Bounded-Energy-Input Convergent-State Property of Dissipative Nonlinear Systems

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Bounded-energy-input convergent-state property of dissipative nonlinear systems: an iISS approach

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***Abstract*—For a class of dissipative nonlinear systems, it is shown that an *i*ISS gain can be computed directly from the corresponding supply function. The result is used to prove the convergence to zero of the state whenever the input signal has bounded energy, where the energy functional is determined by the supply function.**

***Index Terms*—Integral input-to-state stability, dissipative nonlinear systems.**

I. INTRODUCTION

FOr a linear system $\dot{x} = Ax + Bu$, with A Hurwitz, the following property is elementary: if x is a solution on $\mathbb{R}_+ := [0,\infty)$ corresponding to an input $u \in L^p$ for some $p \in [1,\infty)$ (an input of bounded energy), then $x(t) \to 0$ as $t \to \infty$. The question of nonlinear counterparts arises: to what extent (and for which measures of energy) does the bounded-energy-input/convergent-state (BEICS) property hold in the context of a finite-dimensional nonlinear system $\dot{x} = f(x,u)$ under the 0-GAS hypothesis (that is, the assumption that 0 is a globally asymptotically stable equilibrium of the associated autonomous system $\dot{x} = f(x,0)$)? On the one hand, even in the simplest of nonlinear systems satisfying the latter hypothesis, the BEICS property may fail to hold. In [16], Sontag and Krichman construct an example of a 0-GAS system of the form $\dot{x} = f_0(x) + u$ with the property that, for every $\varepsilon > 0$, there is an integrable function, with L^1 norm $\|u\|_1 < \varepsilon$, such that the system admits an unbounded solution: subsequently, in [17], Teel and Hespanha provide an example of a system of similar structure, but with the stronger property of 0-GES (that is, 0 is a globally exponentially stable equilibrium of $\dot{x} = f_0(x)$) for which an exponentially decaying additive input u, arbitrarily small in L^p, can give rise to an unbounded solution. On the other hand, if $\dot{x} = f(x,u)$, with $f\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ locally Lipschitz and $f(0,0) = 0$, is integral input-to-state stable (iISS) (see, [14]), with associated iISS gain function γ (to be made precise in due course), then it is well known that the system is 0-GAS and has the BEICS property with respect to "integrable" (bounded-energy) inputs, provided that integrability is defined via the energy-like functional $u \mapsto \int_0^\infty \gamma(\|u(t)\|\mathrm{d}t$, in which case we say that the system has the γ-BEICS property. Theorem 1 in [2] (see, also, [1]) subsumes the following: if the system is (a) 0-GAS and (b) dissipative with supply function σ (in short, σ-dissipative) in the sense that there exist a proper, positive-definite C^1 function U of Lyapunov type and a class $\mathcal{K}$ function σ such that $\langle \nabla U(\xi), f(\xi,\nu)\rangle \le \sigma(\|\nu\|)$ for all (ξ,ν), then the system is iISS. The crucial point to bear in mind here is that the latter result is non-constructive: the properties of 0-GAS and σ-dissipativity imply only the *existence* of some iISS gain function – the issue of constructing an iISS gain remains; the supply function σ is not in general an iISS gain function and so one cannot conclude that the system has the σ-BEICS property. The main contribution of the present paper is to show that the following condition

$$\forall \text{compact } K \subset \mathbb{R}^n,\quad \exists c > 0:\quad \|f(\xi,\nu)\| \le c\big(1+\sigma(\|\nu\|)\big) \quad \forall (\xi,\nu) \in K \times \mathbb{R}^m,$$

in conjunction with 0-GAS and σ-dissipativity, ensures that σ is an iISS gain function and so the σ-BEICS property holds.

The computation of iISS gain is pertinent to the stability analysis of interconnected systems which contain iISS systems and to the robustness analysis of closed-loop systems. For example, the papers [1], [6] use the knowledge of iISS gain in its subsystem(s) to conclude the stability property of the interconnected systems. Based on the precursor [9] to the present paper, Wang and Weiss [18] use our main result for computing the iISS gain in a robustness analysis of a controlled wind turbine.

II. PRELIMINARIES

We consider nonlinear systems, with input u, of the form

$$\dot{x} = f(x,u), \quad x(0) = x^0 \in \mathbb{R}^n, \quad f(0,0) = 0, \quad f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \text{ locally Lipschitz.} \tag{1}$$

Throughout, the space of inputs is taken to be $\mathcal{U} := L^\infty_{\mathrm{loc}}(\mathbb{R}_+,\mathbb{R}^m)$, that is, the space of measurable locally essentially bounded functions $\mathbb{R}_+ \to \mathbb{R}^m$.

Definition 2.1: For $u \in \mathcal{U}$, $x^0 \in \mathbb{R}_+$, *a solution of* (1) is an absolutely continuous function $x: [0,\omega) \to \mathbb{R}^n$, $\omega > 0$, such that

$$x(t) - x(0) = \int_0^t f(x(\tau),u(\tau))\mathrm{d}\tau \qquad \forall t \in [0,\omega).$$

A solution is *maximal* if it has no proper right extension that is also a solution. A solution is *global* it it exists on $\mathbb{R}_+$.

The following is a consequence of the standard theory of ordinary differential equations (see, e.g. [13]).

Proposition 2.2: For each $u \in \mathcal{U}$ and $x^0 \in \mathbb{R}^n$, the initial-value problem (1) has unique maximal solution $x: [0,\omega) \to \mathbb{R}^n$.

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The set of continuous, strictly-increasing functions $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$, with $\alpha(0) = 0$, is denoted by $\mathcal{K}$ and $\mathcal{K}_\infty \subset \mathcal{K}$ is the set of unbounded functions in $\mathcal{K}$. The set $\mathcal{KL}$ consists of all functions $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that $\beta(\cdot,t) \in \mathcal{K}$ for all $t \in \mathbb{R}_+$ and, for all $s \in \mathbb{R}_+$, $\beta(s,\cdot)$ is decreasing and $\beta(s,t) \to 0$ as $t \to \infty$. A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be *positive definite* if it is continuous, $\alpha(0) = 0$ and $\alpha(s) > 0$ for all $s > 0$.

