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Abstract— **In this paper, I discuss the concept of control by relaxed input. The method allows for the transformation of a non-affine nonlinear system into an affine one. As a result, various control design methodologies for affine systems can directly be applied. The implementation aspect of relaxed input is also discussed.**

Keywords: relaxed control; local decomposition problem; nonlinear control design;

I. INTRODUCTION

Geometric control theory has laid a strong foundation in the study of nonlinear systems analysis and control. The books [3] and [6] describe the extension of various systems theoretic properties of linear systems to nonlinear systems employing tools from differential geometry. Although it is limited to affine nonlinear systems, geometric control theory has been able to generalize various notions from linear systems, such as, controllability, observability, state decomposition, input-output decoupling and minimum-phase systems.

For example, local decomposition results for affine nonlinear systems show the existence of state transformation that leads to decomposition of the state equations into controllable and non-controllable parts, observable and non-observable parts [3]. In [6], a state-feedback control law can be designed which can transform an affine nonlinear system to a linear one.

In this paper, I discuss the concept of control by relaxed input in order to extend the aforementioned results to non-affine nonlinear systems. I extend the result in [4] by discussing the concept of feedback control via relaxed input, by studying the local decomposition problem and by discussing the implementation aspect of feedback control via relaxed input.

Warga in [10] introduced the concept of relaxed control, where the ordinary input functions are replaced by measure-valued input functions, for relaxing the optimal control problems. Gamkrelidze in [2] discussed a relevant methodology of generalized control. The relaxed control method has been extended subsequently by Artstein in [1] for solving stabilization problem of general nonlinear systems. Necessary and sufficient condition for the nonlinear systems to be stabilizable by relaxed control are given in [1] in the form of a control Lyapunov-type condition. Application of relaxed control to optimal control problem has been discussed in [5].

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Suppose that the nonlinear systems are described by the state equation:

$$\dot{x}(t) = f(x(t), u(t)), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}$. The relaxed control method [1], [10] assumes the input u as a finite positive Radon measure valued function μ with U as its support, where (1) becomes

$$\dot{x}(t) = \int_{w\in\mathbb{R}} f(x(t), w)\ \mathrm{d}\mu(w).$$

In practice, the relaxed control signal resembles the principle of control by pulse-width modulation (PWM) [8]. Dither control introduced by Zames and Shneydor in [12], [11], is also based on a similar concept. In [12], [11], the sector condition for the static nonlinearity is relaxed by using an additional dither signal in the control signal.

In Section II, I discuss the concept of relaxed systems with the corresponding relaxed input. In Section III, I discuss an application of relaxed input to solve local decomposition problem for non-affine nonlinear systems. Finally, in Section IV, I discuss the implementation aspect of relaxed input using ordinary input.

II. RELAXED SYSTEMS

Throughout this paper, I consider nonlinear systems described by

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)), \\ y(t) &= h(x(t)), \end{aligned} \tag{2}$$

where $x(t) \in X \subset \mathbb{R}^n$, $u(t) \in U \subset \mathbb{R}^m$ and $y(t) \in Y \subset \mathbb{R}^p$. The functions f and h are assumed to be locally Lipschitz. I also assume that the origin is an element in both U and X.

Let $\mathrm{rpm}(U)$ be the set of all Radon probability measure defined on U. For a compact metric space $V \subset \mathbb{R}^q$, the space $\mathcal{R}_f(V, \mathrm{rpm}(U))$ is the space of all functions $\mu : v \in V \mapsto \mu(v) \in \mathrm{rpm}(U)$ such that the function

$$f_R : (x, v) \in X \times V \mapsto \int_U f(x, \tau)\ \big(\mu(v)\big)(\mathrm{d}\tau) \tag{3}$$

is locally Lipschitz on $X \times V$. The subscript f in $\mathcal{R}_f$ describes its dependence on the vector field f.

Lemma 2.1: The space $\mathcal{R}_f(V, \mathrm{rpm}(U))$ is non-empty.

Proof: Firstly, let us assume that $m = q$, i.e., the dimension of U and V is equal. In this case, we can define $\big(\mu(v)\big)(E) = \int_E \delta_0(\tau)\mathrm{d}\tau$ for all $v \in V$ and for all $E \subset U$. Using μ, simple computation shows that the function

$$\int_{\tau\in U} f(x, \tau)\ \mathrm{d}\big(\mu(v)\big)(\tau) = f(x, 0),$$

where f is locally Lipschitz by assumption.

