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Published in:
Proceedings of the 50th IEEE Conference on Decision and Control and European Control Conference (CDC-ECC)

2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

A note on the deployment of kinematic agents by binary information

Claudio De Persis, Ming Cao, and Francesca Ceragioli

Abstract— **The problem of deploying continuous-time kinematic agents on a line is considered. To achieve the prescribed formation each agent uses a binary information, namely whether the distance of the agent from a neighbor is below or above the prescribed inter-agent distance. A simple control law which achieves and maintains the formation despite the coarse information available is designed.**

I. Introduction

Much effort has been devoted in recent years to the study of distributed coordination control algorithms. Among the many possible coordination tasks ([4], [21], [17], [28], [27], [18], [26]), two of the basic ones which capture several of the interesting features of formation control problems are consensus and deployment. Although the common assumption is that neighboring agents can exchange information continuously and that such information is perfectly known, in many cases this assumption is unrealistic due to limitations in the communication channel, in the sensing capabilities of the agents or in the hardware needed to implement the control laws. For this reason, researchers have started looking at the problem of achieving consensus in the presence of quantized information ([22], [16], [25], [6], [23] to name a few).

The papers above have focused on discrete-time quantized coordination algorithms, but in many cases of interest the agents' equations of motion are in continuous-time and thus works have started to appear which deal with consensus problems in continuous-time in the presence of quantization: the paper [15] has dealt with the consensus algorithm when the relative distance between neighbors is quantized whereas [8] has focused on consensus algorithms which use quantized absolute position measurements. Moreover, to deal with agents which have more complex dynamics, the paper [13] has investigated a passivity approach to coordinated control in the presence of quantized measurements.

One of the difficulties with quantized algorithms in continuous-time is that the use of quantized measurements makes the system discontinuous and a rigorous analysis of these systems must rely on a suitable notion of solution and tools from nonsmooth control theory. The paper [8] has shown that Krasowskii solutions are appropriate to study this class of problems as opposed to Carathéodory solutions which may not exist. Moreover, a new class of quantizers, namely hysteretic quantizers, has been introduced in [8] to deal with the undesired phenomenon of chattering. Prior to [15], [8], the papers [9], [24], [31] to name a few have already recognized the role of nonsmooth control theory in coordination algorithms (see [10] for a recent survey on the topic with some applications in coordination problems).

The consensus problem is not the only one which has been investigated under limited information. Building up on previous work on the deployment of discrete-time kinematic agents ([26], see also [18] for an early work), the authors of [5] have studied both rendez-vous and deployment under quantized position measurements. A related line of research has studied coordination problems in the presence of coarse sensors. The authors of [32] show rendez-vous of nonholonomic carts equipped with sensors which can only detect whether a neighboring cart enters or leaves its field of view. Another example is triangular formation maintenance using bearing-only measurements ([3]).

General classes of deployment problems such as disk-covering and sphere packing have been studied in [11] providing quite comprehensive solutions. These solutions rely on the definition of globally Lipschitz locational optimization functions and on the design of distributed gradient control laws which steer the dynamical systems to the critical points of the functions. The control laws require full information about the location of the agents' Voronoi neighbors.

The aim of this note is to show that for the particular problem of deployment of kinematic agents on a segment, deployment is achievable even when very coarse proximity sensors are used. As already mentioned, the deployment problem under quantized information has been also studied in [5]. Compared with the latter, a number of differences must be pointed out. First of all, in this paper the equations of the kinematic agents are in continuous-time (these agents can be obtained from the kinematic model of omni-directional mobile robots up to singularities – see [20] for a recent paper and references therein). Moreover, while in [5] the information is an absolute position measurement which is quantized via a uniform quantizer, in our paper the information delivered by the sensors to the agents concerns the relative position between two agents and is a binary information, namely it specifies whether the distance of the agent from a neighbor is below or above the prescribed inter-agent distance. The

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agents then aggregates these measurements by taking the average and use this aggregated information as the control action. Quite interestingly, despite the coarse information and the simplicity of the adopted control law, the agents achieve and maintain the prescribed formation. We stress that the formation is achieved as in the case of infinitely precise sensors, and this is in contrast with what is commonly found in consensus via quantized measurements, where only "practical" consensus can be guaranteed ([15], [8], [5]). Moreover it is achieved in finite time.

Early results on a related problem were presented in [14], [12]: compared with the latter, in this work we remove the formation-size-dependent gains which were present in [14], [12] and we propose a different analysis.

