# Noninteracting control of nonlinear systems based on relaxed control

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# Noninteracting control of nonlinear systems based on relaxed control

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***Abstract*— In this paper, we propose methodology to solve noninteracting control problem for general nonlinear systems based on the relaxed control technique proposed by Artstein. For a class of nonlinear systems which cannot be stabilized by smooth feedback, a state-feedback relaxed control can be designed to decouple the system into several SISO or MIMO systems and simplify the controller design.**

**Keywords: relaxed control; noninteracting control problem; nonlinear control**

## I. Introduction

The noninteracting control problem as defined in Nijmeijer and Schumacher [6] or in Isidori [3] involves the design of state feedback in order to decouple an affine nonlinear system into a set of independent single-input single-output systems. The solvability of this problem allows us to simplify the design of controller for multi-input multi-output systems. By means of geometric approach, Nijmeijer and Schumacher in [6] present local solution of the problem using static state feedback. By using dynamic controller, Battilotti in [2] gives sufficient condition for the solvability of the problem.

The results in [2], [6] are restricted to the class of affine nonlinear systems. The generalization of the result has appeared in [7] using geometric point of view. These works can be used to characterize nonlinear systems whose input-output relationship can be decoupled by static or dynamic feedback law.

In this paper, we exploit relaxed control in order to extend the result to a larger class of nonlinear systems.

Artstein in [1] proposes relaxed control methodology which can overcome the control restriction in the stabilization of general nonlinear systems. The concept of relaxed control replaces the input by a probability measure in order to relax the control design. Several examples are given in [1] where the origin of a nonlinear system can only be stabilized by relaxed control. The paper also gives necessary and sufficient condition for the nonlinear systems to be stabilizable by relaxed control.

Suppose that the nonlinear systems are described by the state equations:

$$\dot{x}(t) = f(x(t), u(t)), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}$. The relaxed control method assumes the input $u$ as a probability measure $\mu$, in which case, (1) becomes

$$\dot{x}(t) = \int_{w\in\mathbb{R}} f(x(t), w)\ \mathrm{d}\mu(w).$$

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In other words, the rate changes of the state 'in the average' is given by the expected value of the vector field function $f$.

In practice, the relaxed control signal resembles the principle of control by pulse-width modulation (PWM) [8]. Dither control introduced by Zames and Shneydor in [12], [11], is also based on a similar concept. In [12], [11], the sector condition for the static nonlinearity is relaxed by using an additional dither signal in the control signal.

As an example of dither signal application, let $\phi(v) = v^3 + v$ be the static nonlinearity in the Lur'e problem which has sector $[1, \infty)$ and the matrices $A, B, C, D$ defines the state equations of the linear system with input $u$ and output $y$. The sector of $\phi$ can be changed by adding a dither signal $w$ to the nonlinearity input such that $v = w+y$ where $y$ is the output of the linear system in the Lur'e problem. Suppose that the probability measure of $w$ at every time instance is given by $\mu(E) = \int_E r(\tau)\mathrm{d}\tau$ where

$$r := 0.5\delta_{-0.5} + 0.5\delta_{0.5},$$

$\delta_\epsilon$ is a delta measure concentrated at $\epsilon$. In this case, the state equations of the closed-loop system becomes

$$\begin{aligned}\dot{x} &= \int Ax + B\phi(w+y)\ \mathrm{d}\mu(w)\\ &= Ax + B\frac{1}{2}\big(\phi(y-0.5) + \phi(y+0.5)\big)\\ &= Ax + B(y^3 + 1.75y) = Ax + B\tilde{\phi}(y),\end{aligned}$$

where $\tilde{\phi}$ is the new static nonlinearity with sector $[1.75, \infty)$.

We present sufficient conditions for the solvability of noninteracting control problem by means of relaxed control.