The concept of *integral input-to-state stability* (*i*ISS), introduced in [14] and further developed in, *inter alia*, [2], [3] (the expository article [15] contains a particularly succinct survey), is central to the present paper.

Definition 2.3: System (1) is said to be *integral input-to-state stable* (*i*ISS) if there exist functions $\alpha \in \mathcal{K}_\infty$, $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ (the latter will be referred to as an *i*ISS *gain*) such that, for every $x^0 \in \mathbb{R}^n$ and for every $u \in \mathcal{U}$, the unique maximal solution x of (1) is global and

$$\alpha(\|x(t)\|) \le \beta(\|x^0\|,t) + \int_0^t \gamma(\|u(s)\|)\mathrm{d}s \quad \forall\, t \in \mathbb{R}_+. \qquad (2)$$

An immediate consequence of this definition is

$$\text{(1) is } i\text{ISS} \quad \Longrightarrow \text{(1) is 0-GAS.} \qquad (3)$$

Furthermore, if system (1) is *i*ISS with gain γ and we define an energy functional on $\mathcal{U}$ by $u \mapsto \int_0^\infty \gamma(\|u(t)\|)\mathrm{d}t$, then (1) has the BEICS property. We record this fact in the next proposition (see [14, Proposition 6]).

Proposition 2.4: Assume (1) is *i*ISS with *i*ISS gain $\gamma \in \mathcal{K}$. Let $u \in \mathcal{U}$ satisfy $\int_0^\infty \gamma(\|u(t)\|)\mathrm{d}t < \infty$. Then, for all $x^0 \in \mathbb{R}^n$, the unique global solution x of (1) satisfies $x(t) \to 0$ as $t \to \infty$.

Definition 2.5: A continuously differentiable function $U : \mathbb{R}^n \to \mathbb{R}_+$ is an *i*ISS*-Lyapunov function* for system (1) if there exist functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty, \sigma \in \mathcal{K}$ and a continuous, positive-definite function α_3 such that the following hold:

$$\alpha_1(\|\xi\|) \le U(\xi) \le \alpha_2(\|\xi\|) \quad \forall \xi \in \mathbb{R}^n, \qquad (4)$$

$$\langle \nabla U(\xi), f(\xi,v)\rangle \le -\alpha_3(\|\xi\|) + \sigma(\|v\|) \quad \forall (\xi,v) \in \mathbb{R}^n \times \mathbb{R}^m. \qquad (5)$$

The concept of *i*ISS admits the following elegant characterization [2]: system (1) is *i*ISS if, and only if, it admits a *smooth* (that is, C^∞) *i*ISS -Lyapunov function. However, in our later analysis, we will not wish to impose C^∞ smoothness on various functions arising therein. With this in mind and reiterating Remark II.3 of [2], existence of an *i*ISS Lyapunov function is a *sufficient condition* for *i*ISS (smoothness is not required): in particular, system (1) is *i*ISS if it admits an *i*ISS-Lyapunov function. We record this and related facts in Proposition 2.6 below, which we preface with some terminology.

With $\sigma \in \mathcal{K}$, we associate an energy functional

$$E_\sigma(u) := \int_0^\infty \sigma(\|u(t)\|)\mathrm{d}t,$$

and write $\mathcal{U}_\sigma := \{u \in \mathcal{U} |\ \ E_\sigma(u) < \infty\}$. System (1) is said to have the BEICS property with respect to the energy functional E_σ (for brevity, σ-BEICS) if, for all $u \in \mathcal{U}_\sigma$ and $x^0 \in \mathbb{R}^n$, the unique global solution x of (1) is such that $x(t) \to 0$ as $t \to \infty$.

Proposition 2.6: Assume that there exist a C^1 function $U : \mathbb{R}^n \to \mathbb{R}_+$, functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty, \sigma \in \mathcal{K}$ and a continuous, positive-definite function α_3 such that (4) and (5) hold. Then

(a) system (1) is *i*ISS with *i*ISS gain $\gamma = \sigma$;
(b) system (1) has the σ-BEICS property.

Proof: The proof of Assertion (a) is implicit in the proof of [2, Theorem 1]; the conjunction of Assertion (a) and Proposition 2.4 gives Assertion (b). ■

Our study now focusses on the case wherein (5) is replaced by the weaker assumption:

$$\langle \nabla U(\xi), f(\xi,v)\rangle \le \sigma(\|v\|) \quad \forall (\xi,v) \in \mathbb{R}^n \times \mathbb{R}^m. \qquad (6)$$

To distinguish this case, we adopt some further terminology.

If there exist a C^1 function $U : \mathbb{R}^n \to \mathbb{R}_+$, functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\sigma \in \mathcal{K}$ such that (4) and (6) hold, then we say that (1) is *dissipative*: we refer to σ as the *supply function* σ and (6) is said to be the associated *dissipation inequality* .

Theorem 1 of [2] and [1, Lemma 1] subsume the following.

Proposition 2.7: If (1) is 0-GAS and dissipative (with supply function σ), then (1) is *i*ISS .

In contrast with Assertion (a) of Proposition 2.6, the supply function σ associated with the hypothesis of dissipativity in Proposition 2.7 is not, in general, an *i*ISS gain γ for (1). So one cannot conclude that (1) has the σ-BEICS property; however, an inspection of the proofs of [2, Theorem 1, Proposition II.5, Lemma IV.10] reveals that σ is indeed an *i*ISS gain if the function f in (1) is such that the following holds:

$$\exists\, c > 0 : \quad \|f(0,v)\| \le c\sigma(\|v\|) \quad \forall v \in \mathbb{R}^m. \qquad (7)$$

We summarise this situation as follows.