For the case $m \neq q$, we can use similar argument as above either by projecting or extending the space of $\mathbb{R}^q$ onto $\mathbb{R}^m$. ■

Remark 2.2: Note that when $V \subset U$, by defining $\big(\mu(v)\big)(E) = \int_E \delta_v(\tau)\mathrm{d}\tau$ for all $v \in V$ and for all $E \subset U$, a routine computation shows that

$$\int_U f(x,\tau)\ \big(\mu(v)\big)(\mathrm{d}\tau) = f(x,v), \tag{4}$$

for all $v \in V$, where f is locally Lipschitz by assumption. ♢

For a given vector field f, a sequence $(u_j) : [0,t] \to U$ is said to be converging to $\nu : [0,t] \to \mathrm{rpm}(U)$ if

$$\lim_j \int_0^t f(x(\lambda), u_j(\lambda))\mathrm{d}\lambda = \int_0^t \int_U f(x(\lambda),\tau)\big(\nu(\lambda)\big)(\mathrm{d}\tau)\mathrm{d}\lambda, \tag{5}$$

for every continuous function $x \in C([0,t], X)$ [10].

Using $\mathcal{R}_f(V, \mathrm{rpm}(U))$, the ordinary input u in (2) can be replaced by $\mu \in \mathcal{R}_f(V, \mathrm{rpm}(U))$ such that the relaxed system is given by

$$\dot{x}(t) = \int_U f(x(t),\tau)\ \big(\mu(v(t))\big)(\mathrm{d}\tau) =: f_R(x(t), v(t)),$$
$$x(0) = x_0, \tag{6}$$

where $v(t) \in V$. The signal v becomes the new input variable in the RHS of (6). The system with the *relaxed input* $\mu \in \mathcal{R}_f(V, \mathrm{rpm}(U))$ as given in (6) is called *relaxed system*. The function f_R is locally Lipschitz by the definition of $\mathcal{R}_f(V, \mathrm{rpm}(U))$.

The solution x of (2) using the relaxed input $\mu(v)$ where $\mu \in \mathcal{R}_f(V, \mathrm{rpm}(U))$ and $v \in L^1(V)$ is the Carathéodory solution x of the relaxed system (6). The following proposition is due to Warga [10] which describes the approximation of the solution to the relaxed systems.

Lemma 2.3: Assume that X is open. Then, for every $\mu \in \mathcal{R}_f(V, \mathrm{rpm}(U))$ and $v \in L^1(V)$ such that the solution $\bar{x}$ of (6) is defined for all $t \geq 0$, and for every sequence of input signal (u_j) defined on U that converges to $\mu(v)$, there exists $j_0 \in \mathbb{N}$ and a sequence $(x_j)_{j \geq j_0}$ defined in X such that (x_j, u_j) is the solution of (2) and

$$\lim_j x_j = \bar{x}.$$

Proof: The proof follows immediately from Lemma VI.1.4 in [10] by taking there $\psi_W = Id$, $T = [0,t]$, $y = x$, $\sigma = \mu(v)$ and by the fact that f is a locally Lipschitz function. ■

The original nonlinear system equation with ordinary input in (1) can be derived back from (6) by taking $V = U$ and $\big(\mu(v)\big)(E) = \int_E \delta_v(\tau)\mathrm{d}\tau$ for all $v \in V$ and $E \subset U$.

Lemma 2.4: Let $k : X \to V$ be a locally Lipschitz function, then the composition of $\mu \circ k$ with $\mu \in \mathcal{R}_f(V, \mathrm{rpm}(U))$ belongs to $\mathcal{R}_f(X, \mathrm{rpm}(U))$.

The proof of the lemma follows from the fact that the composition of two locally Lipschitz functions is also a locally Lipschitz function.

The lemma shows that state-feedback can be applied to the relaxed input and the stability of the closed-loop system can be analyzed via the relaxed system equation

$$\dot{x} = f_R(x, k(x)),$$

where f_R is locally Lipschitz according to Lemma 2.4.