**Notations.** For vector fields $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}^p$, we denote $L_{f(x,u)}h(x) = \frac{\partial h(x)}{\partial x} f(x,u)$ and

$$L^i_{f(x,u)}h(x) = \frac{\partial L^{i-1}_{f(x,u)}h(x)}{\partial x} f(x,u), \qquad \forall i > 1.$$

## II. Relaxed control

Throughout this paper, we consider nonlinear systems described by (1) with locally Lipschitz function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. Let $U_R$ be the family of probability measure $\mu$ defined on the input space $\mathbb{R}^m$. A relaxed input is defined by applying $\mu_v \in U_R$ to the ordinary input in (1) such that the rate changes of the state at every time instant is given by

$$\dot{x} = \int_{u\in\mathbb{R}^m} f(x,u)\ \mathrm{d}\mu_v(u) =: f_R(x,v), \tag{2}$$

where $v \in \mathbb{R}^q$ is a vector of parameters of the probability measure which becomes the new input variable in the RHS of (2). The system with the relaxed input $\mu_v$ as given in (2) is called *relaxed system*.

The nonlinear system equations with ordinary input in (1) can be derived back from (2) by taking $v \in \mathbb{R}^m$, $\mu_v(E) = \int_E r_v(\tau)\mathrm{d}\tau$ where $r_v = \delta_v$.

The result in [1] describes the stabilization of (1) by finding *state-feedback relaxed control* $\mu_{v(x)} \in U_R$ such that the resulting differential equation

$$\dot{x} = f_R(x, v(x))$$

is globally asymptotically stable in the origin. The following theorem is the main result of [1].

*Theorem 2.1:* The system (1) with locally Lipschitz $f$ is stabilizable by a state-feedback relaxed control if and only if there is a continuously differentiable function $V : X \to \mathbb{R}_+$ where $X$ is a neighborhood of 0 such that $V$ is positive definite and

$$\inf_{u\in\mathbb{R}^m} \text{grad } V(x)f(x,u) < 0 \qquad \forall x \in X\backslash\{0\}.$$

It is globally stabilizable by a state-feedback relaxed control if and only if $X = \mathbb{R}^n$ and $V$ is radially unbounded.

The above theorem provides flexibility in designing a smooth state-feedback relaxed control for solving controller design for nonlinear systems which can only be stabilized by non-smooth state feedback control.

As an example, let us consider the following nonlinear systems.

$$\left.\begin{aligned} \dot{x}_1 &= \sin(u) \\ \dot{x}_2 &= \cos(u), \end{aligned}\right\} \qquad (3)$$

where $x_1(t), x_2(t), u(t) \in \mathbb{R}$. This system can not be stabilized at any point by using standard state feedback since there is no equilibrium point associated with a constant input $u$. By using $V(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2)$, routine calculation shows that

$$\inf_{u\in\mathbb{R}} \text{grad } V(x)f(x,u) = \inf_{u\in\mathbb{R}} x_1\sin(u) + x_2\cos(u) < 0,$$

for all $\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \in \mathbb{R}^2\backslash\{0\}$. It follows from Theorem 2.1 that (3) can be globally stabilized by a state-feedback relaxed control. In fact, using the following state-feedback relaxed control with probability measure $\mu_{v_1,v_2}(E) = \int_E r_{v_1,v_2}(\tau)\mathrm{d}\tau$ where

$$r_{v_1,v_2} = v_1\delta_{-\pi/2} + v_2\delta_0 + (0.5 - v_1)\delta_{\pi/2} + (0.5 - v_2)\delta_\pi.$$

where $v_1, v_2 \in [0, 0.5]$ and using (2), we have

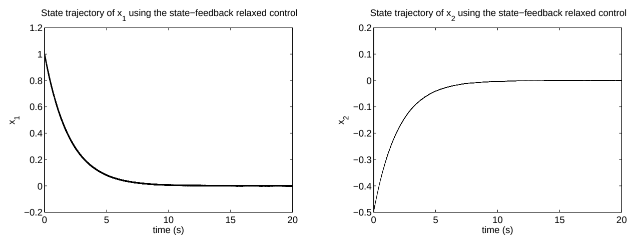
$$\left.\begin{aligned} \dot{x}_1 &= -2v_1 + 0.5 \\ \dot{x}_2 &= 2v_2 - 0.5. \end{aligned}\right\} \qquad (4)$$

By setting $v_1 = 0.25 + 0.25\text{sat}(x_1)$ and $v_2 = 0.25 - 0.25\text{sat}(x_2)$ where sat is the saturation function, the closed-loop relaxed system becomes

$$\left.\begin{aligned} \dot{x}_1 &= -0.25\text{sat}(x_1) \\ \dot{x}_2 &= -0.25\text{sat}(x_2), \end{aligned}\right\} \qquad (5)$$

which is a globally asymptotically stable system.