Proposition 2.8: Assume that (1) is 0-GAS and dissipative with supply function $\sigma \in \mathcal{K}$. Assume further that f and σ are such that (7) holds. Then (1) is *i*ISS with *i*ISS gain $\gamma = \sigma$ and has the σ-BEICS property.

The condition (7) can be restrictive. For example, consider the case where the system (1) is *affine in the control*, that is, for some locally Lipschitz functions $f_0 : \mathbb{R}^n \to \mathbb{R}^n$ (with $f_0(0) = 0$) and $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$,

$$f(\xi,v) = f_0(\xi) + g(\xi)v \quad \forall (\xi,v) \in \mathbb{R}^n \times \mathbb{R}^m. \qquad (8)$$

Assume that $g(0) \ne 0$ and that (1) is 0-GAS and dissipative with supply function $\sigma : s \mapsto s^p$ for some $p \ge 1$. Then (7) holds if, and only if, $p = 1$. In particular, if $p > 1$, then we cannot conclude, via Proposition 2.8, that inputs $u \in L^p$ generate state solutions converging to zero.

III. MAIN RESULT

In the affine-in-the-control system example above with $p > 1$, an application of Young's inequality yields the existence of a positive constant $c_1 > 0$ such that $\|f(0,v)\| = \|g(0)v\| \le \|g(0)\|\|v\| \le c_1(1 + \|v\|^p)$ for all $v \in \mathbb{R}^m$. The main contribution of the paper is to extrapolate this condition and identify a condition on f under which σ is an *i*ISS gain for (1) which, together with Proposition 2.4, ensures the σ-BEICS property: this we do in Theorem 3.1 below. In the context of the above affine-in-the-control system, our main result implies that, for all $p \geq 1$, if the system is 0-GAS and dissipative with supply function $\sigma : s \mapsto s^p$, then inputs $u \in L^p$ do indeed generate state solutions converging to zero (see Corollary 3.6).

Theorem 3.1: Assume that (1) is 0-GAS and dissipative with supply function $\sigma \in \mathcal{K}$. Assume further that f and σ are such that the following holds.

(A) For each compact set $K \subset \mathbb{R}^n$ there exists $c > 0$ such that

$$\|f(\xi,v)\| \leq c\big(1+\sigma(\|v\|)\big) \quad \forall (\xi,v) \in K \times \mathbb{R}^m. \qquad (9)$$

Then (1) is *i*ISS with *i*ISS gain $\gamma = \sigma$ and has the σ-BEICS property.

We preface the proof of Theorem 3.1 with three technical lemmas, wherein $\mathbf{B}_r$ denotes the closed ball in $\mathbb{R}^n$ of radius $r > 0$ and centred at 0.

Lemma 3.2: Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be continuous. Then, for each compact set K, there exists a function $\rho_K \in \mathcal{K}_\infty$ such that

$$\|f(\xi,v) - f(\xi,0)\| \leq \rho_K(\|v\|) \qquad \forall (\xi,v) \in K \times \mathbb{R}^m. \quad (10)$$

Proof: Let $K \subset \mathbb{R}^n$ be compact and define $\tilde{\rho}_K : \mathbb{R}_+ \to \mathbb{R}_+$ by $\tilde{\rho}_K(0) := 0$ and

$$\tilde{\rho}_K(a) := \max\big\{\|f(\xi,v) - f(\xi,0)\| \;\big|\; \xi \in K,\ v \in \mathbf{B}_a\big\} \quad \forall a > 0.$$

By the continuity of f, the function $\tilde{\rho}_K$ is continuous at zero. Clearly, $\tilde{\rho}_K$ is non-decreasing and so, *a fortiori*, is measurable (in fact, it can be shown that $\tilde{\rho}_K$ is upper semicontinuous). Therefore, the function $\rho_K : \mathbb{R}_+ \to \mathbb{R}_+$ is well defined by

$$\rho_K(0) := 0, \qquad \rho_K(a) := a + \frac{1}{a}\int_a^{2a} \tilde{\rho}_K(\tau)\,\mathrm{d}\tau \quad \forall a > 0.$$

It is readily verified that $\rho_K \in \mathcal{K}_\infty$. Moreover, $\rho_K(a) \geq \tilde{\rho}_K(a)$ for all $a \in \mathbb{R}_+$ and so (10) holds. ■

Lemma 3.3: Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be continuous and $\sigma \in \mathcal{K}$. Assume that **(A)** holds. Let $w : \mathbb{R}^n \to \mathbb{R}_+$ be continuous and such that, for some $\alpha \in \mathcal{K}_\infty$,

$$\alpha(\|\xi\|) \leq w(\xi) \quad \forall\ \xi \in \mathbb{R}^n. \qquad (11)$$

Then, for every continuous function $\theta : (0,\infty) \to (0,\infty)$, there exist a continuous function $\delta : (0,\infty) \to (0,\infty)$ such that

$$\|f(\xi,v) - f(\xi,0)\| \leq \theta(w(\xi)) + \delta(w(\xi))\sigma(\|v\|), \quad \forall\ \xi \in \mathbb{R}^n\backslash\{0\} \quad \forall\ v \in \mathbb{R}^m. \quad (12)$$

Proof: By continuity of f and **(A)**, it can be verified that, for every compact set $K \subset \mathbb{R}^n$, there exists $c_K > 0$ such that

$$\|f(\xi,v) - f(\xi,0)\| \leq c_K\big(1+\sigma(\|v\|)\big) \quad \forall\ (\xi,v) \in K \times \mathbb{R}^m.$$

This implies the existence of a strictly increasing sequence (c_k) in $\mathbb{N}$ such that

$$\|f(\xi,v) - f(\xi,0)\| \leq c_k(1+\sigma(\|v\|)) \quad \forall\ (\xi,v) \in \mathbf{B}_k \times \mathbb{R}^m.$$

Let $b : [0,\infty) \to (0,\infty)$ be the continuous function that linearly interpolates the points c_k, $k \in \mathbb{N}$, that is,

$$b(\lambda) := c_k + (c_{k+1} - c_k)(\lambda + 1 - k) \quad \forall \lambda \in [k-1,k) \quad \forall k \in \mathbb{N}.$$