Note that the behavior of the original system with static state-feedback law k can be obtained using the relaxed-input by taking $V = U$ and using μ as constructed in Remark 2.2. Indeed, using (4) and using $v = k(x)$, we get

$$\dot{x} = f(x, k(x)).$$

When we use dynamic state-feedback to the ordinary input, the similar remark also holds.

A similar observation is applicable when a state feedback law with an exogenous signal is used. For example, let $k : X \times W \to V$ be locally Lipschitz function with $W \subset \mathbb{R}^w$ be the space of an exogenous signal $w(t) \in W$. Then, for every $\mu \in \mathcal{R}_f(V, \mathrm{rpm}(U))$, we have $\mu \circ k \in \mathcal{R}_f(X \times W, \mathrm{rpm}(U))$.

The result in [1] describes the stabilization of (1) by designing *state-feedback relaxed control* $\mu \circ k \in \mathcal{R}_f(X, \mathrm{rpm}(U))$ such that the resulting differential equation

$$\dot{x} = f_R(x, k(x))$$

is (locally) asymptotically stable in the origin. The following theorem is the main result of [1].

Theorem 2.5: The system (1) with locally Lipschitz f is locally asymptotically stabilizable by a state-feedback relaxed control if and only if there is a continuously differentiable function $V : X \to \mathbb{R}_+$ where X is a neighborhood of 0 such that V is positive definite and

$$\inf_{u \in \mathbb{R}^m} \mathrm{grad}\ V(x) f(x,u) < 0 \qquad \forall x \in X \backslash \{0\}.$$

It is globally asymptotically stabilizable by a state-feedback relaxed control if and only if $X = \mathbb{R}^n$ and V is radially unbounded.

The above theorem provides flexibility in designing a smooth state-feedback relaxed control for solving controller design for nonlinear systems which can only be stabilized by non-smooth state-feedback control.

III. Decomposition problem

For an affine nonlinear system, the relaxed input $\mu \in \mathcal{R}_f(V, \mathrm{rpm}(U))$ does not yield an advantage over the standard control input. Indeed, let an affine nonlinear system be described by (2) with

$$f(x,u) = f_1(x) + f_2(x)u,$$

where f_1 and f_2 are locally Lipschitz function. For every $\mu \in \mathcal{R}_f(V, \mathrm{rpm}(U))$, the computation of (6) yields

$$\dot{x}(t) = f_1(x(t)) + f_2(x(t)) \int_{\tau \in U} \tau\ \mathrm{d}\big(\mu(v(t))\big)(\tau).$$

It is evident from the above equation that the resulting relaxed system is also an affine nonlinear system with the same f_1 and f_2.

On the other hand, for non-affine nonlinear systems, relaxed input can be designed such that the resulting f_R has a number of useful control properties. For instance, it is possible to transform a non-affine nonlinear system to an affine one.

Proposition 3.1: Consider a non-affine nonlinear system described by (2) with $m = 1$. Let $V = [v_1, v_2]$ where $v_1 < v_2$ are constants. Then there exists $\mu \in \mathcal{R}_f(V, \mathrm{rpm}(U))$ such that the relaxed system in (6) is an affine nonlinear system.

Proof: For the given $v_1, v_2 \in \mathbb{R}$, let the functions f_1, f_2 be defined by

$$f_1(x) = \frac{f(x, v_1)v_2 - f(x, v_2)v_1}{v_2 - v_1},$$

and

$$f_2(x) = \frac{f(x, v_2) - f(x, v_1)}{v_2 - v_1}.$$

For every $v \in V$, define $\big(\mu(v)\big)(E)$ by

$$\big(\mu(v)\big)(E) = \int_E \frac{v - v_1}{v_2 - v_1}\delta_{v_2}(\tau) + \frac{v_2 - v}{v_2 - v_1}\delta_{v_1}(\tau)\mathrm{d}\tau \qquad \forall E \subset U.$$

Using the above μ, it can be shown that

$$\begin{aligned}\dot{x}(t) &= \int_{\tau \in U} f(x(t), \tau)\ \mathrm{d}\big(\mu(v(t))\big)(\tau) \\ &= \frac{v(t) - v_1}{v_2 - v_1} f(x(t), v_2) + \frac{v_2 - v(t)}{v_2 - v_1} f(x(t), v_1) \\ &= f_1(x(t)) + f_2(x(t))v(t),\end{aligned}$$

where f_1 and f_2 are locally Lipschitz functions and $v(t) \in V$. This proves the claim. ■

An extended system to the nonlinear system (2) has also been considered in the literature for constructing an affine system from a non-affine one, by taking the input as the extended state and its time derivative is assigned as the new affine input (see, for example, [6, Chapter 6]). However, designing a stabilizing state-feedback controller for such an extended system can be restrictive. For example, we cannot design a controller for the extended system if the original system can only stabilized by discontinuous state-feedback law.

