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The decentralized implementability problem

Shaik Fiaz*, H.L. Trentelman*

Abstract—This paper deals with the problems of decentralized implementability and decentralized regular implementability in the context of finite-dimensional linear differential system behaviors. Given a plant behavior with a pre-specified partition of the system variable and a desired behavior, the problem of decentralized implementability is to find a controller which is decentralized with respect to the given partition and implements (regularly) the desired behavior with respect to the plant. In this paper we formulate these problems in the behavioral framework, with control as interconnection and we also provide necessary and sufficient conditions for the solvability of these problems.

Keywords: behaviors, implementability, regular interconnection, decentralized controllers.

I. INTRODUCTION

For large scale systems like power networks, digital communication networks, economic systems and flexible manufacturing systems, decentralized control is one of the prominent strategies for control. Such systems are often characterized by geographical separation, large dimension, or consists of many interconnected subsystems. For such systems it is computationally efficient to formulate control laws that use only locally available control variables. As it is easy to implement, and less cost is involved in communication overhead, this approach is also economical. In fact, the decentralized structure is an essential design constraint on controllers in situations where it is prohibited to exchange information between the subsystems. The analysis and the design of such decentralized control has been intensively considered for over three decades. For the vast body of literature on decentralized control in an input-output framework, we refer the reader to the survey papers [10], [11], books [5], [12] and journal articles [2], [3], [13], [16], and references therein.

In this paper we will discuss the problem of decentralized control in the behavioral framework. In contrast to [2], [3] and [16], we work in the generality where we view systems in a behavioral sense, that is, as families of trajectories, and control is viewed as restricting the plant behavior by intersecting it with a controller behavior. In particular we will discuss the problem of implementability by decentralized control. The implementability problem may be considered as a basic question in control: given a plant behavior, together with some 'desired' behavior, the latter is called implementable (sometimes called: achievable) if it can be achieved as controlled behavior by intersecting the plant with a suitable controller. The implementability problem was studied extensively in [18] and [1], and necessary and sufficient conditions were established for the full as well as partial interconnection case, both for general as well as regular interconnections. In this paper we will formulate the problem of decentralized implementability. Here the problem is, for a given plant, to characterize all desired behaviors that can be achieved (implemented) by means of decentralized controllers. A decentralized controller is a controller that only gives 'local' constraints on the control variable. In particular, for a given partition of the control variable into 'local' variables, a controller is called decentralized if it only involves laws on these local variables. In this paper, we will restrict ourselves to the full interconnection version of this problem. We will derive conditions for implementability and regular implementability using such decentralized controllers.

A. Notation and nomenclature

A few words about the notation and nomenclature used. We use standard symbols for the fields of real and complex numbers \( \mathbb{R} \) and \( \mathbb{C} \). \( C^\infty(\mathbb{R}, \mathbb{R}^n) \) denotes the set of infinitely often differentiable functions from \( \mathbb{R} \) to \( \mathbb{R}^n \). \( R[\xi] \) denotes the ring of polynomials in the indeterminate \( \xi \) with real coefficients. We use \( R[\xi] \) to denote the space of matrices with components in \( R[\xi] \). Elements of \( R[\xi] \) are called real polynomial matrices.

For \( n \geq 1 \) we use the notation \( n! \) to represent the set \( \{1,2,\ldots,n\} \). Given matrices \( A_i, i \in \mathbb{N} \), we use the notation blockdiagonal(\( A_1, A_2, \ldots, A_n \)) to represent the block diagonal matrix with diagonal blocks \( A_i \). Finally, we use the notation \( \text{col}(w_1, w_2) \) to represent the column vector formed by stacking \( w_1 \) over \( w_2 \).

II. LINEAR DIFFERENTIAL SYSTEMS AND POLYNOMIAL KERNEL REPRESENTATIONS

In the behavioral approach a dynamical system is given by a triple \( \Sigma = (T, W, \mathcal{B}) \), where \( T \) is the time axis, \( W \) is the signal space, and the behavior \( \mathcal{B} \) is a subset of \( W^T \), the set of all functions from \( T \) to \( W \). A linear differential system is a dynamical system with time axis \( T = \mathbb{R} \), and whose signal space \( W \) is a finite dimensional Euclidean space, say, \( \mathbb{R}^k \). Correspondingly, the manifest variable is then given as \( w = \text{col}(w_1, w_2, \ldots, w_k) \). The behavior \( \mathcal{B} \) is a linear subspace of \( C^\infty(\mathbb{R}, \mathbb{R}^n) \) consisting of all solutions of a set of higher order, linear, constant coefficient differential equations. More precisely, there exists a positive integer \( g \) and a polynomial matrix \( R \in R[\xi] \) such that

\[ \mathcal{B} = \{ w \in C^\infty(\mathbb{R}, \mathbb{R}^n) \mid R(\frac{d}{dt})w = 0 \}. \]

The set of linear differential systems with manifest variable \( w \) taking its value in \( \mathbb{R}^n \) is denoted by \( \Sigma^\xi \).