Section II introduces the problem formulation and Section III a convergence result with some auxiliary lemmas. Section IV studies the deployment problem. Numerical results are illustrated in Section V. Some final comments are given in the Conclusions (Section VI).

II. Problem formulation and model

We study the problem of deploying on a line a group of N agents with kinematic continuous-time model

$$\dot{x}_i = u_i \ , \quad i = 1, \ldots, N \ , \tag{1}$$

with $x_i, u_i \in \mathbb{R}$. The agents are connected through an undirected path graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, with $\mathcal{V} = \{1, 2, \ldots, N\}$ and $\mathcal{E} = \{(1,2), \ldots, (i, i+1), \ldots, (N-1, N)\}$. We recall that a path graph is a connected graph with two vertices of degree 1 and all the other vertices of degree 2. Moreover, we consider a fixed virtual leader and a fixed virtual follower whose positions we denote by $x_0 \in \mathbb{R}$ and $x_{N+1} \in \mathbb{R}$, with $x_0 < x_{N+1}$. Let $d_i > 0$ be the desired inter-agent distance between the agents i and $i+1$, with $i = 0, 1, \ldots, N$. The agents must be deployed within the segment of left end point x_0 and right end point x_{N+1} at the prescribed distance from each other and must preserve the order in such a way that each agent of position x_i precedes the agent of position x_{i-1} and follows the agent of position x_{i+1}. Consistently, we have $x_{N+1} = x_0 + d_0 + \ldots + d_N$.

In addition to deployment and order preservation, the control laws we are interested in use very coarse information. Namely, we consider the scenario in which the agents are endowed with sensors which are capable to detect whether the distance from a neighbor is greater or less than the desired distance and set the control action accordingly. The proposed control law takes the following form:

$$\begin{aligned}
u_1 &= -\kappa \mathrm{sgn}(x_1 - x_0 - d_0) + \kappa \mathrm{sgn}(x_2 - x_1 - d_1) \\
u_i &= -\kappa \mathrm{sgn}(x_i - x_{i-1} - d_{i-1}) + \kappa \mathrm{sgn}(x_{i+1} - x_i - d_i) \\
&\qquad\qquad i = 2, \ldots, N-1 \\
u_N &= -\kappa \mathrm{sgn}(x_N - x_{N-1} - d_{N-1}) + \\
&\qquad\qquad \kappa \mathrm{sgn}(x_{N+1} - x_N - d_N),
\end{aligned} \tag{2}$$

where $\kappa > 0$ is a parameter and $\mathrm{sgn} : \mathbb{R} \to \{-1, +1\}$ is the sign function: $\mathrm{sgn}(y) = +1$ if $y \geq 0$ and $\mathrm{sgn}(y) = -1$ if $y < 0$. From (2) it is seen that each agent computes the average of the signs of the distances from the neighbors measured by the binary sensors and use it as the control law.

The resulting closed-loop system is:

$$\begin{aligned}
\dot{x}_1 &= -\kappa \mathrm{sgn}(x_1 - x_0 - d_0) + \kappa \mathrm{sgn}(x_2 - x_1 - d_1) \\
\dot{x}_i &= -\kappa \mathrm{sgn}(x_i - x_{i-1} - d_{i-1}) + \kappa \mathrm{sgn}(x_{i+1} - x_i - d_i) \\
&\qquad\qquad i = 2, \ldots, N-1 \\
\dot{x}_N &= -\kappa \mathrm{sgn}(x_N - x_{N-1} - d_{N-1}) + \\
&\qquad\qquad \kappa \mathrm{sgn}(x_{N+1} - x_N - d_N),
\end{aligned} \tag{3}$$

The system has the following interesting interpretation. If the distance between agents i and $i+1$ is larger than or equal to the desired distance d_i then $\mathrm{sgn}(x_{i+1} - x_i - d_i) = +1$ and this contributes a positive value to the velocity of the agent i. Otherwise $\mathrm{sgn}(x_{i+1} - x_i - d_i) = -1$, and this results in a negative term in the velocity of agent i. Analogously for the distance between agent i and $i-1$. As a consequence, if both agents $i-1$ and $i+1$ are too close to agent i (i.e. $0 < x_{i+1} - x_i < d_i$ and $0 < x_i - x_{i-1} < d_{i-1}$) or too far from agent i (i.e. $x_{i+1} - x_i > d_i$ and $x_i - x_{i-1} > d_{i-1}$) then the agent i does not move. In fact, the agent cannot move in any direction to get away from or get closer to either agent $i-1$ or agent $i+1$, and its only possible choice is to stay still. On the other hand if one of the agents, say $i-1$ is close, and the other one is far, then $\dot{x}_i = 2\kappa > 0$, i.e. the agent i moves away from agent $i-1$ and closer to agent $i+1$. If it is the agent $i+1$ to be close and the agent $i-1$ to be far, then $\dot{x}_i = -2\kappa$ and the agent i moves towards $i-1$ and away from $i+1$. The paper [11] has discussed analogous relations between the control laws for disk-covering and sphere-packing and behavior-based robotics rules.