The following corollaries are consequences of Proposition 3.1.

Corollary 3.2: Consider a non-affine nonlinear system described by (2) with $m = 1$. Suppose that there exists $v_1, v_2 \in \mathbb{R}$ such that $v_1 < 0$, $v_2 > 0$ and the system

$$\dot{x} = f_1(x)$$

where

$$f_1(x) = \frac{f(x, v_1)v_2 - f(x, v_2)v_1}{v_2 - v_1},$$

is locally asymptotically stable (at the origin), then using the relaxed input

$$\mu(E) = \int_E \frac{-v_1}{v_2 - v_1}\delta_{v_2}(\tau) + \frac{v_2}{v_2 - v_1}\delta_{v_1}(\tau)\mathrm{d}\tau,$$

the corresponding relaxed system is locally asymptotically stable.

Corollary 3.3: Consider a non-affine nonlinear system described by (2) with $m = 1$. Suppose that there exists $v_1, v_2 \in \mathbb{R}$ such that $v_1 < 0$, $v_2 > 0$ and the system

$$\dot{x} = f_1(x) + f_2(x)v \tag{7}$$

where

$$f_1(x) = \frac{f(x, v_1)v_2 - f(x, v_2)v_1}{v_2 - v_1}, \quad f_2(x) = \frac{f(x, v_2) - f(x, v_1)}{v_2 - v_1}.$$

is locally controllable (i.e., its linearization at the origin is controllable), then using the relaxed input $\mu \circ Kx$ where $K \in \mathbb{R}^{1\times n}$ is the locally stabilizing state-feedback gain of (7) and

$$\big(\mu(v)\big)(E) = \int_E \frac{v - v_1}{v_2 - v_1}\delta_{v_2}(\tau) + \frac{v_2 - v}{v_2 - v_1}\delta_{v_1}(\tau)\mathrm{d}\tau \qquad \forall v \in V,\ \forall E \subset U,$$

the corresponding relaxed system is locally exponentially stable.

Example 3.4: Consider the system

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix} x_1u + x_1x_3u^2 + x_2e^{x_2} \\ u + x_3 \\ x_4 - x_2x_3u^4 \\ x_3u + \sin(\pi u/2)(x_3^2 + x_2x_4 - x_2^2x_3)\end{bmatrix}. \tag{8}$$

Note that this system is a modified form of [3, Example 8.2].

By taking $v_1 = -1$ and $v_2 = 1$, and using the construction as in the proof of Proposition 3.1, we can obtain an affine nonlinear system given by

$$\dot{x} = f_1(x) + f_2(x)v \tag{9}$$

where $v \in (-1, 1)$,

$$f_1(x) = \frac{f(x, v_1)v_2 - f(x, v_2)v_1}{v_2 - v_1} = \begin{bmatrix} x_1x_3 + x_2e^{x_2} \\ x_3 \\ x_4 - x_2x_3 \\ x_3^2 + x_2x_4 - x_2^2x_3\end{bmatrix}$$

and

$$f_2(x) = \frac{f(x, v_2) - f(x, v_1)}{v_2 - v_1} = \begin{bmatrix} x_1 \\ 1 \\ 0 \\ x_3\end{bmatrix}.$$

$\diamondsuit$

The transformation of non-affine form into an affine one by relaxed input allows us to extend the result from the geometric control theory. Recall the following result from Isidori [3, Proposition 7.1].