Then, for all $(\xi,v) \in \mathbb{R}^n \times \mathbb{R}^m$,

$$\|f(\xi,v) - f(\xi,0)\| \leq b(\|\xi\|)(1+\sigma(\|v\|)). \qquad (13)$$

By Lemma 3.2, there exists $\rho_1 \in \mathcal{K}_\infty$ such that

$$\|f(\xi,v) - f(\xi,0)\| \leq \rho_1(\|v\|) \quad \forall (\xi,v) \in \mathbf{B}_1 \times \mathbb{R}^m. \qquad (14)$$

Let $\theta : (0,\infty) \to (0,\infty)$ be continuous. Denote by $\chi_1 \in \mathcal{K}_\infty$ the inverse of the function $\rho_1 \in \mathcal{K}_\infty$ and write $\tilde{b} = b \circ \alpha^{-1}$. Define the continuous function $\delta_1 : (0,1] \to (0,\infty)$ by

$$\delta_1(a) := \tilde{b}(a) + \frac{\|\tilde{b}(a) - \theta(a)\|}{\sigma(\chi_1(\theta(a)))} \quad \forall a \in (0,1].$$

If $\xi \in \mathbf{B}_1\backslash\{0\}$ and $\|v\| \leq \chi_1(\theta(w(\xi)))$ then $\rho_1(\|v\|) \leq \theta(w(\xi))$ and so, by (14),

$$\|f(\xi,v) - f(\xi,0)\| \leq \theta(w(\xi)) \leq \theta(w(\xi)) + \delta_1(w(\xi))\sigma(\|v\|).$$

If $\xi \in \mathbf{B}_1\backslash\{0\}$ and $\|v\| > \chi_1(\theta(w(\xi)))$ then, by (11) and (13),

$$\begin{aligned}\|f(\xi,v) - f(\xi,0)\| &\leq b(\|\xi\|)(1+\sigma(\|v\|))\\ &\leq \tilde{b}(w(\xi))(1+\sigma(\|v\|))\\ &\leq \theta(w(\xi)) + \|\tilde{b}(w(\xi)) - \theta(w(\xi))\|\\ &\quad + \tilde{b}(w(\xi))\sigma(\|v\|)\\ &\leq \theta(w(\xi)) + \delta_1(w(\xi))\sigma(\|v\|).\end{aligned}$$

This establishes that

$$\|f(\xi,v) - f(\xi,0)\| \leq \theta(w(\xi)) + \delta_1(w(\xi))\sigma(\|v\|) \quad \forall (\xi,v) \in (\mathbf{B}_1\backslash\{0\}) \times \mathbb{R}^m. \quad (15)$$

For every $k \in \mathbb{N}$, $k \geq 2$, let C_k denote the compact set

$$C_k := \{\xi \in \mathbb{R}^n \,|\, 1 \leq w(\xi) \leq k\}.$$

By Lemma 3.2, for each $k \geq 2$, there exists $\rho_k \in \mathcal{K}_\infty$ such that

$$\|f(\xi,v) - f(\xi,0)\| \leq \rho_k(\|v\|) \quad \forall (\xi,v) \in C_k \times \mathbb{R}^m.$$

For every $k \geq 2$, let $\chi_k \in \mathcal{K}_\infty$ denote the inverse of $\rho_k \in \mathcal{K}_\infty$ and define the continuous function $\delta_k : [1,k] \to (0,\infty)$ by

$$\delta_n(a) := \tilde{b}(a) + \frac{\|\tilde{b}(a) - \theta(a)\|}{\sigma(\chi_k(\theta(a)))}.$$

Then an argument analogous to that leading to (15) gives

$$\|f(\xi,v) - f(\xi,0)\| \leq \theta(w(\xi)) + \delta_k(w(\xi))\sigma(\|v\|) \quad \forall (\xi,v) \in C_k \times \mathbb{R}^m, \quad k = 2,3,\ldots. \quad (16)$$

Now, define

$$\delta_1^* := \delta_1(1), \quad \delta_k^* := \max\{\max_{a\in[1,k]} \delta_k(a), \delta_{k-1}^*\}, \quad k = 2,3,\ldots.$$

The sequence $(\delta_k^*)_{k\in\mathbb{N}}$ so constructed is non-decreasing. Finally, define the function $\delta : (0,\infty) \to (0,\infty)$ as follows

$$\delta(a) := \begin{cases} \delta_1(a) + \delta_2^* - \delta_1^*, & a \in (0,1]\\ \delta_{k+1}^* + (\delta_{k+2}^* - \delta_{k+1}^*)(a-k), & a \in (k,k+1], k \in \mathbb{N}\end{cases}$$

The function δ is continuous, with the properties

$$\delta(a) \geq \delta_1 \quad \forall a \in (0,1], \qquad \text{and}$$
$$\delta(a) \geq \delta_k(a) \quad \forall a \in [1,k],\ k = 2,3,\ldots.$$

In view of (15) and (16), it follows that (12) holds. ■

Lemma 3.4: Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be locally Lipschitz with $f(0,0) = 0$ and $\sigma \in \mathcal{K}$. Assume **(A)** holds and (1) is 0-GAS.

For every $\varepsilon > 0$, there exists a continuous positive-definite function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ and a C^1 function $W : \mathbb{R}^n \to \mathbb{R}_+$ such that $W(0) = 0$, $W(x) > 0$ for $x \neq 0$ and, for all $(\xi, v) \in \mathbb{R}^n \times \mathbb{R}^m$,

$$\langle W(\xi), f(\xi,v)\rangle \le -\alpha(\|\xi\|) + \varepsilon\sigma(\|v\|). \tag{17}$$

Remark 3.5: The function W in Lemma 3.4 is not necessarily proper; that is, its sublevel sets are not necessarily compact.