Let \( R \in R[g][\xi] \) be a polynomial matrix. If the behavior \( \mathcal{B} \) is represented by \( R(\frac{d}{dt})w = 0 \) then we call this a kernel representation of \( \mathcal{B} \). Further, a kernel representation is said to be minimal if every other kernel representation of \( \mathcal{B} \) has at least \( g \) rows. A given kernel representation, \( R(\frac{d}{dt})w = 0 \), is minimal if and only if the polynomial matrix \( R \) has full row rank (see [8], Theorem 3.6.4). The number of rows in any minimal polynomial kernel representation of \( \mathcal{B} \) is equal to the output cardinality of \( \mathcal{B} \), denoted by \( p(\mathcal{B}) \). This number corresponds to the number of outputs in any input/output representation of \( \mathcal{B} \).

We speak of a system as the behavior \( \mathcal{B} \), one of whose representations is given by \( R(\frac{d}{dt})w = 0 \) or just \( \mathcal{B} = \ker(R) \). The \( \frac{d}{dt} \) is often suppressed to enhance readability.

The controllable part of a behavior \( \mathcal{B} \) is defined as the largest controllable sub-behavior of \( \mathcal{B} \). This is denoted by \( \mathcal{B}_{\text{cont}} \) (see [8]).

Definition 2.1: Let \( \mathcal{B} \in \Sigma^\xi \) with system variable \( w \) partitioned as \( w = (w_1, w_2) \). We will call \( w_2 \) free in \( \mathcal{B} \) if, for any \( w_2 \in C^\infty(\mathbb{R}, \mathbb{R}^2) \), there exists \( w_1 \) such that \( (w_1, w_2) \in \mathcal{B} \). We
call \( w_2 \) maximally free if it is free, and we cannot enlarge this set with components from \( w_1 \) and still continue to have freeness for this enlarged set of variables.

The following result was shown in [8]:

**Proposition 2.2:** Let \( \mathfrak{B} \in \mathbb{L}^{x_1+x_2} \) with system variable \((w_1, w_2)\). Let a minimal kernel representation of \( \mathfrak{B} \) be given by
\[
R_1 \begin{pmatrix} R_1 & R_2 \end{pmatrix} \begin{pmatrix} w_1 \mid w_2 \end{pmatrix} = 0.
\]
Then \( w_2 \) is free in \( \mathfrak{B} \) if and only if the polynomial matrix \( R_1 \) has full row rank.

We now review some facts on elimination of variables. Let \( \mathfrak{B} \in \mathbb{L}^{x_1+x_2} \) with system variable \( w = (w_1, w_2) \). Let \( P_{w_1} \) denote the projection onto the \( w_1 \)-component. Then the set \( P_{w_1} \mathfrak{B} \) consisting of all \( w_1 \) for which there exists \( w_2 \) such that \((w_1, w_2) \in \mathfrak{B} \) is again a linear differential system. We denote \( P_{w_1} \mathfrak{B} \) by \((\mathfrak{B})_{w_1} \), and call it the behavior obtained by eliminating \( w_2 \) from \( \mathfrak{B} \).

If \( \mathfrak{B} = \ker \begin{pmatrix} R_1 & R_2 \end{pmatrix} \), then a representation for \((\mathfrak{B})_{w_1} \) is obtained as follows: choose a unimodular matrix \( U \) such that
\[
UR_2 = \begin{pmatrix} R_{12} \\ 0 \end{pmatrix},
\]
with \( R_{12} \) full row rank, and conformably partition
\[
UR_1 = \begin{pmatrix} R_{11} \\ R_{21} \end{pmatrix}.
\]
Then \((\mathfrak{B})_{w_1} = \ker(R_{21}) \) (see [8], section 6.2.2).

**Theorem 3.1:** Let \( \mathfrak{X} \in \mathbb{L}^x \) be a given behavior, to be interpreted with respect to \( \mathfrak{B} \), and \( \mathfrak{c} \) is the system whose behavior is the intersection \( \mathfrak{P} \cap \mathfrak{c} \). This controlled behavior is again a linear differential system. Indeed, if \( \mathfrak{P} = \ker(R) \) and \( \mathfrak{c} = \ker(C) \), then \( \mathfrak{P} \cap \mathfrak{c} = \ker \begin{pmatrix} R \\ C \end{pmatrix} \) since \( \mathfrak{B} \) is free in \( \mathfrak{P} \).