Figure 1 shows a numerical example of the implementation of the above state-feedback relaxed control. The probability measure $\mu_{(v_1,v_2)}(u)$ is implemented by a multi-level PWM signal where the width at each duty cycle is determined by $v_1, v_2$ and the levels are $-\pi/2, 0, \pi/2$ and $\pi$.




(a) (b)

Fig. 1. The simulation of the closed-loop system for system (3) using state-feedback relaxed control: (a). The trajectory of $x_1$; (b). The trajectory of $x_2$;

## III. Noninteracting control problem

The system (3) with the output $y = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$ defines a single-input multi-output system. It has been shown before that the system cannot be stabilized by using state-feedback law which is assigned to its input $u$. This problem can be solved when a relaxed input is implemented to (3). In this case, the system (3) becomes (4) which is two independent single-input single-output (SISO) systems. The decoupling of the system into a number of independent SISO systems has simplified the design of the state-feedback controller.

Throughout this section, we assume the nonlinear system in (1) with locally Lipschitz $f$, input $u \in \mathbb{R}^m$, state $x \in \mathbb{R}^n$ and with the output given by $y = h(x)$ where $y \in \mathbb{R}^p$ and $h$ is a locally Lipschitz function. Adopting the definition in [3] for general class of nonlinear systems, the noninteracting control problem is defined as follows.

*Definition 3.1:* The feedback $u = k(x, v)$ with $v \in \mathbb{R}^q$, $q \le p$, solves the *noninteracting control problem* if the closed-loop systems can be decomposed into $q$ independent input-output subsystems. □

In other words, the system described by

$$\dot{x} = f(x, k(x, v)), \qquad y = h(x), \qquad (6)$$

has the property that, for any given initial conditions $x(0)$, if two input signals $v_1$ and $v_2$ which are equal for almost all $t$ but the $i$-th component, are applied in (6), then the output signals are almost equal but the $i$-th component(s).

For an affine nonlinear systems with $m = p = q$, the following theorem establishes the sufficient and necessary conditions for the solvability of the problem using static feedback law $u = \alpha(x) + \beta(x)v$ without using the relaxed input.

*Theorem 3.2:* [3, Proposition 3.2.] Consider an affine nonlinear systems with $m$ inputs and $m$ outputs, i.e.,

$$\left.\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x), \end{aligned}\right\} \qquad (7)$$

where $x \in \mathbb{R}^n$, $y, u \in \mathbb{R}^m$, $f, g, h$ are smooth functions on $\mathbb{R}^n$ with an initial state $x_0$. The noninteracting control problem is solvable with the state feedback law of the form $u = \alpha(x) + \beta(x)v$ if and only if the system has a vector relative degree $\{r_1, \ldots, r_m\}$ at $x_0$, i.e.,

$$L_{g_j} L_f^k h_i(x) = 0,$$

for all $1 \le j \le m$, for all $1 \le i \le m$, for all $k < r_i - 1$ and for all $x$ in a neighborhood of $x_0$, where $g_j$ is the $j$-th column vector of $g$ and $h_i$ is the $i$-th row vector of $h$, and the matrix

$$\begin{bmatrix} L_{g_1} L_f^{r_1-1} h_1(x_0) & \cdots & L_{g_m} L_f^{r_1-1} h_1(x_0) \\ L_{g_1} L_f^{r_2-1} h_2(x_0) & \cdots & L_{g_m} L_f^{r_2-1} h_2(x_0) \\ \cdots & \cdots & \cdots \\ L_{g_1} L_f^{r_m-1} h_m(x_0) & \cdots & L_{g_m} L_f^{r_m-1} h_m(x_0) \end{bmatrix}$$

is nonsingular.

This result can be generalized to affine systems with $p < m$ where the input can be partitioned into $p$ disjoint sets (see also, Remark 3.3 in [3]). The extension of the work to the non-affine systems can be found in [7].