Proposition 3.5: Consider an affine nonlinear systems given by

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i. \tag{10}$$

Assume that there exists a nonsingular involutive distribution Δ (with dimension d) such that it is invariant under the vector fields f, g_i, $i = 1, \ldots m$. If the distribution span$\{g_1, \ldots g_m\}$ is contained in Δ, then for every x_0 there exists a neighborhood X and a local coordinates transformation $z = \Phi(x)$ defined on X such that

$$\begin{bmatrix} \dot{z}_1 \\ \vdots \\ \dot{z}_d \end{bmatrix} = f_1(z) + \sum_{i=1}^{m} \tilde{g}_i(z)u_i \tag{11}$$

$$\begin{bmatrix} \dot{z}_{d+1} \\ \vdots \\ \dot{z}_n \end{bmatrix} = f_2(z_{d+1}, \ldots, z_n). \tag{12}$$

Using this proposition and Proposition 3.1, it is possible to transform non-affine systems into affine systems that can locally be decomposed into controllable and non-controllable parts.

Corollary 3.6: Assume that there exists a relaxed input $\mu \in \mathcal{R}_f(V, \mathrm{rpm}(U))$ such that the non-affine nonlinear system (2) can be transformed into an affine form (10) that admits a nonsingular involutive distribution Δ as in Proposition 3.5. Then (2) can be locally decomposed using the relaxed input μ.

Example 3.7: Consider again Example 3.4. It has been shown in Example 3.4 that the implementation of relaxed input allows us to transform a non-affine nonlinear system (8) into an affine nonlinear system (9). The functions f_1 and f_2 are the same the functions f and g as in [3, Example 8.2]. Following the same construction as in [3, Example 8.2], using $z_1 = x_1$, $z_2 = x_2$, $z_3 = x_3$ and $z_4 = x_4 - x_2x_3$, we can decompose (9) into

$$\dot{z} = \begin{bmatrix} z_1z_3 + z_2e^{z_2} \\ z_3 \\ z_4 \\ 0 \end{bmatrix} + \begin{bmatrix} z_1 \\ 1 \\ 0 \\ 0 \end{bmatrix} v.$$

$\diamondsuit$

Remark 3.8: Proposition 3.5 describes the local decomposition of affine nonlinear systems into controllable and non-controllable components. It has been used to address the reachability concept for affine systems, see for example, [3, Theorem 8.13]. Therefore, similar to the Corollary 3.6, we can also obtain a weaker notion of reachability for non-affine systems by using relaxed input, e.g., reachable via relaxed input. $\diamondsuit$

We conclude the section by noting that the possibility of transforming non-affine nonlinear systems to affine ones, has allowed the extension of various control theoretic results from geometric control theory which have been derived for affine nonlinear systems.

IV. Implementation of the Relaxed Input

In practice, the relaxed input μ must be implemented using the ordinary input u of (2). Here, I will discuss the approximation of the relaxed input μ for the case $m = 1$ and $U = [a, b]$ where $a, b \in \mathbb{R}$.

Suppose that we have a relaxed input $\mu \in \mathcal{R}_f(V, \mathrm{rpm}(U))$ with $U = [a, b]$ and let $v \in C(\mathbb{R}_+, V)$. The approximation is based on the discrete-time sampling of $\mu(v)$ which is then projected to the continuous-time approximating input u_j. Roughly speaking, if the sequence of time sample is given by (t_k), the approximating input u_j is obtained by concatenating sequences of $(u_{j,k})$ where each $u_{j,k} : [t_k, t_{k+1}) \to U$ is defined such that the time proportion of $u_{j,k}$ spent on any subset $A \subset U$ is equal to $\mu(v(t_k))(A)$. Before I discuss the sequence of approximating input (u_j) which converges to $\mu(v)$ on every compact set $[0, t]$, I describe first the construction of $u_{j,k}$.

Let $\Delta > 0$ be the sampling period, i.e., $\Delta = t_{k+1} - t_k$ for all k, and for simplicity of notation in the later description of approximating input sequence, I also denote $j := 1/\Delta$. Let us construct $u_{j,k} : [t_k, t_{k+1}) \to U$, which is based on $\mu(v(t_k))$, as follows. Define $\gamma_k := w \in [a, b] \mapsto \big(\mu(v(t_k))\big)([a, w])$ which is a non-decreasing function. Let $\Gamma_k \subset [0, 1]$ be defined by $\Gamma_k := \{\gamma_k(w) \mid w \in [a, b]\}$ which can be a union of disjoint sets due to Dirac measures in $\mu(v(t_k))$. Let N denote the number of Dirac measures in $\mu(v(t_k))$.

