{aligned}$$

with

$$\tilde{k}_{i+1} = \begin{cases} k_2 & i=1 \\ 1+k_{i+1} & i=2,\ldots,n-2 \\ 1 & i=n-1\ .\end{cases}$$

Then

$$\begin{aligned}\nabla V(z)\cdot v &= \sum_{i\in\mathcal{I}_2(z)} z_i(z_i^2-d_i^2)v_i \\ &\leq \sum_{i\in\mathcal{I}_2(z)} -(k_i+1)|z_i||z_i^2-d_i^2| \\ &\quad +\tilde{k}_{i+1}|z_i||z_i^2-d_i^2||2\lambda_j-1|\ .\end{aligned}$$

By (4), $k_i+1-\tilde{k}_{i+1} \geq 1$ for all i, and therefore, if $z \in \mathcal{D}\setminus\mathcal{E}$, then

$$\nabla V(z)\cdot v \leq -\sum_{i\in\mathcal{I}_2(z)} |z_i||z_i^2-d_i^2| < 0\ ,$$

for all $v \in \mathcal{K}(f(z))$. This shows that for all $z \in \mathcal{D}\setminus\mathcal{E}$, either $\max\dot{\overline{V}}(z) < 0$ or $\dot{\overline{V}}(z) = \emptyset$. In summary, for all $z \in \mathbb{R}^{n-1}$, either $\max\dot{\overline{V}}(z) \leq 0$ or $\dot{\overline{V}}(z) = \emptyset$, and $0 \in \dot{\overline{V}}(z)$ if and only if $z \in \mathcal{E}$.

It is known (Lemma 1 in [16]) that if $\varphi(t)$ is a solution of the differential inclusion $\dot{z} \in \mathcal{K}(f(z))$, then $\frac{d}{dt}V(\varphi(t))$ exists almost everywhere and $\frac{d}{dt}V(\varphi(t)) \in \dot{\overline{V}}(\varphi(t))$. We conclude that $V(\varphi(t))$ is non-increasing. Let $z_0 \in S$, with $S \subset \mathbb{R}^{n-1}$ a compact and strongly invariant set for (3). For any z_0, such a set exists and includes the point $(d_1,d_2,\ldots,d_{n-1}) \in \mathcal{E}$ (hence $S\cap\mathcal{E} \neq \emptyset$), by definition of $V(z)$ and because $V(z)$ is non-increasing along the solutions of (3). Since $\max\dot{\overline{V}}(z) \leq 0$ or $\dot{\overline{V}}(z) = \emptyset$ for all $z \in \mathbb{R}^{n-1}$, then by the LaSalle invariance principle for differential inclusions [16], [17], any solution $\varphi(t)$ to the differential inclusion converges to the largest weakly invariant set in $S\cap\overline{\mathcal{E}} = S\cap\mathcal{E}$ ($\mathcal{E}$ is closed). Since the choice (4) yields that the gains k_i's satisfy the condition in Lemma 1, $\mathcal{E}$ is the set of equilibria of (3) (and therefore it is weakly invariant) and since $S\cap\mathcal{E} \neq \emptyset$, we conclude that any solution $\varphi(t)$ converges to the set of points $S\cap\mathcal{E}$. □

Since the equilibrium set $\mathcal{E}$ contains those points for which two agents coincide with each other, it is of interest to characterize those initial conditions under which the asymptotic positions of some of the agents become coincident. In the next subsection, we use a three-agent formation as an example to show how such analysis can be carried out.

C. Trajectory based analysis

We specialize the rigid formation examined before to the case $n = 3$. Letting $k_1 = k_2 = 1$, the one-dimensional rigid formation becomes:

$$\begin{aligned}\dot{x}_1 &= -\mathrm{sgn}(x_1-x_2)\mathrm{sgn}(|x_1-x_2|-d_1) \\ \dot{x}_2 &= \mathrm{sgn}(x_1-x_2)\mathrm{sgn}(|x_1-x_2|-d_1)- \\ &\quad \mathrm{sgn}(x_2-x_3)\mathrm{sgn}(|x_2-x_3|-d_2) \\ \dot{x}_3 &= \mathrm{sgn}(x_2-x_3)\mathrm{sgn}(|x_2-x_3|-d_2)\ .\end{aligned} \tag{5}$$