Proof: The 0-GAS property implies that there exist a smooth $V : \mathbb{R}^n \to \mathbb{R}_+$, $\nabla V(0) = 0$ and functions $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ such that

$$\left.\begin{aligned} \alpha_1(\|\xi\|) \le V(\xi) \le \alpha_2(\|\xi\|) \\ \langle \nabla V(\xi), f(\xi,0)\rangle \le -\alpha_3(\|\xi\|) \end{aligned}\right\} \quad \forall \xi \in \mathbb{R}^n \tag{18}$$

(see, for example, [10]). Define $\tilde{\alpha}_4 : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\tilde{\alpha}_4(a) = \max\{\|\nabla V(\xi)\| \mid \xi \in \mathbb{R}^n,\ V(\xi) \le a\} \quad \forall a \in \mathbb{R}_+.$$

By the continuity of ∇V, the function $\tilde{\alpha}_4$ is continuous at zero. The function $\tilde{\alpha}_4$ is non-decreasing and so we may define a continuous function $\alpha_4 : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\alpha_4(0) = 0, \qquad \alpha_4(a) = \frac{1}{a}\int_a^{2a} \tilde{\alpha}_4(\tau)\,\mathrm{d}\tau \quad \forall a > 0.$$

Moreover, α_4 is non-decreasing with $\alpha_4(a) \ge \tilde{\alpha}_4(a)$ for all $a \in \mathbb{R}_+$ and $\|\nabla V(\xi)\| \le \alpha_4(V(\xi))$ for all $\xi \in \mathbb{R}^n$. Now define the continuous function $\theta : (0,\infty) \to (0,\infty)$ by

$$\theta(a) = \min\left\{a, \frac{\alpha_3(\alpha_2^{-1}(a))}{2\alpha_4(a)}\right\} \quad \forall a \in (0,\infty),$$

in which case, we have

$$\begin{aligned} \|\nabla V(\xi)\|\theta(V(\xi)) \le \alpha_4(V(\xi))\theta(V(\xi)) \le \frac{1}{2}\alpha_3(\alpha_2^{-1}(V(\xi))) \\ \le \frac{1}{2}\alpha_3(\|\xi\|)\ \forall \xi \in \mathbb{R}^n. \end{aligned} \tag{19}$$

By Lemma 3.3, there exists a continuous function $\delta : (0,\infty) \to (0,\infty)$ such that, for all $(\xi, v) \in (\mathbb{R}^n\backslash\{0\}) \times \mathbb{R}^m$,

$$\|f(\xi,v) - f(\xi,0)\| \le \theta(V(\xi)) + \delta(V(\xi))\sigma(\|v\|). \tag{20}$$

Let $\varepsilon > 0$ and define a continuous function $\kappa \in \mathcal{K}$ by

$$\kappa(0) = 0, \quad \kappa(a) = \min\left\{a, \frac{\varepsilon}{\alpha_4(a)\delta(a)}\right\} \quad \forall a \in (0,\infty).$$

It follows that, for all $\xi \in \mathbb{R}^n\backslash\{0\}$,

$$\kappa(V(\xi))\|\nabla V(\xi)\|\delta(V(\xi)) < \frac{\varepsilon\|\nabla V(\xi)\|\delta(V(\xi))}{\alpha_4(V(\xi))\delta(V(\xi))} \le \varepsilon. \tag{21}$$

Define the function $W : \mathbb{R}^n \to \mathbb{R}_+$ by $W(\xi) := \int_0^{V(\xi)} \kappa(\tau)\,\mathrm{d}\tau$, which is C^1. Since $V(0) = 0$, $\nabla V(0) = 0$ and $\kappa(0) = 0$, it follows that $W(0) = 0$ and $\nabla W(0) = 0$. Since κ and V are positive definite, it follows that $W(\xi) > 0$ for all $\xi \neq 0$. Invoking (18), (19), (20) and (21), we have

$$\begin{aligned} &\langle \nabla W(\xi), f(\xi,v)\rangle = \kappa(V(\xi))\langle \nabla V(\xi), f(\xi,v)\rangle \\ &\le \kappa(V(\xi))\|\nabla V(\xi)\|\|f(\xi,v) - f(\xi,0)\| - \kappa(V(\xi))\alpha_3(\|\xi\|) \\ &\le \kappa(V(\xi))\|\nabla V(\xi)\|\left[\theta(V(\xi)) + \delta(V(\xi))\sigma(\|v\|)\right] \\ &\qquad - \kappa(V(\xi))\alpha_3(\|\xi\|) \\ &\le -\frac{1}{2}\kappa(V(\xi))\alpha_3(\|\xi\|) + \varepsilon\sigma(\|v\|) \end{aligned} \tag{22}$$

for all $(\xi, v) \in \mathbb{R}^n\backslash\{0\} \times \mathbb{R}^m$.

Since $\langle \nabla W(0), f(0,v)\rangle = 0$ for all $v \in \mathbb{R}^m$, (22) holds for all $(\xi, v) \in \mathbb{R}^n \times \mathbb{R}^m$. The proof is completed by setting

$$\alpha(a) := \frac{1}{2}\min\left\{\kappa(V(\eta))\alpha_3(\|\eta\|) \mid \|\eta\| \le a,\ \eta \in \mathbb{R}^n\right\}.$$

■

PROOF OF THEOREM 3.1. By dissipativity of (1), there exist a C^1 function $V : \mathbb{R}^n \to \mathbb{R}_+$, $\tilde{\alpha}_1, \tilde{\alpha}_2 \in \mathcal{K}_\infty$ and $\sigma \in \mathcal{K}$ such that

$$\left.\begin{aligned} \tilde{\alpha}_1(\|\xi\|) \le V(\xi) \le \tilde{\alpha}_2(\|\xi\|) \quad \forall \xi \in \mathbb{R}^n, \\ \langle \nabla V(\xi), f(\xi,v)\rangle \le \sigma(\|v\|) \quad \forall (\xi,v) \in \mathbb{R}^n \times \mathbb{R}^m. \end{aligned}\right\} \tag{23}$$

By Lemma 3.4, there exists a continuous positive-definite function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ and $W \in C^1(\mathbb{R}^n, \mathbb{R}_+)$ such that $W(0) = 0$, $W(x) > 0$ for $x \neq 0$ and (17) holds.