For every $\nu \in [0, 1]$, let

$$\eta(\nu) := \min\{\xi \in \Gamma_k \mid \xi - \nu \geq 0\}. \tag{13}$$

In other words, $\eta(\nu)$ is the closest point in Γ_k to ν from above and $\eta(\xi) = \xi$ for all $\xi \in \Gamma_k$. Let $\gamma_k^{-1} := \xi \in \Gamma_k \mapsto \min\{w \in [a, b] \mid \gamma_k(w) = \xi\}$. Using these functions, we can define $u_{j,k}(t_k + \xi) = \gamma_k^{-1} \circ \eta(\xi/\Delta)$ for all $\xi \in (0, \Delta]$. The signal $u_{j,k}$ is non-decreasing.

For every $A \subset [a, b]$ and by defining $T_{j,k}(A) := \{t \in [t_k, t_{k+1}] \mid u_{j,k}(t) \in A\}$, we have that

$$\mu_{j,k}(A) := \frac{\int_{T_{j,k}(A)} \mathrm{d}\tau}{\Delta} = \mu(v(t_k))(A). \tag{14}$$

Based on the above construction, the concatenation of $u_{j,k}$ gives the approximating input sequence (u_j) which converges to $\mu(v)$ as $\Delta \to 0$. More precisely, the approximating u_j is given by

$$u_j(t) = \begin{cases} u_{j,1}(t) & 0 < t \leq \Delta \\ u_{j,2}(t) & \Delta < t \leq 2\Delta \\ \vdots & \vdots \\ u_{j,k}(t) & (k-1)\Delta < t \leq k\Delta, \end{cases} \tag{15}$$

where $k \in \mathbb{N}$. Moreover, (14) holds for every k where $t_k = (k-1)\Delta$ (or, equivalently, $t_k = (k-1)/j$).

In the following, I will show that the approximating input (u_j) converges to $\mu(v)$ as $\Delta \to 0$ (or as $j \to \infty$).

Let u_j be the sequence of the approximating input as constructed above with $\Delta = 1/j$. For simplicity, let $t \in \mathbb{N}$ and $x \in C([0,t], X)$, in which case,

$$\begin{aligned}&\int_0^t f(x(\lambda), u_j(\lambda))\mathrm{d}\lambda \\ &= \sum_{k=1}^{jt} \int_{[(k-1)/j,k/j]} f(x(\lambda), u_{j,k}(\lambda))\mathrm{d}\lambda.\end{aligned}$$

For a sufficiently large j (or, equivalently, small Δ), x can be approximated by a piecewise-constant signal on each interval $[(k-1)/j, k/j]$ and we obtain

$$\begin{aligned}&\int_0^t f(x(\lambda), u_j(\lambda))\mathrm{d}\lambda \\ &= \sum_{k=1}^{jt} \int_{[(k-1)/j,k/j]} f(x((k-1)/j), u_{j,k}(\lambda))\mathrm{d}\lambda \\ &= \sum_{k=1}^{jt} \Delta \int_U f(x((k-1)/j), \tau)\mu_{j,k}(\mathrm{d}\tau) \\ &= \sum_{k=1}^{jt} \frac{1}{j} \int_U f(x((k-1)/j), \tau)\mu(v((k-1)/j))(\mathrm{d}\tau).\end{aligned}$$

By taking the limit $j \to \infty$, we get

$$\begin{aligned}&\lim_j \int_0^t f(x(\lambda), u_j(\lambda))\mathrm{d}\lambda \\ &= \lim_j \sum_{k=1}^{jt} \frac{1}{j} \int_U f(x((k-1)/j), \tau)\mu(v((k-1)/j))(\mathrm{d}\tau) \\ &= \int_0^t \int_U f(x(\lambda), \tau)\mu(v(\lambda))(\mathrm{d}\tau)\mathrm{d}\lambda,\end{aligned}$$

which shows that (5) holds and Lemma 2.3 can be used to conclude the approximation to the solution of the relaxed systems (6).