Observe that the model also incorporates the case in which two agents are at a distance which is larger than the range of the sensors. In fact, if agent i is too far from agent $i+1$, so that the sensor on agent i cannot measure the distance $x_{i+1} - x_i$, then it can set this quantity to any quantity which is strictly greater than d_i. Hence, $\mathrm{sgn}(x_{i+1} - x_i - d_i) = +1$. Similarly, if agent $i-1$ is beyond the range of the sensor of agent i, then it can set $x_i - x_{i-1} > d_{i-1}$, with $\mathrm{sgn}(x_i - x_{i-1} - d_{i-1}) = +1$.

In the next section we analyze the closed-loop system introduced above. Before doing this, we first observe that since the system (3) is discontinuous its solutions are to be intended in a generalized sense. In this paper, we consider Krasowskii solutions to (3), namely solutions to the differential inclusion

$$\dot{x} \in \mathcal{K}(f(x)) \ , \tag{4}$$

where $f(x)$ denotes the right-hand side of (3), and where

$$\mathcal{K}(f(x)) = \bigcap_{\delta > 0} \overline{\mathrm{co}} \left(f(B(x, \delta)) \right) \ , \tag{5}$$

with $\overline{\mathrm{co}}$ the closed convex hull of a set, and $B(x, \delta)$ the ball centered at x and of radius δ. The reason for considering Krasowskii solutions rather than other notions of solutions such as Carathéodory is that, similarly to [8], one can show by simple examples that Carathéodory solutions may not exist. On the other hand, Krasowskii solutions can be proven to always exist. In fact, it can be shown that there exist solutions which can only slide along discontinuity surfaces. In the following we say that the problem of *deploying kinematic agents by binary information* is solvable if we can prove that for any pair of constant parameters x_0, x_{N+1}, any gain $\kappa > 0$ and any initial condition $\overline{x} \in \mathbb{R}^N$ such that $x_0 < \overline{x}_1 < \overline{x}_2 < \ldots < \overline{x}_N < x_{N+1}$, every Krasowskii solution $x(t)$ to (3) originating from $\overline{x}$ satisfies $x_0 < x_1(t) < x_2(t) < \ldots < x_N(t) < x_{N+1}$ for all $t \geq 0$ and converges in finite time to $x_* := (x_0+d_0, x_0+d_0+d_1, \ldots, x_0+d_0+\ldots+d_{N-1})$.

Basic notions of nonsmooth control theory Before ending the section, we recall a few basic notions from the theory of nonsmooth control systems which will be used throughout the paper (see [1], [7], [10] for more details). $x_0 \in \mathbb{R}^N$ is a Krasowskii equilibrium for (4) if the function $x(t) = x_0$ is a Krasowskii solution to (4) starting from the initial condition x_0, namely if $\mathbf{0} \in \mathcal{K}(f(x_0))$. A set $\mathcal{S}$ is weakly (strongly) invariant for (4) if for any initial condition $\overline{x} \in \mathcal{S}$ at least one (all the) Krasowskii solution $x(t)$ starting from $\overline{x}$ belongs (belong) to $\mathcal{S}$ for all t in the domain of definition of $x(t)$. Let $V : \mathbb{R}^N \to \mathbb{R}$ be a locally Lipschitz function. Then by Rademacher's theorem the gradient of V exists almost everywhere. Let R be the set of measure zero where $\nabla V(x)$ does not exist. Then the Clarke generalized gradient of V at x is the set $\partial V(x) = \mathrm{co}\{\lim_{i \to +\infty} \nabla V(x_i) : x_i \to x,\ x_i \notin S,\ x_i \notin R\}$ where S is any set of measure zero in $\mathbb{R}^N$. We define the set-valued derivative of V at x with respect to (4) the set $\dot{\overline{V}}(x) = \{a \in \mathbb{R} : \exists v \in \mathcal{K}(f(x)) \text{ s.t. } a = p \cdot v,\ \forall p \in \partial V(x)\}$. The definition of regular functions used in the following nonsmooth LaSalle invariance principle can be found e.g. in [1, p. 363]:

Theorem 1 ([1, Th. 3] [9, Th. 2]) *Let $V : \mathbb{R}^N \to \mathbb{R}$ be a locally Lipschitz and regular function. Let $\overline{x} \in \mathcal{S}$, with $\mathcal{S}$ compact and strongly invariant for (4). Assume that for all $x \in \mathcal{S}$ either $\dot{\overline{V}}(x) = \emptyset$ or $\dot{\overline{V}}(x) \subseteq (-\infty, 0]$. Then any Krasowskii solution to (4) starting from $\overline{x}$ converges to the largest weakly invariant subset contained in $\mathcal{S} \cap \{x \in \mathbb{R}^N : \mathbf{0} \in \dot{\overline{V}}(x)\}$, with $\mathbf{0}$ the null vector in $\mathbb{R}^N$.*

III. A CONVERGENCE RESULT

In this section we study a slightly more general problem than the one formulated at the end of the previous section. Namely, we show that, given any choice of the constant parameters $x_0, x_{N+1}, d_0, \ldots, d_N$ such that $x_{N+1} = x_0 + d_0 + \ldots + d_N$ and $d_0, \ldots, d_N > 0$, any $\kappa > 0$, any Krasowskii solution to (3) converges in finite time to x_* from *any* initial condition in $\mathbb{R}^N$. In other words, we do not assume any initial order of the agents neither that their initial positions are within the segment $[x_0, x_{N+1}]$. Although the problem may not be always physically feasible (for instance, if agent i is not adjacent to the agent $i+1$ it is impossible for it to measure the distance $\mathrm{sgn}(x_{i+1} - x_i - d_i)$), the convergence result in Theorem 2 is used in the next section to prove that the problem of deploying kinematic agents by binary information is solvable.

The convergence result rests on a basic yet fundamental fact, namely that the differential inclusion (4) can be written as a gradient differential inclusion. More precisely, let $V(x) = \sum_{j=0}^{N} |x_{j+1} - x_j - d_j|$. Thanks to [29], Theorem 1, first statement, and [19], Lemma 2.8, one has $\mathcal{K}(f(x)) = -\kappa \partial V(x)$, with $\partial V(x)$ the Clarke generalized gradient. Moreover the unique Krasowskii equilibrium of (3) is x_*, as proved in the following lemma:

Lemma 1 $\mathbf{0} \in \mathcal{K}(f(x))$ *if and only if $x = x_*$.*

Proof: Clearly x_* is a minimum for the function V then $\mathbf{0} \in \partial V(x_*)$ ([11], Proposition 2.3) and $\mathbf{0} \in \mathcal{K}(f(x_*))$.

On the other hand, if $\mathbf{0} \in \mathcal{K}(f(x))$, then $\mathbf{0} \in \partial V(x)$ and since $V(x)$ is convex then x is a minimum for V (see [30], Theorem 10.1). Since x_* is the unique minimum of V one gets $x = x_*$. □

We are now able to prove the desired convergence result. The result is fundamentally the convergence of a gradient system (see Proposition 2.9 in [11]), but we report its proof here for the sake of completeness:

Theorem 2 *Any Krasowskii solution to (3) converges to x_* in finite time.*

Proof: $x(\cdot) : \mathbb{R}_+ \to \mathbb{R}^N$ is a Krasowskii solution to (3) if it is absolutely continuous and satisfies $\dot{x} \in \mathcal{K}(f(x))$ for a.e. $t \geq 0$, with $f(x)$ the vector field on the right-hand side of (3). Let $V(x) = \displaystyle\sum_{j=0}^{N} |x_{j+1} - x_j - d_j|$. As recalled above $\mathcal{K}(f(x)) = -\kappa \partial V(x)$. $V(x)$ is a convex function and as such it is regular (see e.g. [1, p. 364]).
Consider the set-valued derivative of V, $\dot{\overline{V}}(x) = \{a \in \mathbb{R} : \exists v \in \mathcal{K}(f(x)) \text{ s.t. } a = p \cdot v,\ \forall p \in \partial V(x)\}$. For points x where $f(x)$ is continuous, $\dot{\overline{V}}(x) = \{-\kappa |\nabla V(x)|^2\}$, with

$$|\nabla V(x)|^2 = \sum_{j=1}^{N} |\mathrm{sgn}(x_j - x_{j-1} - d_{j-1}) - \mathrm{sgn}(x_{j+1} - x_j - d_j)|^2$$