Let us express the system in the coordinates z_1, z_2, so as to obtain:

$$\begin{aligned}\dot{z}_1 &= -2\mathrm{sgn}(z_1)\mathrm{sgn}(|z_1|-d_1) + \mathrm{sgn}(z_2)\mathrm{sgn}(|z_2|-d_2) \\ \dot{z}_2 &= \mathrm{sgn}(z_1)\mathrm{sgn}(|z_1|-d_1) - 2\mathrm{sgn}(z_2)\mathrm{sgn}(|z_2|-d_2)\ .\end{aligned} \tag{6}$$

We study the solutions of the system above. In what follows, it is useful to distinguish between two sets of points:

$$\begin{aligned}\mathcal{E}_1 &= \{z \in \mathbb{R}^2 : |z_i| = d_i,\ i=1,2\}, \\ \mathcal{E}_2 &= \{z \in \mathbb{R}^2 : |z_i| = d_i \text{ or } |z_i| = 0,\ i=1,2\}\ .\end{aligned}$$

Clearly, $\mathcal{E}_1 \subset \mathcal{E}_2$. We now prove that all the solutions converge to the desired set $\mathcal{E}_1$ except for solutions which originates on the z_1- or the z_2-axis:

Theorem 2: All Krasowskii solutions of (6) converge in finite time to the set $\mathcal{E}_2$. In particular, the solutions which converge to the points $\{(d_1,0),(0,d_2),(-d_1,0),(0,-d_2)\}$ must originate from the set of points $\{z : z_1\cdot z_2 = 0, z \neq 0\}$. Moreover, the only solution which converges to $(0,0)$ is the trivial solution which originates from $(0,0)$.

Proof: Because of the symmetry of $f(z)$, it suffices to study the solutions which originate in the first quadrant only. In the first quadrant we distinguish four regions: (i) $\mathcal{R}_1 = \{z \in \mathbb{R}^2 : z_i \geq d_i, i=1,2\}$, (ii) $\mathcal{R}_2 = \{z \in \mathbb{R}^2 : 0 \leq z_1 < d_1,\ z_2 \geq d_2\}$, (iii) $\mathcal{R}_3 = \{z \in \mathbb{R}^2 : 0 \leq z_i < d_i,\ i=1,2\}$, (iv) $\mathcal{R}_4 = \{z \in \mathbb{R}^2 : z_1 \geq d_1,\ 0 \leq z_2 < d_2\}$. Now we examine the solutions originating in these regions.

(i) $z(0) \in \mathcal{R}_1$. If both $z_1(0) > d_1$ and $z_2(0) > d_2$, then the system equations become

$$\dot{z}_1 = -1\ , \quad \dot{z}_2 = -1$$

and the solution satisfies $z_2(t) = z_1(t) + z_2(0) - z_1(0)$. In other words, the solution evolves along the line of slop $+1$ and intercept $z_2(0)-z_1(0)$. If $z_2(0)-z_1(0) = d_2-d_1$, then the solution $z(t)$ converges to the point $z = (d_1,d_2)$ in finite time. In particular $z(t_1) = (d_1,d_2)$ with $t_1 = z_1(0)-d_1 = z_2(0)-d_2$. If $z_2(0)-z_1(0) > d_2-d_1$, then $z(t)$ converges in finite time to the semi-axis $\{z : z_1 = d_1,\ z_2 > d_2\}$. This is a set of points at which $f(z)$ is discontinuous, since for $z_1 \geq d_1$, $f(z) = (-1,-1)$, and for $z_1 < d_1$, $f(z) = (3,-3)$. Since at these points $F(z) = \mathrm{co}\{(-1,-1),(3,-3)\}$,[1] and vectors in $F(z)$ intersect the tangent space at the semi-axis in those points, a sliding mode along the semi-axis must occur. Since $\dot{z}(t) \in F(z(t))$, we conclude that the sliding mode must satisfy the equations

$$\dot{z}_1(t) = 0,\ \dot{z}_2(t) = -\frac{3}{2}\ ,$$

[1]Here $\mathrm{co}\{v_1,\ldots,v_m\}$ denotes the smallest closed convex set which contains $v_1,\ldots,v_m$.