For the system (3), it is possible to control independently each state by applying the input $u(t) \in \{-\pi/2, \pi/2\}$ for controlling $x_1$ or $u(t) \in \{0, \pi\}$ for controlling $x_2$. However, we cannot assign a state-feedback $u = k(x, v)$ in order to get two independent input signal $v = \begin{bmatrix} v_1 & v_2 \end{bmatrix}^T$ such that we have two independent SISO systems. We will deal with this problem using relaxed input.

In order to formalize the problem, we give below the definition of noninteracting control problem using relaxed input.

*Definition 3.3:* The relaxed input $\mu_{(v,x)} \in U_R$ with $v \in \mathbb{R}^q$ solves *the noninteracting control problem with relaxed input* if the relaxed system (2) with the output $y = h(x)$ consists of $q$ independent input-output subsystems. □

*Proposition 3.4:* The noninteracting control problem is solvable with relaxed input if for every $i \in \{1, \ldots, q\}$ there exist $u_i(x), w_i(x) \in \mathbb{R}^m$ such that

$$\begin{aligned} &L_{f(x,u_i(x))} h_j(x) = 0 \qquad \forall j \neq i \\ &L_{f(x,w_i(x))} h_j(x) = 0 \qquad \forall j \neq i \\ &L_{f(x,w_i(x))} h_i(x) < L_{f(x,u_i(x))} h_i(x), \end{aligned}$$

hold for every $x \in \mathbb{R}^n$.

*Proof:* Let $i \in \{1, \ldots, q\}$ and take $x \in \mathbb{R}^n$. We denote $\beta_i(x) = L_{f(x,u_i(x))} h_i(x)$, $\alpha_i(x) = L_{f(x,w_i(x))} h_i(x)$ and $I_i(x) = [L_{f(x,w_i(x))} h_i(x), L_{f(x,u_i(x))} h_i(x)]$.

Define relaxed input $\mu_{(v_i,x)} \in U_R$ by

$$\mu_{v_i,x}(E) = \int_E r_{v_i,x}(\tau) \mathrm{d}\tau \tag{8}$$

where

$$r_{v_i,x} = \frac{\beta_i(x) - v_i}{\beta_i(x) - \alpha_i(x)} \delta_{w_i(x)} + \frac{v_i - \alpha_i(x)}{\beta_i(x) - \alpha_i(x)} \delta_{u_i(x)}.$$

where $v_i \in I_i(x)$.

Using this relaxed input, it follows that

$$\begin{aligned} \dot{y}_i &= \int_{u \in \mathbb{R}^m} \frac{\partial h(x)}{\partial x} f(x, u) \, \mathrm{d}\mu_{(v_i,x)}(u) \\ &= \frac{\beta_i(x) - v_i}{\beta_i(x) - \alpha_i(x)} \alpha_i(x) \\ &\quad + \frac{v_i - \alpha_i(x)}{\beta_i(x) - \alpha_i(x)} \beta_i(x) \\ &= v_i, \end{aligned} \tag{9}$$

where $v_i \in I_i(x)$. Also,

$$\begin{aligned} \dot{y}_j &= \int_{u \in \mathbb{R}^m} \frac{\partial h(x)}{\partial x} f(x, u) \, \mathrm{d}\mu_{(v_i,x)}(u) \\ &= \frac{\beta_i(x) - v_i}{\beta_i(x) - \alpha_i(x)} 0 \\ &\quad + \frac{v_i - \alpha_i(x)}{\beta_i(x) - \alpha_i(x)} 0 \\ &= 0, \end{aligned} \tag{10}$$

for all $j \neq i$. In other words, the relaxed input $\mu_{(v_i,x)}$ only affect the $i$-th output $y_i$ but not the rest of the output $y_j$, $j \neq i$.

The same construction can be used for every $i \in \{1, \ldots, q\}$, to construct the relaxed input $\mu_{(v_i,x)}$, $i = 1, \ldots, q$. The combined relaxed input is then given by

$$\mu_{(v,x)} = \sum_{i=1}^{q} \frac{1}{q} \mu_{(v_i,x)}. \tag{11}$$

■

Note that (11) is one of the solutions to the noninteracting control problem using relaxed input. The convex combination of $\mu_{(v_1,x)}, \ldots, \mu_{(v_q,x)}$ where $\mu_{(v_i,x)}$ are as in the proof of Proposition 3.4, gives the family of relaxed inputs which solve the problem.