Example 4.1: Let $U = [-10, 10]$ and consider $\mu(v(t))$ which is given by

$$\big(\mu(v(t))\big)(E) = \int_E v(t)\delta_{-5}(E) + (1 - v(t))\delta_5(E) \qquad \forall E \subset [-10, 10],\ v(t) \in [0, 1].$$

Following the construction of $u_{j,k}$ as before, we first compute γ_k which is given by

$$\gamma_k(w) = \begin{cases} 0 & w \in [-10, -5) \\ v(t_k) & w \in [-5, 5) \\ 1 & w \in [5, 10]. \end{cases}$$

The set $\Gamma_k = \{0, v(t_k), 1\}$ and it is straightforward to check that

$$u_{j,k}(t_k + \xi) = \begin{cases} -5 & 0 \le \xi < v(t_k)/j \\ 5 & v(t_k)/j \le \xi < 1/j. \end{cases}$$

$\diamondsuit$

It is worth to note that the implementation of relaxed input in the Example 4.1 has been used widely for implementing control signal using pulse width modulation signal where the width of the pulse is modified according to v. The following examples describe the approximation of relaxed input which are given by a uniform probability measure valued function and by a conditional Gaussian probability measure valued function.

Example 4.2: Let $U = [a, b]$. Let $\mu(v)$ be a uniform probability measure valued function defined on U and be given by

$$\big(\mu(v(t))\big)(E) = \frac{\big|E \cap [v(t), b]\big|}{b - v(t)} \quad \forall E \subset [a, b],$$

where $|\cdot|$ is the Lebesgue measure. In this case, $v(t) \in [a, b]$ defines the lower interval of the uniform probability measure valued function.

Using the same construction of $u_{j,k}$ as before,

$$\gamma_k(w) = \begin{cases} 0 & w \in [a, v(t_k)) \\ \frac{w - v(t_k)}{b - v(t_k)} & w \in [v(t_k), b]. \end{cases}$$

The set $\Gamma_k = [0, 1]$ and $u_{j,k}$ is given by

$$u_{j,k}(t_k + \xi) = v(t_k) + j\xi(b - v(t_k)) \quad \forall \xi \in [0, 1/j).$$

$\diamondsuit$

Example 4.3: Let $U = [a, b]$ and $\mu(v)$ be a conditional Gaussian probability measure valued function on $[a, b]$. Suppose that for every $E \subset [a, b]$, $(\mu(v))(E)$ is defined by

$$\mu(v(t))(E) = \frac{\int_E e^{-\frac{(x - v(t))^2}{2}}\mathrm{d}x}{\int_{[a,b]} e^{-\frac{(x - v(t))^2}{2}}\mathrm{d}x}.$$

Using the same construction of $u_{j,k}$ as before,

$$\gamma_k(w) = \frac{\mathrm{erf}\left(\frac{w - v(t_k)}{\sqrt{2}}\right) - \mathrm{erf}\left(\frac{a - v(t_k)}{\sqrt{2}}\right)}{\mathrm{erf}\left(\frac{b - v(t_k)}{\sqrt{2}}\right) - \mathrm{erf}\left(\frac{a - v(t_k)}{\sqrt{2}}\right)},$$

where erf is the Gauss error function. The set $\Gamma_k = [0, 1]$ and the function γ_k is invertible. Thus, $u_{j,k}$ is given by

$$u_{j,k}(t_k + \xi) = \gamma_k^{-1}(j\xi) \quad \forall \xi \in [0, 1/j).$$

$\diamondsuit$

Remark 4.4: In some cases, it may be desirable that the approximating u (i.e., the concatenation of $u_{\Delta,t}$) is a continuous signal. For example, if there are design requirements that restrict high frequency noises due to an implementation of discontinuous signals. As a result, the approximating u using the concatenation of $u_{\Delta,t}$ as in (15) can be undesirable since it gives result to discontinuous signal u. However, if γ is continuous for every $v(t)$ then we can design an approximating u which is continuous. For example, such an approximating input is given by

$$u_j(t) = \begin{cases} u_{j,1}(t) & t \in (0,1]\Delta \\ u_{j,2}(2\Delta - t) & t \in (1,2]\Delta \\ \vdots & \vdots \\ u_{j,k}(t) & t \in (k,k+1]\Delta \\ u_{j,(k+1)}((k+2)\Delta - t) & t \in (k+1,k+2]\Delta \\ \vdots & \vdots \end{cases} \tag{16}$$