and therefore, after a finite time, the solution converges to the point (d_1, d_2). On the other hand, if $z_2(0) - z_1(0) < d_2 - d_1$, then the solution reaches the ray $\{z : z_1 > d_1,\ z_2 = d_2\}$. Similar considerations as before can show that a sliding mode occurs along the ray and that it satisfies the equations

$$\dot{z}_1(t) = -\frac{3}{2},\ \dot{z}_2(t) = 0\ ,$$

and again convergence in finite time to (d_1, d_2) is inferred. Finally we examine the case $z(0) = (d_1, d_2)$. At the point (d_1, d_2),

$$F(d_1, d_2) = \mathrm{co}\{(-1, -1), (3, -3), (1, -1), (1, 1)\}\ ,$$

i.e. $\mathbf{0} \in F(d_1, d_2)$ and (d_1, d_2) is an equilibrium point. Similarly as before, one shows that the solution which originates from (d_1, d_2) must stay in (d_1, d_2).

(ii) $z(0) \in \mathcal{R}_2$. If $z_1(0) > 0$ and $z_2(0) > d_2$, then the map $f(z)$ is equal to the vector $(3, -3)$ and the solution $z(t)$ satisfies $z_2(t) = -z_1(t) + z_1(0) + z_2(0)$. If $z_1(0) + z_2(0) = d_1 + d_2$, then $z(t)$ converges to (d_1, d_2), while if $z_1(0) + z_2(0) = d_1 + d_2$, it first converges to the semi-axis $\{z : z_1 = d_1,\ z_2 > d_2\}$, and then it slides towards (d_1, d_2). When $z_1(0) + z_2(0) = d_1 + d_2$, the solution reaches the segment $\{z : 0 < z_1 < d_1,\ z_2 = d_2\}$. On this segment, $F(z) = \mathrm{co}\{(3, -3), (1, 1)\}$, and since this intersects the tangent space at the segment, a sliding mode occurs. The sliding mode obeys the equations

$$\dot{z}_1(t) = \frac{3}{2},\ \dot{z}_2(t) = 0\ ,$$

which show that the state reaches (d_1, d_2).

If $z_1(0) = 0$ and $z_2(0) > d_2$, then the initial condition lies on another discontinuity surface of $f(z)$. Observe that, for those points such that $-d_1 < z_1 \leq 0$ and $z_2 > d_2$, $f(z) = (-1, -1)$. Hence, $F(z) = \mathrm{co}\{(-1, -1), (3, -3)\}$ intersects the tangent space at the semi-axis in those points, and the solutions can slide along the semi-axis until they reach the point $(0, d_2)$ and stop, or can enter the region $\mathcal{R}_2 \setminus \{z : z_1 = 0, z_2 > d_2\}$, and then converge to (d_1, d_2), or they can enter the region $\{z : -d_1 < z_1 < 0, z_2 > d_2\}$ and converge to the point $(-d_1, d_2)$.

The point $(0, d_2)$ is an equilibrium, and if $z(0) = (0, d_2)$, solutions stay at the equilibrium.

We review the remaining cases succinctly, as they are qualitatively similar to the cases examined above.

(iii) $z(0) \in \mathcal{R}_3$. If $z_i(0) > 0$ for $i = 1, 2$, then the solutions converge to (d_1, d_2) possibly sliding along the segments $\{z : 0 < z_1 \leq d_1, z_2 = d_2\}$ or $\{z : z_1 = d_1, 0 < z_2 \leq d_2\}$. If $z_1(0) = 0$ and $z_2(0) > 0$, then the solution can converge to the points $(-d_1, d_2)$, $(0, d_2)$ or (d_1, d_2). If $z_1(0) > 0$ and $z_2(0) = 0$, then the solutions can converge to (d_1, d_2), $(d_1, 0)$ or $(d_1, -d_2)$. Finally, if $z_i(0) = 0$ for $i = 1, 2$, the solutions can converge to any of the points in $\mathcal{E}_2$. In particular, a possible solution is the one which remains in $(0, 0)$.