**Proposition 3.2:** Let \( \mathfrak{P} \in \mathbb{L}^x \) and \( \mathfrak{K} \subseteq \mathbb{L}^x \). Let \( \mathfrak{P} = \ker(R) \) and \( \mathfrak{K} = \ker(K) \) be kernel representations. Then the following are equivalent:

1. \( \mathfrak{K} \) is implementable with respect to \( \mathfrak{P} \) by \( \mathfrak{B} \).
2. \( \mathfrak{P} \subseteq \mathfrak{K} \).

In that case, we also call the controller \( \mathfrak{c} \) regular (with respect to \( \mathfrak{P} \)).

**Proposition 3.3:** The interconnection of \( \mathfrak{P} \) and \( \mathfrak{c} \) is called regular if
\[
p(\mathfrak{P}) + p(\mathfrak{c}) = p(\mathfrak{P} \cap \mathfrak{c}),
\]
in other words, if the output cardinalities of the plant and the controller add up to the output cardinality of the controlled behavior.

In terms of kernel representations this condition can be expressed as follows. Let \( \mathfrak{P} = \ker(R) \) and \( \mathfrak{c} = \ker(C) \) be minimal kernel representations of plant and controller, respectively. Then
\[
\mathfrak{P} \cap \mathfrak{c} = \ker \begin{pmatrix} R \\ C \end{pmatrix}
\]
is a kernel representation of the controlled behavior. Since the output cardinality of a behavior is equal to the rank of the polynomial matrices in any of its kernel representations, the interconnection of \( \mathfrak{P} \) and \( \mathfrak{c} \) is regular if and only if \( \ker \begin{pmatrix} R \\ C \end{pmatrix} \) has full row rank, equivalently yields a minimal kernel representation of \( \mathfrak{P} \cap \mathfrak{c} \).

**Definition 3.3:** Given \( \mathfrak{P} \in \mathbb{L}^x \), a given behavior \( \mathfrak{K} \subseteq \mathbb{L}^x \) is called regularly implementable by full interconnection (with respect to \( \mathfrak{P} \)) if there exists a regular controller \( \mathfrak{c} \in \mathbb{L}^x \) that implements \( \mathfrak{K} \) by full interconnection.

The following result from [1] gives a characterization of all regularly implementable behaviors.

**Proposition 3.4:** Let \( \mathfrak{P} \in \mathbb{L}^x \), \( \mathfrak{K} \subseteq \mathbb{L}^x \), and \( \mathfrak{P}_{\text{cont}} \) be its controllable part. Let \( \mathfrak{K} \subseteq \mathbb{L}^x \). Then the following statements are equivalent:

1. \( \mathfrak{K} \) is regularly implementable by full interconnection with respect to \( \mathfrak{P} \).
2. \( \mathfrak{K} + \mathfrak{P}_{\text{cont}} = \mathfrak{P} \).

The previous result does not use representations of the behaviors involved. The following result characterizes regular implementability in terms of kernel representations (see [9]):

**Proposition 3.5:** Let \( \mathfrak{P} \in \mathbb{L}^x \), \( \mathfrak{K} \subseteq \mathbb{L}^x \), and \( \mathfrak{P} = \ker(R) \) and \( \mathfrak{K} = \ker(K) \) be minimal kernel representations of plant and desired behavior. Then the following are equivalent:

1. \( \mathfrak{K} \) is regularly implementable by full interconnection with respect to \( \mathfrak{P} \).
2. \( R \) has full row rank for all \( \lambda \in \mathbb{C} \).
Clearly, since \( \begin{pmatrix} F(\lambda) \\ L(\lambda) \end{pmatrix} \) has full column rank for all \( \lambda \in \mathbb{C} \), we have \( \ker \begin{pmatrix} R \\ C \end{pmatrix} = \ker(K) \).

**Lemma 3.8:** Let \( \mathcal{P} \in \mathcal{L}^n \) and let \( \mathcal{K} \in \mathcal{L}^m \). Assume that \( \mathcal{K} \) is regularly implementable with respect to \( \mathcal{P} \). Let \( \mathcal{P} = \ker(R) \) and \( \mathcal{K} = \ker(K) \) be minimal kernel representations and let \( F \) be a polynomial matrix with \( F(\lambda) \) full row rank for all \( \lambda \in \mathbb{C} \) such that \( R = FK \). Then the following statements are equivalent:

1) \( \mathcal{C} = \ker(C) \) regularly implements \( \mathcal{K} \) by full interconnection and \( \ker(C) \) is a minimal representation of \( \mathcal{C} \).
2) there exists a polynomial matrix \( L \) such that \( C = LK \), where \( \begin{pmatrix} F \\ L \end{pmatrix} \) is unimodular.

**Proof:**

(1) \( \Rightarrow \) (2)

If \( \ker(C) \) regularly implements \( \ker(K) \) by full interconnection, then \( \ker \begin{pmatrix} R \\ C \end{pmatrix} = \ker(K) \). Since the interconnection is regular, both kernel representations are minimal. Hence there exists a unimodular matrix \( U = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \) such that \( \begin{pmatrix} R \\ C \end{pmatrix} = UK \), which implies that \( R = U_1K \) and \( C = U_2K \). It follows that \( U_1 = F \). Define \( L := U_2 \). Then \( \begin{pmatrix} F \\ L \end{pmatrix} \) is unimodular.