Define the C^1 function $U : \mathbb{R}^n \to \mathbb{R}_+$ by $U(\xi) = \frac{1}{2}\big(V(\xi) + W(\xi)\big)$ for all $\xi \in \mathbb{R}^n$. Setting $\alpha_3 = \alpha/2$ and invoking (17) together with the second of inequalities (23), we conclude that (5) holds. For each $a \in \mathbb{R}_+$, define $\alpha_4(a) := \min\{W(\xi) \mid \|\xi\| \le a\}$ and $\alpha_5(a) := \max\{W(\xi) \mid \|\xi\| \le a\}$. Then $\alpha_1 := (\tilde{\alpha}_1 + \alpha_4)/2$ and $\alpha_2 := (\tilde{\alpha}_2 + \alpha_5)/2$ are $\mathcal{K}_\infty$ functions satisfying (4). An application of Proposition 2.6 completes the proof. □

Recalling Proposition 2.8 and the discussion in the paragraph thereafter, the following fact is known: if the system (1) is affine in the control, 0-GAS and dissipative with supply function $\sigma\colon s \mapsto s$, then (1) is *i*ISS and has the BEICS property with respect to the L^1 energy functional $u \mapsto \int_0^\infty \|u(t)\|\mathrm{d}t$. The following corollary extends this result and, as a special case, establishes the BEICS property with respect to the L^p energy functional for all $1 < p < \infty$.

Corollary 3.6: Assume that system (1) is affine in the control, 0-GAS and dissipative with supply function $\sigma\colon s \mapsto \int_0^s \vartheta(z)\mathrm{d}z$ for some $\vartheta \in \mathcal{K}$. Then (1) is *i*ISS with *i*ISS gain $\gamma = \sigma$ and has the σ-BEICS property.

Proof: In view of Theorem 3.1, it suffices to show that property **(A)** holds. Since (1) is affine in the control, there exist locally Lipschitz $f_0 : \mathbb{R}^n \to \mathbb{R}^n$, with $f_0(0) = 0$, and $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ such that (8) holds. Let the compact set $K \subset \mathbb{R}^n$ be arbitrary. Then there exists $c_0 > 0$ such that $\|f_0(\xi)\| \le c_0$ and $\|g(\xi)\| \le c_0$ for all $\xi \in K$. Therefore, $\|f(\xi,v)\| \le c_0(1 + \|v\|)$ for all $(\xi,v) \in K \times \mathbb{R}^m$.

Let $c_1 > 0$. Then $\|f(\xi,v)\| \le c_0\big(1 + c_1\big)$ for all $(\xi,v) \in K \times \mathbf{B}_{c_1}$. On the other hand, if $\|v\| > c_1$, we have

$$\begin{aligned} \|v\| &\le c_1 + \frac{1}{\vartheta(c_1)}\int_{c_1}^{\|v\|} \vartheta(z)\ \mathrm{d}z \\ &\le c_1 + \frac{1}{\vartheta(c_1)}\sigma(\|v\|). \end{aligned}$$

Thus, using $c = c_0(1 + c_1 + 1/\vartheta(c_1))$,

$$\|f(\xi,v)\| \le c\big(1 + \sigma(\|v\|)\big) \quad \forall (\xi,v) \in K \times \mathbb{R}^m.$$

This completes the proof. ■

Example 3.7: Consider system (1) with

$$f : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2,\ (\xi, v) = (\xi_1, \xi_2, v) \mapsto \begin{bmatrix} -\xi_2 \\ \xi_1 - \xi_2^3 + \xi_2 v \end{bmatrix}.$$

For $U\colon \xi \mapsto 2\|\xi\|^2$, we have

$$\langle \nabla U(\xi), f(\xi,v)\rangle = -4\xi_2^4 + 4\xi_2^2 v \le v^2 \quad \forall (\xi,v) \in \mathbb{R}^2 \times \mathbb{R}.$$

Thus, the system is dissipative with supply function $\sigma\colon s \mapsto s^2$.

Moreover, an application of LaSalle's invariance principle confirms that the system is 0-GAS. By Corollary 3.6, it follows that the system is *i*ISS with *i*ISS gain $\gamma = \sigma$ and has the BEICS property with respect to the L^2 energy functional $u \mapsto \int_0^\infty u^2(t)\mathrm{d}t$. We remark that it is not clear if one can invoke Proposition 2.7 to arrive at the same conclusion.

Next, we highlight further consequences of Theorem 3.1.

IV. WEAKLY ZERO-DETECTABLE SYSTEMS

Here, we investigate a situation which, in essence, is intermediate between satisfaction of the *i*ISS inequality (5) and the dissipation inequality (6) (see (24) below).

Let $h : \mathbb{R}^n \to \mathbb{R}^l$ be continuous, with $h(0) = 0$. As in [2], system (1) is said to be *weakly zero-detectable with respect to h* if the following holds: if x is a global solution of $\dot{x} = f(x,0)$ with the property that $h(x(t)) = 0$ for all $t \in \mathbb{R}_+$, then $\lim_{t\to\infty} x(t) = 0$.

Corollary 4.1: Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be locally Lipschitz and $h : \mathbb{R}^n \to \mathbb{R}^l$ continuous, with $f(0,0) = 0 = h(0)$. Assume that there exist functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\sigma \in \mathcal{K}$, a continuous positive-definite function α and a C^1 function $U : \mathbb{R}^n \to \mathbb{R}_+$ such that (4) holds and, for all $(\xi,v) \in \mathbb{R}^n \times \mathbb{R}^m$,

$$\langle \nabla U(\xi), f(\xi,v)\rangle \le -\alpha(\|h(\xi)\|) + \sigma(\|v\|). \tag{24}$$

Assume further that f and σ satisfy **(A)** and that (1) is weakly zero-detectable with respect to h. Then (1) is *i*ISS with *i*ISS gain $\gamma = \sigma$ and has the σ-BEICS property.