We remark that using the relaxed input $\mu_{(v,x)}$ as in the proof of Proposition 3.4, the input $v$ of the relaxed system may not be defined in a proper input space. At every state $x$, the input $v$ is defined in $I(x) = \frac{1}{q} (I_1(x) \times I_2(x) \times \ldots \times I_q(x))$ and there is no guarantee that there exists an input space $V$ such that $V \subset \cap_{x \in \mathbb{R}^n} I(x)$. This can complicate the controller design using $v$ and we deal with this in the following proposition.

*Proposition 3.5:* If there exist constants $a < b$ such that for every $i = 1, \ldots, q$ there exist $u_i(x), w_i(x) \in \mathbb{R}^m$ such that

$$\begin{aligned} &L_{f(x,u_i(x))} h_j(x) = 0 \qquad \forall j \neq i \\ &L_{f(x,w_i(x))} h_j(x) = 0 \qquad \forall j \neq i \\ &L_{f(x,u_i(x))} h_i(x) > b \\ &L_{f(x,w_i(x))} h_i(x) < a, \end{aligned}$$

hold for every $x \in \mathbb{R}^n$, then the noninteracting control problem is solvable with relaxed input $\mu_{(v,x)}$ (as in (11) and (8)) where $v \in \frac{1}{q}[a, b]^q$.

The proof of Proposition 3.5 follows a similar line as that of Proposition 3.4 using the fact that $[a,b] \subset \cap_{x\in\mathbb{R}^n} I_i(x)$ for all $i \in \{1,\ldots,q\}$.

*Remark 3.6:* The requirement for the same constants $a$ and $b$ for every $i \in \{1,\ldots,q\}$ can be weakened by allowing different $a_i$ and $b_i$ for each $i$.

The previous propositions give sufficient conditions for systems which can be transformed by relaxed input into systems with relative degree of one. The natural generalization of the results is given in the following propositions.

*Proposition 3.7:* The noninteracting control problem is solvable with relaxed input if for every $i = 1,\ldots,q$ there exist $u_i(x), w_i(x) \in \mathbb{R}^m$ and $r_i \in \mathbb{N}$ such that

$$\begin{array}{ll} L_{f(x,u_i(x))} h_j(x) = 0 & \forall j \neq i \\ L_{f(x,w_i(x))} h_j(x) = 0 & \forall j \neq i \\ L^k_{f(x,w_i(x))} h_i(x) = L^k_{f(x,u_i(x))} h_i(x) & \forall k < r_i \\ L^{r_i}_{f(x,w_i(x))} h_i(x) < L^{r_i}_{f(x,u_i(x))} h_i(x), & \end{array}$$

hold for every $x \in \mathbb{R}^n$.

*Proof:* The proof of the proposition is similar to that of Proposition 3.4. Let $i \in \{1,\ldots,q\}$ and take $x \in \mathbb{R}^n$. If $r_i = 1$, then the proof is the same as that of Proposition 3.4. Otherwise, we denote $\beta_{i,k}(x), \alpha_{i,k}(x)$ by

$$\begin{aligned} \beta_{i,k}(x) &= L^k_{f(x,u_i(x))} h_i(x), \\ \alpha_{i,k}(x) &= L^k_{f(x,w_i(x))} h_i(x). \end{aligned}$$

With this notation, $\beta_{i,1}(x)$ and $\alpha_{i,1}(x)$ are the same as $\beta_i(x)$ and $\alpha_i(x)$ defined in the proof of Proposition 3.4. The hypotheses of the proposition imply that $\beta_{i,k}(x) = \alpha_{i,k}(x)$ for all $k < r_i$ and for all $x \in \mathbb{R}^n$. We define $I_i(x) = [\alpha_{i,r_i}(x), \beta_{i,r_i}(x)]$ which is a non-empty set.

We now use the relaxed input $\mu_{(v_i,x)} \in U_R$ defined by $\mu_{v_i,x}(E) = \int_E r_{v_i,x}(\tau) \mathrm{d}\tau$ where

$$r_{v_i,x} = \frac{\beta_{i,r_i}(x) - v_i}{\beta_{i,r_i}(x) - \alpha_{i,r_i}(x)} \delta_{w_i(x)} + \frac{v_i - \alpha_{i,r_i}(x)}{\beta_{i,r_i}(x) - \alpha_{i,r_i}(x)} \delta_{u_i(x)}.$$

where $v_i \in I_i(x)$.