(2) \( \Rightarrow \) (1)

Assume \( C = LK \). We have \( \begin{pmatrix} R \\ C \end{pmatrix} = \begin{pmatrix} F \\ L \end{pmatrix} K \).

Clearly, since \( \begin{pmatrix} F \\ L \end{pmatrix} \) is unimodular, we have \( \ker \begin{pmatrix} R \\ C \end{pmatrix} = \ker(K) \), so \( \ker(C) \) implements \( \mathcal{K} \). Also, the interconnection is regular since \( \begin{pmatrix} R \\ C \end{pmatrix} \) has full row rank. □

**IV. DECENTRALIZED IMPLEMENTABILITY**

Let \( \mathcal{P} \in \mathcal{L}^n \) be a given plant behavior, with system variable \( w \). Let \( \mathcal{K} \in \mathcal{L}^m \) be a desired behavior. In this section we will deal with the problem to find decentralized controllers that implement \( \mathcal{K} \) by full interconnection. A decentralized controller is a controller that only gives 'local' constraints on the control variable \( w \). In particular, for a given partition of the variable \( w \) into

\[
w = (w_1, w_2, w_3, \ldots, w_n)
\]

with \( w_i \) taking values in \( \mathbb{R}^{w_i} \) (\( i \in \mathbb{N} \)) a controller is called decentralized if it only involves laws on the local variables \( w_i \). More precisely:

**Definition 4.1:** Let \( \mathcal{C} \in \mathcal{L}^n \), with system variable \( w \), be interpreted as a controller. Let \( w \) be partitioned as \( w = (w_1, w_2, \ldots, w_n) \) with \( w_i \) of dimension \( w_i, w = \sum_{i=1}^{n} w_i \). Then \( \mathcal{C} \) is called decentralized with respect to the partition of the system variable if for all \( i \in \mathbb{N} \) there exists \( \mathcal{C}_i \in \mathcal{L}^{w_i} \) with system variable \( w_i \) such that \( \mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2 \times \cdots \times \mathcal{C}_n \).

The following proposition gives for a given behavior the property of being decentralized:

**Proposition 4.2:** Let \( \mathcal{C} \in \mathcal{L}^n \) with system variable \( w \) partitioned as \( w = (w_1, w_2, \ldots, w_n) \) with \( w_i \) of dimension \( w_i \). Then the following statements are equivalent:

1) \( \mathcal{C} \) is decentralized with respect to the given partition.
2) there exists polynomial matrices \( C_i \in \mathbb{R}^{w_i \times w_i} \) such that \( \mathcal{C} \) admits a kernel representation \( \mathcal{C} = \ker(C) \), where \( C = \text{blockdiag}(C_1, C_2, \ldots, C_n) \).
3) \( \mathcal{C} \) has full row rank.

**Proof:** From Definition 4.1 the equivalence between statements 1) and 2) is straightforward by defining \( \mathcal{C}_i := \ker(C_i) \). We now prove the equivalence of statements 2) and 3) of the Proposition.

(2) \( \Rightarrow \) (3)

If \( \mathcal{C} = \ker(\text{blockdiag}(C_1, C_2, \ldots, C_n)) \) then we have \( \ker(C_i) = \ker(C_i) = \ker(C_i) = \ker(C_i) \).

Therefore we have \( \ker(C_i) = \ker(C_i) \).

(3) \( \Rightarrow \) (2)

We prove this implication for \( n = 2 \). For the case \( n > 2 \) the proof can be given by induction. Let \( \mathcal{C} \in \mathcal{L}^{w_1 \times w_2} \) with system variable \( (w_1, w_2) \). Let \( \mathcal{C} = \ker(C_1, C_2) \) be a minimal kernel representation. Then there exists a unimodular matrix \( U_1 \) such that \( U_1C_1 = \begin{pmatrix} C_{11} & 0 \\ C_{21} & C_{22} \end{pmatrix} \) such that \( C_{22} \) has full row rank. Then we have \( \ker(C_1) = \ker(C_1) \), \( \ker(C_2) = \ker(C_2) \), and \( \ker(C_2) = \ker(C_2) \). As \( \ker(C_1) = \ker(C_1) \) and \( \ker(C_2) = \ker(C_2) \), there exists a unimodular matrix \( \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} \) such that \( \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} \) is a unimodular matrix.

(2) \( \Rightarrow \) (1).