(iv) $z(0) \in \mathcal{R}_4$. Solutions which start from initial conditions such that $z_1(0) > d_1$ and $z_2(0) > 0$ converge to (d_1, d_2). If $z_1(0) = d_1$ and $z_2(0) > 0$, then the solution converge to (d_1, d_2) possibly sliding on the segment $\{z : z_1 = d_1, 0 < z_2 < d_2\}$. If $z_1(0) > d_1$ and $z_2(0) = 0$, the solutions can converge to one of the three possible points: $(d_1, -d_2)$, $(d_1, 0)$, (d_1, d_2). □

A few comments are in order:

- Sliding modes arise naturally for those situations in which, for instance, the state reaches the semi-axis $\{z : z_1 > d_1, z_2 = d_2\}$. This forces us to consider Krasowskii solutions rather than Carathéodory solutions. On the other hand, the set of Krasowskii solutions may be too large in some cases, as it is evident for instance for those solutions which start on the z_1- or z_2-axis.
- The occurrence of sliding modes are not acceptable in practice as they would require fast information transmission. A mechanism to prevent sliding modes in the system (5) can be introduced following [12].

IV. Simulation results

In this section, we present simulation results for the guided formation control with coarsely quantized information. We consider a formation consisting of 6 agents, labeled by $1, \ldots, 6$. The distance constraints are $|x_i - x_{i+1}| = 1$, $i = 1, \ldots, 5$. The initial positions of agents 1 to 6 are 0, 0.5, 1, 2, 4 and 5 respectively. Then the shape of the initial formation is shown in figure 1. We choose $k_1 = 6$, $k_2 = 5$, $k_3 = 4$, $k_4 = 3$ and $k_5 = 2$ and simulate the agents' motion under the control laws (1). In figure 2, we show the shape of the final formation. To see how the shape evolves with time, we present the curve of the Lyapunov function $V(z) = \frac{1}{4}\sum_{i=1}^{5}(z_i^2 - d_i^2)^2$ in figure 3. Since our analysis has been carried out using Krasowskii solutions, when we further look into the dynamics of z, it is clear that the sliding mode may still happen when the Krasowskii solution converges. But this effect due to the system's non-smoothness is within an acceptable level as shown in figure 4 which presents the curve of z_1.

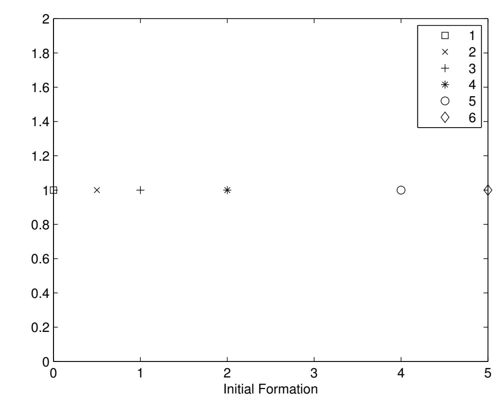

Fig. 1. The initial shape of the 6-agent formation.

V. Concluding remarks

In this paper, we have studied the problem of controlling a one-dimensional guided formation using coarsely quantized

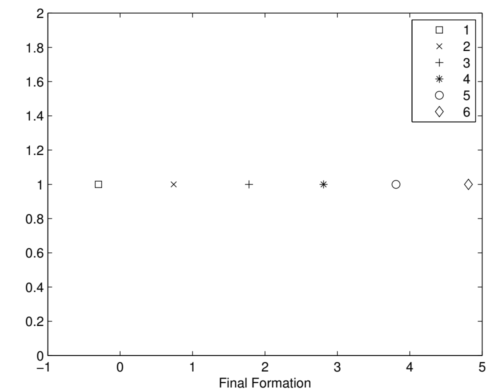

Fig. 2. The final shape of the 6-agent formation.

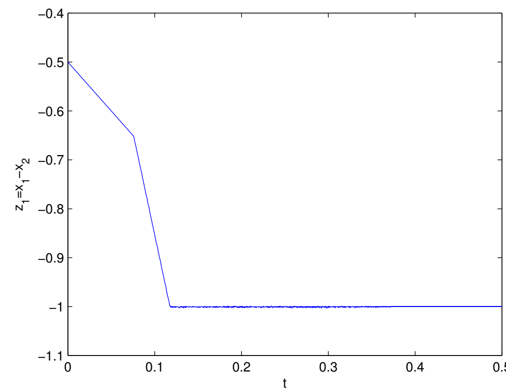

Fig. 4. The dynamics of z_1.