Using this relaxed input, it follows that for every $v_i \in I_i(x)$

$$\begin{aligned} \dot{y}_i &= \int_{u\in\mathbb{R}^m} \frac{\partial h(x)}{\partial x} f(x,u) \ \mathrm{d}\mu_{(v_i,x)}(u) \\ &= \frac{\beta_{i,r_i}(x) - v_i}{\beta_{i,r_i}(x) - \alpha_{i,r_i}(x)} \alpha_{i,1}(x) \\ &\quad + \frac{v_i - \alpha_{i,r_i}(x)}{\beta_{i,r_i}(x) - \alpha_{i,r_i}(x)} \beta_{i,1}(x) \\ &= \alpha_{i,1}(x), \end{aligned} \tag{12}$$

where the last inequality is due to $\alpha_{i,1}(x) = \beta_{i,1}(x)$.

Using (12), it can be computed that

$$\begin{aligned} \ddot{y}_i &= \int_{u\in\mathbb{R}^m} \frac{\partial \alpha_{i,1}(x)}{\partial x} f(x,u) \ \mathrm{d}\mu_{(v_i,x)}(u) \\ &= \frac{\beta_{i,r_i}(x) - v_i}{\beta_{i,r_i}(x) - \alpha_{i,r_i}(x)} \alpha_{i,2}(x) \\ &\quad + \frac{v_i - \alpha_{i,r_i}(x)}{\beta_{i,r_i}(x) - \alpha_{i,r_i}(x)} \beta_{i,2}(x) \\ &= \alpha_{i,2}(x), \end{aligned} \tag{13}$$

for all $x \in \mathbb{R}^n$. By induction, it follows that for every $k \in \{1,\ldots,r_i - 1\}$

$$y_i^{(k)} = \alpha_{i,k}(x) \qquad \forall x \in \mathbb{R}^n. \tag{14}$$

Finally, we also obtain that

$$\begin{aligned} y_i^{(r)} &= \int_{u\in\mathbb{R}^m} \frac{\partial \alpha_{i,r_i-1}(x)}{\partial x} f(x,u) \ \mathrm{d}\mu_{(v_i,x)}(u) \\ &= \frac{\beta_{i,r_i}(x) - v_i}{\beta_{i,r_i}(x) - \alpha_{i,r_i}(x)} \alpha_{i,r_i}(x) \\ &\quad + \frac{v_i - \alpha_{i,r_i}(x)}{\beta_{i,r_i}(x) - \alpha_{i,r_i}(x)} \beta_{i,r_i}(x) \\ &= v_i, \end{aligned} \tag{15}$$

for every $x \in \mathbb{R}^n$. Hence, the new input $v_i$ appears on the $r_i$-th derivative of $y_i$.

On the other hand, using similar technique as in the proof of Proposition 3.4, (10) holds for all $x \in \mathbb{R}^n$ and for all $j \neq i$.

The same construction can be used for every $i \in \{1,\ldots,q\}$, to construct the relaxed input $\mu_{(v_i,x)}$, $i = 1,\ldots,q$. The combined relaxed input is then given by (11). This completes the proof. ■

*Proposition 3.8:* If there exist constants $a < b$ such that for every $i = 1,\ldots,q$ there exist $u_i(x), w_i(x) \in \mathbb{R}^m$ and $r_i \in \mathbb{N}$ such that

$$\begin{array}{ll} L_{f(x,u_i(x))} h_j(x) = 0 & \forall j \neq i \\ L_{f(x,w_i(x))} h_j(x) = 0 & \forall j \neq i \\ L^k_{f(x,w_i(x))} h_i(x) = L^k_{f(x,u_i(x))} h_i(x) & \forall k < r_i \\ L^{r_i}_{f(x,w_i(x))} h_i(x) < a & \\ L^{r_i}_{f(x,u_i(x))} h_i(x) > b, & \end{array}$$

hold for every $x \in \mathbb{R}^n$, then the noninteracting control problem is solvable with relaxed input $\mu_{(v,x)}$ where $v \in \frac{1}{q}[a,b]^q$.