Proof: In view of Theorem 3.1, it suffices to show that (1) is 0-GAS. From (4) and (24), we may infer that the zero state is a stable equilibrium of $\dot{x} = f(x,0)$ and so, for each x^0, the unique maximal solution x of the initial-value problem is global. It remains to show that the zero state is a globally attractive equilibrium of $\dot{x} = f(x,0)$: this is a consequence of (24) in conjunction with weak zero-detectability hypothesis and the LaSalle invariance principle [11]. ■

The next result identifies a situation in which one may conclude the *i*ISS and BEICS properties without positing dissipativity *a priori*.

Corollary 4.2: Assume that system (1) is affine in the control, that is, for some locally Lipschitz functions $f_0 : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, (8) holds. Let $\vartheta \in \mathcal{K}_\infty$, and define $\sigma \in \mathcal{K}_\infty$ and $\psi \in \mathcal{K}_\infty$ by $\sigma(s) := \int_0^s \vartheta(z)\mathrm{d}z$ and $\psi(s) := \int_0^s \vartheta^{-1}(z)\mathrm{d}z$. Assume that there exist functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and a C^1 function $U : \mathbb{R}^n \to \mathbb{R}_+$ such that (4) holds and

$$\langle \nabla U(\xi), f_0(\xi)\rangle + \psi\big(\|h(\xi)\|\big) \le 0 \quad \forall \xi \in \mathbb{R}^n \tag{25}$$

where $h : \mathbb{R}^n \to \mathbb{R}^m$ is given by $h(\xi) = \big(\nabla U(\xi)\big)^T g(\xi)$. Assume further that (1) is weakly zero-detectable with respect to h. Then, system (1) is *i*ISS with *i*ISS gain $\gamma = \sigma$ and has the BEICS property with respect to the energy functional E_σ.

Proof: By the argument (*mutatis mutandis*) used in the proof of Corollary 4.1, it follows, via (4), (25) and the weak zero-detectability hypothesis, that (1) is 0-GAS. To see that (1) is dissipative with supply function $\sigma = \gamma$, note that

$$\begin{aligned}\langle \nabla U(\xi), f(\xi,v)\rangle &= \langle \nabla U(\xi), f_0(\xi)\rangle + \langle \nabla U(\xi), g(\xi)v\rangle \\ &\le \langle \nabla U(\xi), f_0(\xi)\rangle + \|h(\xi)\|\,\|v\| \\ &\le \langle \nabla U(\xi), f_0(\xi)\rangle + \psi\big(\|h(\xi)\|\big) + \sigma(\|v\|) \\ &\le \sigma(\|v\|) \quad \forall (\xi,v) \in \mathbb{R}^n \times \mathbb{R}^m,\end{aligned}$$

wherein generalized Young's inequality is used to obtain the second inequality and (25) ensures the last inequality. Therefore, (1) is dissipative with storage function σ. Invoking Corollary 3.6, the result follows. ■

Example 4.3: Consider again the system in Example 3.7, with $f(\xi,v) = f_0(\xi) + g(\xi)v$ and

$$f_0\colon \mathbb{R}^2 \to \mathbb{R}^2,\;\; \xi = (\xi_1,\xi_2) \mapsto \begin{bmatrix} -\xi_2 \\ \xi_1 - \xi_2^3 \end{bmatrix},$$
$$g\colon \mathbb{R}^2 \to \mathbb{R}^2,\;\; \xi = (\xi_1,\xi_2) \mapsto \begin{bmatrix} 0 \\ \xi_2 \end{bmatrix}.$$

Let $U\colon \xi \mapsto 2\|\xi\|^2$ and $h\colon (\xi_1,\xi_2) = \xi \mapsto \langle \nabla U(\xi), g(\xi)\rangle = 4\xi_2^2$. Then it is evident that the system is weakly zero-detectable with respect to h. Moreover,

$$\langle \nabla U(\xi), f_0(\xi)\rangle = -4\xi_2^4 = -\big|h(\xi)\big|^2/4 \quad \forall \xi \in \mathbb{R}^2,$$

and so (25) holds with $\psi\colon s \mapsto s^2/4$.

Invoking Corollary 4.2, we arrive at the same conclusion as in Example 3.7: the system is *i*ISS with *i*ISS gain $\sigma : s \mapsto s^2$ and has the σ-BEICS property

The final result establishes that, if (8) holds with bounded g and globally Lipschitz f_0 and 0 is a globally *exponentially* stable stable equilibrium of $\dot{x} = f_0(x)$, then, for each $p \in (1,\infty)$, the system has the BEICS property with respect to the L^p energy functional $u \mapsto \int_0^\infty \|u(t)\|^p\mathrm{d}t$.

Corollary 4.4: Let system (1) be affine in the control, that is, for some functions $f_0 : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, (8) holds. Assume further that f_0 is globally Lipschitz, g is locally Lipschitz and bounded, and the system is 0-GES (that is, 0 is a globally exponentially stable equilibrium of the system $\dot{x} = f_0(x)$). Then, for each $p \in (1,\infty)$, (1) is *i*ISS with *i*ISS gain $\sigma : s \mapsto s^p$ and has the σ-BEICS property.