Clearly, $-|\nabla V(x)|^2 < 0$ because otherwise $\mathrm{sgn}(x_j - x_{j-1} - d_{j-1}) = \mathrm{sgn}(x_{j+1} - x_j - d_j)$ for all $j \in \{1, \ldots, N\}$, and this would cause x to violate the condition $x_{N+1} = x_0 + d_0 + \ldots + d_N$.
Suppose now that x is a state where $f(x)$ is discontinuous. Let $\dot{\overline{V}}(x) \neq \emptyset$ and $a \in \dot{\overline{V}}(x)$. Then there must exist $v \in \mathcal{K}(f(x))$ such that $a = p \cdot v$ for all $p \in \partial V(x)$. Since $\mathcal{K}(f(x)) = -\kappa \partial V(x)$, $a = p \cdot v$ must be true also for $p = -\frac{1}{\kappa} v$, which implies that $a = -\frac{1}{\kappa}|v|^2$. Hence, if $\dot{\overline{V}}(x) \neq \emptyset$, then $\dot{\overline{V}}(x) = \{a \in \mathbb{R} : \exists v \in \mathcal{K}(f(x)) \text{ s.t. } a = -\frac{1}{\kappa}|v|^2\}$, which implies $\dot{\overline{V}}(x) \subseteq (-\infty, 0]$. Since $\dot{V}(x(t))$ exists for almost every t and $\dot{V}(x(t)) \in \dot{\overline{V}}(x(t))$ for almost every t ([7], [9]), $V(x(t))$ can not increase. The definition of $V(x)$ implies that there exists a strongly invariant set $\mathcal{S} \subset \mathbb{R}^N$ for (3) which includes both $x(0)$ and x_*. Then, the nonsmooth LaSalle invariance principle ([1], [9]) implies that any Krasowskii solution must converge to the largest

weakly invariant set contained in the intersection of $\mathcal{S}$ and the set of points x such that $\mathbf{0} \in \dot{\overline{V}}(x)$. We have proven earlier that if $\mathbf{0} \in \mathcal{K}(f(x))$ then $x = x_*$. This implies that for $x \neq x_*$ either $\dot{\overline{V}}(x) = \emptyset$ or $\mathbf{0} \notin \dot{\overline{V}}(x)$. Hence, the intersection reduces to x_* and since x_* is trivially a weakly invariant point (Lemma 1), we conclude that any Krasowskii solution must converge to x_*.

Thanks to Proposition 4 in [9], in order to prove finite time convergence to x_*, it is enough to prove that there exists $\epsilon > 0$ such that $\max \dot{\overline{V}}(x) \leq -\epsilon$ for all $x \neq x_*$, i.e. there exists $\epsilon > 0$ such that $|v|^2 \geq \frac{1}{\kappa}\epsilon$ for any $v \in \partial V(x)$, $x \neq x_*$. Let us first of all remark that, if V is differentiable at x, then $v = \nabla V(x)$ and $|v|^2 \geq 4$. Then we observe that $\nabla V(x)$ takes a finite number of values over $\mathbb{R}^N \backslash \{x_*\}$. As a consequence, also the set-valued map $\partial V(x)$ admits a finite nuber of set-values in $2^{\mathbb{R}^N}$ over $\mathbb{R}^N \backslash \{x_*\}$, that we denote by $\mathcal{V}_1, ..., \mathcal{V}_M$. By Lemma 1, $\mathbf{0} \in \partial V(x)$ if and only if $x = x_*$. Then, for all $v \in \mathcal{V}_i, i = 1, ..., M$, one has $|v| > 0$. On the other hand, for all $i = 1, ..., M$, the set $\mathcal{V}_i$ is compact and then there exists $\min\{|v|^2, v \in \mathcal{V}_i\} > 0$. By taking $\epsilon = \kappa \min\{\min\{|v|^2, v \in \mathcal{V}_i\}, i = 1, ..., M\}$, we get that $\max \dot{\overline{V}}(x) \leq -\epsilon$ for all $x \neq x_*$. ☐