The proof of the proposition is similar to that of Proposition 3.5 and 3.7.

## IV. Controller design for stabilization problem

In the previous section, we can design relaxed input which approximately solves noninteracting control problem. Based on the result from previous section, we explore the application of relaxed input in order to solve stabilization problem.

*Corollary 4.1:* In addition to the assumptions in Proposition 3.5, suppose that $a < 0$ and $b > 0$ then state-feedback relaxed control can be designed such that $y(t) \to 0$ as $t \to \infty$.

*Proof:* Let $i \in \{1, \ldots, q\}$ and construct the same relaxed input $\mu_{(v_i,x)}$ as in the proof of Proposition 3.4 which gives us (9) and (10). Since $[a, b] \subset I_i(x)$ for all $x$, we prove the corollary by designing the feedback law for $v_i$ which satisfies $v_i \in [a, b]$ and $y_i(t) \to 0$ as $t \to \infty$.

We define $c := \min\{-a, b\}$ and let $v_i = -c\ \text{sat}(y_i)$. Using this feedback law, it can be checked that $v_i \in [a, b]$ for all $y_i$. It follows from (9) that

$$\dot{y}_i = -c\ \text{sat}(y_i), \tag{16}$$

Using $V_i = \int_0^{\text{sat}(y_i)} \sigma\ \mathrm{d}\sigma$, we have

$$\dot{V}_i = -(\text{sat}(y_i))^2 \leq 0.$$

From this inequality, we have that $|y_i(t)| \leq |y_i(0)|$ for all $t \in \mathbb{R}_+$ and $\text{sat}(y_i) \in L_2(\mathbb{R}_+)$. Since $\text{sat}(y_i) \in L_2(\mathbb{R}_+)$, (16) implies that also $\dot{y}_i \in L_2(\mathbb{R}_+)$. By using Barbalat's lemma, $\dot{y}_i, \text{sat}(y_i) \in L_2(\mathbb{R}_+) \Rightarrow y_i(t) \to 0$ as $t \to \infty$.

We use the same arguments for all $i \in \{1, \ldots, q\}$ to conclude the proof. ■

In the Corollary 4.1, the constructed relaxed input produces relaxed system whose input signal $v$ is defined in a compact set $[a, b]^q$. This enforces limitation in the design of feedback law and we use saturation function in the proof of Corollary 4.1. The works of Teel in [10] and Kaliora and Astolfi in [4] are relevant in this respect which provides controller design with bounded input signal for the relaxed systems described by (2) or (9).

*Corollary 4.2:* In addition to the assumptions in Proposition 3.8, suppose that $a < 0$ and $b > 0$ then state-feedback relaxed control can be designed such that $y(t) \to 0$ as $t \to \infty$.

*Proof:* Let $i \in \{1, \ldots, q\}$ and construct the same relaxed input $\mu_{(v_i,x)}$ as in the proof of Proposition 3.7 which gives us (14), (15) and (10). Since $[a, b] \subset I_i(x)$ for all $x$, we prove the corollary by designing the feedback law for $v_i$ whose domain is $[a, b]$, such that $y_i(t) \to 0$ as $t \to \infty$.

For the case $r_i = 1$, the feedback law constructed in the proof of Corollary 4.1 can be used to ensure $y_i(t) \to 0$ as $t \to \infty$.

We will evaluate the case when $r_i > 1$. Let us denote by $z_1 = h(x)$, $z_2 = \alpha_{i,1}(x)$, $\ldots$, $z_{r_i} = \alpha_{i,r_i-1}$. It follows from (14) and (15) that

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ &\vdots \\ \dot{z}_{r_i} &= v_i. \end{aligned}$$

By an application of Proposition 4 in [4], we can design a stabilizing controller for the chain integrator form above with saturated control signal. Using this controller, Proposition 4 in [4] ensures that $y_i(t)$ converges to zero as $t \to \infty$.

We use the same arguments for all $i \in \{1, \ldots, q\}$ to conclude the proof. ■

## V. Conclusion

This paper presents methodology to decouple input and output for nonlinear systems by using relaxed input. It has been shown that for a certain class of nonlinear systems, the proposed method can simplify the controller design.

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