As \( C_{11} \) has full row rank we have \( \ker(C_{11}) \) and \( \ker(C_{22}) \) which implies that \( V_{22} \) is a unimodular matrix. It is easy to verify that

\[
\begin{pmatrix} I & -V_{12}V_{22}^{-1} \\ 0 & V_{22}^{-1} \end{pmatrix}
\]

is a unimodular matrix and

\[
\begin{pmatrix} I & -V_{12}V_{22}^{-1} \\ 0 & V_{22}^{-1} \end{pmatrix} = \begin{pmatrix} C_{11} & 0 \\ C_{21} & C_{22} \end{pmatrix}
\]

Therefore from Equations (2) and (3) we have

\[
\ker(C_{11}) \circ \ker(C_{22})
\]

(4)

Given a plant \( \mathcal{P} \) together with a partition (1) of its variable, and a given desired behavior \( \mathcal{K} \) we now deal with the question whether \( \mathcal{K} \) can be implemented by means of a decentralized controller. We give the following definitions:

**Definition 4.3:** Let \( \mathcal{K} \in \mathcal{L}^m \). Assume the system variable \( w \) is partitioned as in (1). We call \( \mathcal{K} \) decentralized implementable with respect to \( \mathcal{P} \) if there exists a decentralized controller \( \mathcal{C} \in \mathcal{L}^n \) such that \( \mathcal{K} = \mathcal{P} \circ \mathcal{C} \).

**Definition 4.4:** Let \( \mathcal{K} \in \mathcal{L}^m \). Assume the system variable \( w \) is partitioned as in (1). We call \( \mathcal{K} \) decentralized regularly implementable with respect to \( \mathcal{P} \) if there exists a decentralized regular controller \( \mathcal{C} \in \mathcal{L}^n \) such that \( \mathcal{K} = \mathcal{P} \circ \mathcal{C} \).

In the following we want to establish conditions for a given desired behavior \( \mathcal{K} \) to be decentralized (regularly) implementable with respect to \( \mathcal{P} \). For simplicity, we assume that the system variable \( w \) is partitioned into two parts, \( w = (w_1, w_2) \). The following theorem gives necessary and sufficient conditions for a behavior \( \mathcal{K} \) to be decentralized implementable with respect to \( \mathcal{P} \):

**Theorem 4.5:** Let \( \mathcal{P} \in \mathcal{L}^n \) with variable \( w \) partitioned as \( w = (w_1, w_2) \), and with minimal kernel representation \( \mathcal{P} = \ker(R) \). Let \( \mathcal{K} \in \mathcal{L}^m \) with minimal kernel representation \( \mathcal{K} = \ker(K) \),

\[
K = \begin{pmatrix} K_1 & K_2 \end{pmatrix}
\]

Assume that \( \mathcal{K} \) is implementable by full
interconnection with respect to $\mathcal{P}$ and let $F$ be a polynomial matrix such that $R = FK$. Then $X$ is decentralized implementable with respect to $\mathcal{P}$ if and only if there exist polynomial matrices $L_1, L_2$ such that
\[
\begin{pmatrix}
F(\lambda) \\
L_1(\lambda) \\
L_2(\lambda)
\end{pmatrix}
\]
has full column rank for all $\lambda \in \mathbb{C}$ and $L_1 K_2 = 0$, $L_2 K_1 = 0$. In this case a decentralized controller is given by $\mathcal{C} = \ker \left( \begin{smallmatrix} L_1 K_1 & 0 \\ 0 & L_2 K_2 \end{smallmatrix} \right)$.

Proof: A proof follows immediately from Lemma 3.7 and Proposition 4.2.

Along the same lines, decentralized regular implementability is dealt with in the next theorem:

**Theorem 4.6:** Let $\mathcal{P} \in \mathcal{L}^w$ with variable $w$ partitioned as $w = (w_1, w_2)$. Let $X \in \mathcal{L}^w$ with minimal kernel representation $X = \ker(R)$. Let $\mathcal{K} \in \mathcal{L}^c$ with minimal kernel representation $\mathcal{K} = \ker(K)$, $K = (K_1 \ K_2)$. Assume that $\mathcal{X}$ is regularly implementable with respect to $\mathcal{P}$ and let $F$ be a polynomial matrix with $F(\lambda)$ full row rank for all $\lambda \in \mathbb{C}$ such that $R = FK$. Then $X$ is decentralized regularly implementable with respect to $\mathcal{P}$ if and only if there exist polynomial matrices $L_1, L_2$ such that
\[
\begin{pmatrix}
F \\
L_1 \\
L_2
\end{pmatrix}
\]
is unimodular and $L_1 K_2 = 0$, $L_2 K_1 = 0$. In this case a decentralized regular controller is given by $\mathcal{C} = \ker \left( \begin{smallmatrix} L_1 K_1 & 0 \\ 0 & L_2 K_2 \end{smallmatrix} \right)$.

Proof: Again, a proof follows immediately from Lemma 3.8 and Proposition 4.2.