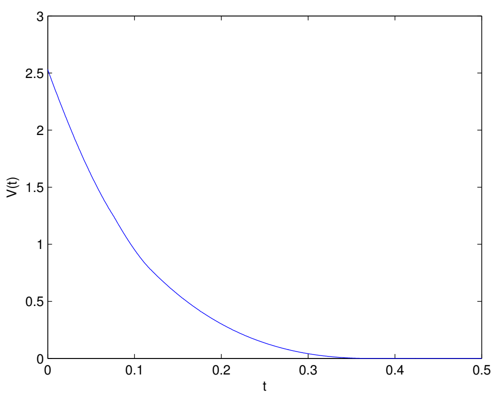

Fig. 3. The curve of the Lyapunov function V.

information. It has been shown that even when the guidance system adopts quantizers that return only the one-bit sign information about the quantized signal, the formation can still converge to the desired equilibrium under the proposed control law.

The point model we have used throughout the analysis is a simplified description of vehicle dynamics. When more detailed models are taken into consideration, we need to deal with collision avoidance and other practical issues as well. So it is of great interest to continue to study the same problem with more sophisticated vehicle models and more physical constraints from the applications. It is also worth pointing out that when different assumptions are made about the type of quantizers to be utilized, the convergence behavior of the system may change.

REFERENCES

[1] M. Broucke and P. Varaiya. The automated highway system: A transportation technology for the 21st century. *Control Engineering Practice*, 5:1583–1590, 1997.

[2] R. Bishop. A survey of intelligent vehicle applications worldwide. In *Proc. of the 2000 IEEE Intelligent Vehicles Symposium*, pages 25–30, 2000.

[3] S. E. Shladover. PATH at 20 - History and major milestones. *IEEE Transactions on Intelligent Transportation Systems*, 8:584–592, 2007.

[4] E. Tuci, R. Grob, V. Trianni, F. Mondada, M. Bonani, and M. Dorigo. Cooperation through self-assembly in multi-robot systems. *ACM Transactions on Autonomous and Adaptive Systems*, 1:115–150, 2006.

[5] B. D. O. Anderson, C. Yu, B. Fidan, and J. Hendrickx. Rigid graph control architectures for autonomous formations. *IEEE Control Systems Magazine*, 28:48–63, 2008.

[6] M. Cao, A. S. Morse, C. Yu, B. D. O. Anderson, and S. Dasgupta. Controlling a triangular formation of mobile autonomous agents. In *Proc. of the 45th IEEE CDC*, pages 3603–3608, 2007.

[7] L. Krick, M. E. Broucke, and B. A. Francis. Stabilization of infinitesimally rigid formations of multi-robot networks. *International Journal of Control*, 82:423–439, 2009.

[8] A. Kashyap, T. Basar, and R. Srikant. Quantized consensus. *Automatica*, 43:1192–1230, 2007.

[9] P. Frasca, R. Carli, F. Fagnani, and S. Zampieri. Average consensus on networks with quantized communication. *International Jouranl of Robust and Nonliner Control*, 19:1787–1816, 2009.

[10] D. V. Dimarogonas and K. H. Johansson. Stability analysis for multi-agent systems using the incidence matrix: Quantized communication and formation control. *Automatica*, 46:695–700, 2010.

[11] G. N. Nair, F. Fagnani, S. Zampieri, and R. J. Evans. Feedback control under data rate constraints: An overview. *Proceedings of IEEE*, 95:108–137, 2007.

[12] F. Ceragioli, C. De Persis, and P. Frasca. Discontinuities and hysteresis in quantized average consensus. In *ArXiv:1001.2620v2*, 2010. Available online at `http://arxiv.org/abs/1001.2620`.

[13] F. Ceragioli and C. De Persis. Discontinuous stabilization of nonlinear systems: Quantized and switching controls. *Systems and Control Letters*, 56:461–473, 2007.

[14] C. De Persis and M. Cao. Guiding one-dimensional formations of mobile agents with coarsely quantized information. In *Proc. of the 2nd IFAC Workshop on Distributed Estimation and Control in Networked Systems*, 2010.

[15] B. Paden and S. S. Sastry. A calculus for computing Filippov's differential inclusion with application to the variable structure control of robot manipulators. *IEEE Transactions on Circuits and Systems*, 34:73–82, 1987.

[16] A. Bacciotti and F. Ceragioli. Stability and stabilization of discontinuous systems and nonsmooth Liapunov functions. *ESAIM: Control, Optimisation & Calculus of Variations*, 4:361–376, 1999.

[17] J. Cortes. Distributed algorithms for reaching consensus on general functions. *Automatica*, 44:726–737, 2008.