$\diamondsuit$

Example 4.5: Let us consider a nonlinear system described by

$$\left.\begin{aligned} \dot{x}_1 &= \sin(u) \\ \dot{x}_2 &= \cos(u), \end{aligned}\right\} \tag{17}$$

where $x_1(t), x_2(t), u(t) \in \mathbb{R}$. Consider the following relaxed input $\mu(v)$ with $v = \left[\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right]$

$$\begin{aligned} \mu(v)(E) = \int_E & v_1 \chi_{[0,\pi]}(\tau) \\ & +(0.5 - v_1)\chi_{[\pi,3\pi/2]\cup[-\pi/2,0]}(\tau) \\ & +v_2 \chi_{[-\pi/2,\pi/2]}(\tau) + (0.5 - v_2)\chi_{[\pi/2,3\pi/2]}(\tau)\mathrm{d}\tau \end{aligned} \tag{18}$$

for all $E \subset \mathbb{R}$ where $\chi_{[a,b]}$ denotes the indicator function on the interval $[a,b]$ and $v_1, v_2 \in [0, 0.5]$. Note that the relaxed input μ defined above is constructed based on a uniform probability measure. Using μ, the relaxed system of (17) is given by

$$\left.\begin{aligned} \dot{x}_1 &= 4v_1 - 1 \\ \dot{x}_2 &= 4v_2 - 1. \end{aligned}\right\} \tag{19}$$

By setting $v_1 = 0.25 - 0.25\text{sat}(x_1)$ and $v_2 = 0.25 - 0.25\text{sat}(x_2)$, the closed-loop relaxed system is given by

$$\left.\begin{aligned} \dot{x}_1 &= -\text{sat}(x_1) \\ \dot{x}_2 &= -\text{sat}(x_2), \end{aligned}\right\} \tag{20}$$

which is globally asymptotically stable. Figure 1 shows the implementation of the relaxed input $\mu(v)$ by an ordinary input u as constructed before with different Δ. $\triangle$

Remark 4.6: The implementation of the relaxed input using an approximating input u_j by concatenating $\{u_{j,k}\}$ as given in (15) or (16) uses a fixed sampling time $\Delta = 1/j$. However, by construction, each $u_{j,k}$ is designed to approximate the relaxed input at any sampled time t_k with the corresponding inter-sampling time to the next sampled time t_{k+1}. The inter-sampling time $t_{k+1} - t_k =: \Delta_{j,k}$ does not have to be a fixed constant as long as it satisfies $\lim_j \Delta_{j,k} = 0$. Thus, we can use different $\Delta_{j,k}$ that may depend on the design requirement such as computational complexity, hardware bandwidth, communication constraint and stability. $\diamondsuit$

V. Conclusions

In this paper, I discuss the control design by relaxed input. It enables the transformation of non-affine nonlinear systems into another form that is amenable to stability analysis and controller design.

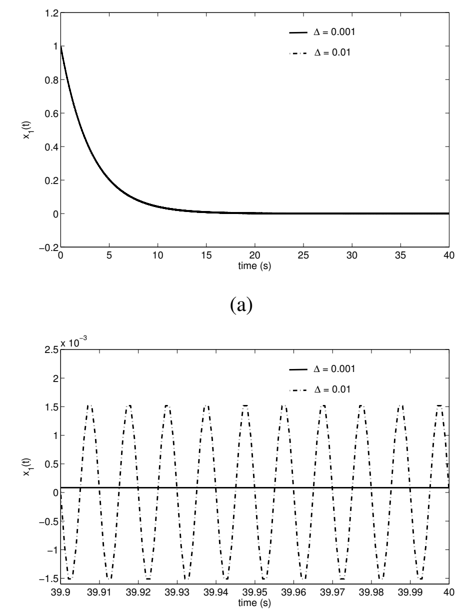

Fig. 1. (a). The plot of $x_1(t)$ of (17) using state-feedback relaxed control where the relaxed input is a uniform probability measure valued function and is implemented by an approximating input u_j (15) with $\Delta = 0.001$ and $\Delta = 0.01$; (b). The close-up of the state behaviour at 40s;

The implementation of relaxed input that is based on delta measure leads to the Pulse Width Modulation (PWM) control signal. By considering different type of relaxed inputs, we can generalize the PWM control signal and, in particular, it is possible to have an approximating input signal which is continuous.

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