The following corollaries are immediate consequences of the foregoing:

**Corollary 4.7:** Let $\mathcal{P} \in \mathcal{L}^w$ with variable $w$ partitioned as $w = (w_1, w_2)$. Let $X \in \mathcal{L}^w$ with minimal kernel representation $X = \ker(K)$, $K = (K_1 \ K_2)$. Denote $\kappa := p(\mathcal{K})$. Then $X$ is decentralized implementable with respect to $\mathcal{P}$ if and only if
1) $\mathcal{K} \subset \mathcal{P}$,
2) there exist behaviors $\mathcal{H}_1 \in \mathcal{L}^c$, $\mathcal{H}_2 \in \mathcal{L}^c$ such that $\ker(K_1) \supset \ker(\mathcal{H}_1)$, $\ker(K_2) \supset \ker(\mathcal{H}_2)$, $\ker(\mathcal{H}_1) \cap \ker(\mathcal{H}_2) = 0$.

Proof: Assume $X$ is decentralized implementable with respect to $\mathcal{P}$. Then clearly 1) holds. Let $\ker(R)$ be a minimal kernel representation of $\mathcal{P}$. Define $\mathcal{H}_1 := \ker(L_1)$ and $\mathcal{H}_2 := \ker(L_2)$. Then $\mathcal{H}_1 \supset \ker(\mathcal{H}_1)$ and $\mathcal{H}_2 \supset \ker(\mathcal{H}_2)$. Conversely, let $\mathcal{P} \supset \ker(R)$ be a minimal kernel representation of $\mathcal{P}$. By 1) there exists $F$ such that $R = FK$. Let $L_1$ and $L_2$ be such that $\ker(L_1) \supset \ker(\mathcal{H}_1)$ and $\ker(L_2) \supset \ker(\mathcal{H}_2)$. Then $L_1 K_2 = 0$, $L_2 K_1 = 0$, $L_1 K_1 = 0$, and $L_2 K_2 = 0$ (5) has full column rank for all $\lambda \in \mathbb{C}$. By Theorem 4.5 this yields that $X$ is decentralized implementable with respect to $\mathcal{P}$.

In a similar way we can characterize decentralized regular implementability.

**Corollary 4.8:** Let $\mathcal{P} \in \mathcal{L}^w$ with variable $w$ partitioned as $w = (w_1, w_2)$. Let $X \in \mathcal{L}^w$ with minimal kernel representation $X = \ker(K)$, $K = (K_1 \ K_2)$. Denote $\kappa := p(\mathcal{X})$. Then $X$ is decentralized regularly implementable with respect to $\mathcal{P}$ if and only if
1) $\mathcal{K} + \mathcal{P}_{\text{cont}} \subset \mathcal{P}$,
2) there exist behaviors $\mathcal{H}_1 \in \mathcal{L}^c$, $\mathcal{H}_2 \in \mathcal{L}^c$ such that $\ker(K_1) \supset \ker(\mathcal{H}_1)$, $\ker(K_2) \supset \ker(\mathcal{H}_2)$, and $\ker(\mathcal{H}_1) \cap \ker(\mathcal{H}_2) = 0$.

Proof: A proof follows immediately from Lemma 3.7 and Proposition 4.2.

In addition to full interconnection, in [18] and [1] results have been established on implementability by partial interconnection (see also [6], [7], [14]). In this section, we will establish necessary conditions for decentralized implementability by full interconnection (as introduced in the previous section) in terms of concepts around partial interconnection.

We will first briefly review implementability by partial interconnection. In control by partial interconnection, only a pre-specified subset of the plant variables is available for interconnection. Let $F \in \mathcal{L}^{w \times c}$ be a linear differential system, with system variable $(w, c)$, where $w$ takes its values in $\mathbb{R}^n$ and $c \in \mathbb{R}^m$. Before the controller acts, there are two behaviors of the plant that are relevant: the behavior $F \in \mathcal{L}^{w \times c}$ (the full plant behavior) of the variables $w$ and $c$ combined, and the behavior $(F \in \mathcal{L}^{w \times c})_w$ of the to-be-controlled variables $w$ (with the interconnection variable $c$ eliminated). Hence
\[
(F \in \mathcal{L}^{w \times c})_w = \{w \in \mathcal{L}^{\infty}(\mathbb{R}, \mathbb{R}^n) \mid \exists c \in \mathcal{L}^{\infty}(\mathbb{R}, \mathbb{R}^m) \text{ such that } (w, c) \in F \in \mathcal{L}^{w \times c}\}.
\]