Proof: By the global Lipschitz property of f_0 and global exponential stability of $\dot{x} = f_0(x)$, there exist a C^1 function $V : \mathbb{R}^n \to \mathbb{R}_+$ and positive constants $a_1, a_2, a_3, a_4 > 0$ such that

$$\left.\begin{aligned} &a_1\|\xi\|^2 \le V(\xi) \le a_2\|\xi\|^2, \\ &\langle \nabla V(\xi), f_0(\xi)\rangle \le -a_3 V(\xi) \\ &\|\nabla V(\xi)\| \le a_4\sqrt{V(\xi)} \end{aligned}\right\} \quad \forall \xi \in \mathbb{R}^n$$

(see, for example, [4]). Invoking boundedness of g, we may infer the existence of $a_5 > 0$ such that

$$\big\|\big(\nabla V(\xi)\big)^T g(\xi)\big\| \le a_5\sqrt{V(\xi)} \quad \forall \xi \in \mathbb{R}^n.$$

Let $p \in (1,\infty)$ be arbitrary and define $a_6 := a_3^{p-1}/a_5^p$. Now define the function $U : \mathbb{R}^n \to \mathbb{R}_+$ by $U(\xi) := \frac{2a_6}{p}\big(V(\xi)\big)^{p/2}$ in which case, (4) holds with

$$\alpha_1 : s \mapsto \left(\frac{2a_6 a_1^{p/2}}{p}\right) s^p, \quad \alpha_2 : s \mapsto \left(\frac{2a_6 a_2^{p/2}}{p}\right) s^p.$$

The function U is C^1 with

$$\nabla U(0) = 0,$$
$$\nabla U(\xi) = a_6\big(V(\xi)\big)^{(p-2)/2}\nabla V(\xi) \quad \forall \xi \neq 0,$$
$$\|\nabla U(\xi)\| \leq a_4 a_6\big(V(\xi)\big)^{(p-1)/2} \quad \forall \xi \in \mathbb{R}^n.$$

Moreover, for all $\xi \in \mathbb{R}^n$, we have

$$\langle \nabla U(\xi), f_0(\xi)\rangle \leq -a_3 a_6\big(V(\xi)\big)^{p/2} = -\left(\frac{a_3\sqrt{V(\xi)}}{a_5}\right)^p$$

and, on defining $h:\mathbb{R}^n \to \mathbb{R}^m$ by $h(\xi) := \big(\nabla U(\xi)\big)^T g(\xi)$,

$$\|h(\xi)\| \leq a_5 a_6\big(V(\xi)\big)^{(p-1)/2} = \left(\frac{a_3\sqrt{V(\xi)}}{a_5}\right)^{p-1} \quad \forall \xi \in \mathbb{R}^n.$$

Therefore, we arrive at

$$\langle \nabla U(\xi), f_0(\xi)\rangle + \|h(\xi)\|^{\frac{p}{p-1}} \leq 0 \quad \forall \xi \in \mathbb{R}^n.$$

Defining $\vartheta \in \mathcal{K}_\infty$ by $\vartheta(s) := ps^{p-1}$, the functions σ and ψ in Corollary 4.2 are $\sigma: s \mapsto s^p$ and $\psi: s \mapsto (1/p)^{\frac{p}{p-1}}(p-1)s^{\frac{p}{p-1}}$. Since $p > 1$, $\psi(\|h(\xi)\|) \leq \|h(\xi)\|^{\frac{p}{p-1}}$ for all $\xi \in \mathbb{R}^n$. This implies that (25) holds.

Noting that the 0-GES property trivially implies weak zero-detectability with respect to h, an application of Corollary 4.2 establishes the iISS property with iISS gain $\sigma: s \mapsto s^p$. ■

V. DISCUSSION

In view of recent results on L^p-input state-convergence, we conclude with some remarks on the various assumptions on f that are used in [7], [8], [12], in relation to **(A)**.

In [7], [8], using arguments based on infinite-dimensional systems theory, it is shown that if (1) is 0-GAS and satisfies (24) with $\alpha = \sigma : s \mapsto s^p$ then (1) has the BEICS property with respect to $\mathcal{U}_\sigma = L^p$ inputs, provided that f satisfies:

(A1) For each compact set $K \subset \mathbb{R}^n$, there exist $c_1, c_2 > 0$ such that, for all $\xi, \eta \in K, v \in \mathbb{R}^m$,

$$\|f(\xi,v) - f(\eta,v)\| \leq (c_1 + c_2\|v\|^p)\|\xi - \eta\| \qquad (26)$$

(A2) For each fixed $\eta \in \mathbb{R}^n$, there exist $c_3, c_4 > 0$ such that

$$\|f(\eta,v)\| \leq c_3 + c_4\|v\|^p \quad \forall v \in \mathbb{R}^m. \qquad (27)$$

This result is subsumed by Corollary 4.1 since **(A1)** and **(A2)** imply **(A)**. Indeed, let $K \subset \mathbb{R}^n$ be compact and fix $\eta \in K$. Using Assumptions **(A1)** and **(A2)**, there exist constants $c_1, c_2, c_3, c_4 > 0$ such that, for all $(\xi, v) \in K \times \mathbb{R}^m$,

$$\begin{aligned}\|f(\xi,v)\| &\leq \|f(\xi,v) - f(\eta,v)\| + \|f(\eta,v)\| \\ &\leq (c_1 + c_2\|v\|^p)\|\xi - \eta\| + (c_3 + c_4\|v\|^p),\end{aligned}$$

whence **(A)**. On the other hand, it is clear that **(A)** does not imply **(A1)** and **(A2)**.

Interpreted in the restricted context of systems of form (1), in [12] the assumption imposed on f takes the form:
(A3) For each compact set $K \subset \mathbb{R}^n$ there exists $k > 0$ such that

$$\|f(\xi,v) - f(\xi,0)\| \leq k\|v\| \quad \forall (\xi,v) \in K \times \mathbb{R}^m. \qquad (28)$$

Under this assumption on f and imposing the 0-GAS hypothesis, the following is implicit in the main result of [12]: if $u \in L^p$, $1 \leq p < \infty$ and the unique maximal solution x of (1) is global with non-empty ω-limit set, then $x(t) \to 0$ as $t \to \infty$ (we remark that the latter assumption of non-emptiness of the ω-limit set does not hold in the case of the counter-example constructed in [17]). Clearly, **(A3)** is more restrictive than **(A)**: it is readily verified that **(A3)** implies **(A)** (with $\sigma = \mathrm{id}$) and it is clear that **(A)** does not imply **(A3)**. However, it is difficult to make direct comparisons between the main result of the present paper (Theorem 3.1) and that of [12] because dissipativity of (1) is not posited in the latter.

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