By the elimination theorem, $(F \in \mathcal{L}^{w \times c})_w \subset \mathcal{L}^w$. Let $c \in \mathcal{L}^c$. The controller $\mathcal{C}$ restricts the interconnection variables $c$. The full controlled behavior $F \in \mathcal{L}^{w \times c} \cap \mathcal{C}$ is obtained by the interconnection of $F \in \mathcal{L}^{w \times c}$ and $\mathcal{C}$ through the variable $c$ and is defined as:
\[
(F \in \mathcal{L}^{w \times c}) \cap \mathcal{C} = \{(w, c) \mid (w, c) \in (F \in \mathcal{L}^{w \times c}) \text{ and } c \in \mathcal{C}\}.
\]

Eliminating $c$ from the full controlled behavior, we obtain its restriction $(F \in \mathcal{L}^{w \times c})_w \cap \mathcal{C}$, to the behavior of the to-be-controlled variable $w$, defined by
\[
(F \in \mathcal{L}^{w \times c})_w \cap \mathcal{C} = \{w \in \mathcal{L}^{\infty}(\mathbb{R}, \mathbb{R}^n) \mid \exists c \in \mathcal{C} \text{ such that } (w, c) \in (F \in \mathcal{L}^{w \times c}) \cap \mathcal{C}\}.
\]

Note that, again by the elimination theorem, $(F \in \mathcal{L}^{w \times c})_w \cap \mathcal{C} \subset \mathcal{L}^w$.

**Definition 5.1:** Given $F \in \mathcal{L}^{w \times c}$ and $K \in \mathcal{L}^w$, we say that $\mathcal{C} \in \mathcal{L}^c$ implements $K$ through $c$ if $K = (F \in \mathcal{L}^{w \times c})_w \cap \mathcal{C}$.

The (partial interconnection) implementability problem is to characterize, for given $F \in \mathcal{L}^{w \times c}$, all $K \in \mathcal{L}^w$ for which there exists a $\mathcal{C} \in \mathcal{L}^c$ that implements $K$ through $c$. This problem has a very simple and elegant solution: it depends only on the projected full plant behavior $(F \in \mathcal{L}^{w \times c})_w$ and on the hidden behavior $N_w(F \in \mathcal{L}^{w \times c})$ given by
\[
N_w(F \in \mathcal{L}^{w \times c}) = \{w \mid (w, 0) \in (F \in \mathcal{L}^{w \times c})_w\}.
\]

**Theorem 5.2:** [18] Let $F \in \mathcal{L}^{w \times c}$ be the full plant behavior. Then $K \in \mathcal{L}^w$ is implementable with respect to $F \in \mathcal{L}^{w \times c}$ by a controller $\mathcal{C} \in \mathcal{L}^c$ acting on the interconnection variable $c$ if and only if $N_w(F \in \mathcal{L}^{w \times c}) \subset K \subset (F \in \mathcal{L}^{w \times c})_w$.

Theorem 5.2 shows that $K$ can be any behavior of $\mathcal{L}^w$ that is wedged in between the given behaviors $N_w(F \in \mathcal{L}^{w \times c})$ and $(F \in \mathcal{L}^{w \times c})_w$. The implementability problem was also studied in [6], [15] and [9]. In particular, the question when a particular controlled behavior can
be implemented by a feedback processor remains a very important one, and was discussed e.g. in [17] and [14].

Next, we turn to regular implementability by partial interconnection.

Definition 5.3: Let $\mathcal{P}_{\text{full}} \in \mathcal{L}^{<e}$ and $\mathcal{C} \in \mathcal{L}^{e}$. The interconnection of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$ through $c$ is called regular if

$$p(\mathcal{P}_{\text{full}} \wedge c) = p(\mathcal{P}_{\text{full}}) + p(c),$$

i.e., the output cardinalities of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$ add up to that of the full controlled behavior $\mathcal{P}_{\text{full}} \wedge c$. In that case we also call the controller $\mathcal{C}$ regular.

Definition 5.4: A given $\mathcal{K} \in \mathcal{L}^{e}$ is called regularly implementable through $c$ with respect to $\mathcal{P}_{\text{full}}$ if there exists a $\mathcal{C} \in \mathcal{L}^{e}$ such that $\mathcal{K}$ is implemented by $\mathcal{C}$, and the interconnection of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$ is regular.

Similar to implementability by full interconnection, an important question is under what conditions a given behavior $\mathcal{K}$ is regularly implementable through $c$ with respect to $\mathcal{P}_{\text{full}}$. The following theorem from [1] provides a solution to this problem:

Theorem 5.5: Let $\mathcal{K} \in \mathcal{L}^{e}$. Let $\mathcal{P}_{\text{full}}$ and $\mathcal{K}$ be the corresponding projected plant behavior and hidden behavior, respectively. Let $\mathcal{P}_{\text{full}} \wedge c$ be the controllable part of $\mathcal{P}_{\text{full}}$. Let $\mathcal{K} \in \mathcal{L}^{e}$. Then $\mathcal{K}$ is regularly implementable with respect to $\mathcal{P}_{\text{full}}$ by interconnection through $c$ if and only if the following two conditions are satisfied:

- $\mathcal{K}_{\text{w}}(\mathcal{P}_{\text{full}}) \subset \mathcal{K} \subset (\mathcal{P}_{\text{full}})_{\text{w}}$
- $\mathcal{K} + (\mathcal{P}_{\text{full}})_{\text{w}} \wedge c = (\mathcal{P}_{\text{full}})_{\text{w}}$

The above theorem has two conditions. The first one is exactly the condition for implementability through $c$. The second condition formalizes the notion that the autonomous part of $\mathcal{P}_{\text{full}}$ is taken care of by $\mathcal{K}$. While the autonomous part of $(\mathcal{P}_{\text{full}})_{\text{w}}$ is not unique, $(\mathcal{P}_{\text{full}})_{\text{w}} \wedge c$ is. This makes verifying the regular implementability of a given $\mathcal{K}$ computable. As a consequence of this theorem, note that if $(\mathcal{P}_{\text{full}})_{\text{w}}$ is controllable, then $\mathcal{K} \in \mathcal{L}^{e}$ is regularly implementable with respect to $\mathcal{P}_{\text{full}}$ by interconnection through $c$ if and only if it is implementable with respect to $\mathcal{P}_{\text{full}}$ by interconnection through $c$. We now return to the decentralized regular implementability problem (by full interconnection). The following theorem gives necessary conditions:

Theorem 5.6: Let $\mathcal{P}, \mathcal{K} \in \mathcal{L}^{e}$, with system variable $w$ partitioned as $w = (w_1, w_2, \ldots, w_n)$ with $w_i$ of dimension $\mathbb{R}^{r_i}$. Then $\mathcal{K}$ is decentralized regular implementable with respect to $\mathcal{P}$ by full interconnection only if the following conditions hold:

1) $\mathcal{K} + \mathcal{P}_{\text{cont}} = \mathcal{P}$
2) $w_i$ is not free in $\mathcal{K}$ for all $i \in \mathbb{N}$, and
3) for every $i \in \mathbb{N}$ there exists a $\mathcal{S}_i \in \mathcal{L}^{(n)}$ such that
   a) $\mathcal{S}_i$ is regular implementable with respect to $\mathcal{P}$ by interconnection through $w_i$.
   b) $(\mathcal{K})_w$ regularly implementable by full interconnection with respect to $\mathcal{S}_i$.

Proof: Clearly $\mathcal{K}$ regularly implementable by full interconnection with respect to $\mathcal{P}$ is a necessary condition. From Proposition 3.5 we have $\mathcal{K} + \mathcal{P}_{\text{cont}} = \mathcal{P}$. Let $\mathcal{P} = \ker \begin{pmatrix} R_1 & R_2 & \ldots & R_n \\ C_1 & 0 & \ldots & 0 \\ 0 & C_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & C_n \end{pmatrix}$. Therefore we have

$$\mathcal{K} = \mathcal{P} \cap \mathcal{C} = \ker \begin{pmatrix} R_1 & R_2 & \ldots & R_n \\ C_1 & 0 & \ldots & 0 \\ 0 & C_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & C_n \end{pmatrix}. \quad (6)$$

From (6) and using Proposition 2.2, for all $i \in \mathbb{N}$, $w_i$ is not free in $\mathcal{K}$. For $i \in \mathbb{N}$ there exists unimodular matrices $U_i$ such that

$$U_i = \begin{pmatrix} R_1 & R_2 & \ldots & R_n \\ C_1 & 0 & \ldots & 0 \\ 0 & C_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & C_n \end{pmatrix} = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ C_1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & S_n \end{pmatrix}, \quad (7)$$

and $L_1 \ldots L_{i-1} L_i L_{i+1} \ldots L_n$ has full row rank. Therefore we have

$$(\mathcal{K})_{w_i} = \ker \begin{pmatrix} C_i \\ S_i \end{pmatrix}. \quad (8)$$

Define $S_i := \ker (S_i)$. From (7) it is evident that for all $i \in \mathbb{N}$, $S_i$ is regularly implementable with respect to $\mathcal{P}$ by interconnection through $(w_1, w_2, \ldots, w_{i-1}, w_{i+1}, \ldots, w_n)$. From (8) it is clear $(\mathcal{K})_{w_i}$ is regularly implementable with respect to $S_i$.

VI. CONCLUSIONS

In this paper we have introduced the problems of decentralized implementability and decentralized regular implementability. Given a plant behavior and a desired behavior, the problem is to give conditions for the existence of a decentralized controller that (regularly) implements the desired behavior. In the first part of this paper we have established necessary and sufficient conditions in terms of geometric properties of the desired behavior. In the second part of the paper we have obtained a set of necessary conditions, expressed in terms of regular implementability by partial interconnection.

REFERENCES