IV. A SOLUTION TO THE DEPLOYMENT PROBLEM

The main result of the previous section states that all the solutions to (3) converge to the prescribed formation from *any* initial position of the agents. However, with no further specification, the control problem modeled by (3) becomes unrealistic. In a deployment problem the agents' positions satisfy suitable conditions. First of all, the agents are initially within the segment where they must deploy. Moreover, without loss of generality we can assume that $x_0 < x_1(0) < x_2(0) < \ldots < x_N(0) < x_{N+1}$, where the inequalities are strict to take into account the physical dimensions of the agents. Then we can say that a solution $x(\cdot)$ to (3) is feasible if the order remains preserved for any time, i.e. $x_0 < x_1(t) < x_2(t) < \ldots < x_N(t) < x_{N+1}$ for all $t \geq 0$. In what follows we show that any Krasowskii solution to (3) is feasible. To this purpose, it is convenient to add the trivial components $\dot{x}_0 = 0$ and $\dot{x}_{N+1} = 0$ to the system (3) and rewrite the entire system in the new coordinates:

$$z_0 = x_0 \,, \quad z_i = x_i - x_{i-1}, \; i = 1, 2, \ldots, N+1 \,, \tag{6}$$

as:

$$\begin{array}{rcl} \dot{z}_0 & = & 0 \\ \dot{z}_1 & = & -\kappa\mathrm{sgn}(z_1 - d_0) + \kappa\mathrm{sgn}(z_2 - d_1) \\ \dot{z}_i & = & \kappa\mathrm{sgn}(z_{i-1} - d_{i-2}) - 2\kappa\mathrm{sgn}(z_i - d_{i-1}) + \\ & & \qquad \kappa\mathrm{sgn}(z_{i+1} - d_i) \quad i = 2, \ldots, N \\ \dot{z}_{N+1} & = & \kappa\mathrm{sgn}(z_N - d_{N-1}) - \kappa\mathrm{sgn}(z_{N+1} - d_N). \end{array} \tag{7}$$

Before going on, let us first denote the change of coordinates (6) as $z = Sx$, where by a slight abuse of notation we set $x = (x_0 \; x_1 \ldots x_N \; x_{N+1})^T$, $\dot{x} = f(x)$, and the system (7) as $\dot{z} = g(z)$. Then, we would like to make sure that the analysis of the original system (3) (with $\dot{x}_0 = 0$ and $\dot{x}_{N+1} = 0$) can actually be reduced to the study of (7). To this purpose we state the following simple fact:

Lemma 2 *For any Krasowskii solution $x(t)$ to (3), there exists a Krasowskii solution $z(t)$ to (7) such that $z(t) = Sx(t)$ for all $t \geq 0$.*

In other words for any solution $x(t)$ to (3) there is a solution $z(t)$ to (7) which is uniquely determined by $x(t)$. Hence statements about the solutions to (7) can be used to infer properties of the solutions to (3).

Proof: If we can prove that the function $\bar{z}(t) = Sx(t)$ is a Krasowskii solution to (7), then the thesis holds taking $z(t) = \bar{z}(t)$. Observe that $\bar{z}(t)$ satisfies $\dot{\bar{z}}(t) = S\dot{x}(t) \in S\mathcal{K}(f(x(t))) = S\mathcal{K}(f(S^{-1}\bar{z}(t)))$. Since $Sf(S^{-1}z) = g(z)$, by [29], Theorem 1, property 5, $S\mathcal{K}(f(S^{-1}\bar{z}(t))) = \mathcal{K}(Sf(S^{-1}\bar{z}(t))) = \mathcal{K}(g(\bar{z}(t)))$ and this shows $\dot{\bar{z}}(t) \in \mathcal{K}(g(\bar{z}(t)))$, that is $\bar{z}(t)$ is a Krasowskii solution to (7). ☐

We are now ready to prove that if a Krasowskii solution z starts in the positive orthant $\mathbb{R}^N_+$ then it can never leave it:

Lemma 3 *Any Krasowskii solution to (7) with initial condition such that $z_i(0) > 0$ for all $i = 1, 2, \ldots, N+1$, satisfies $z_i(t) > 0$ for all $i = 1, 2, \ldots, N+1$ and for all $t \geq 0$.*

Proof: Suppose that the thesis is not true. Then there must exist an open interval of time (t_1, t_2) and an index i such that $0 < z_i(t) < d_{i-1}$ and $\dot{z}_i(t) < 0$ for all $t \in (t_1, t_2)$. In what follows we will exploit the property that if $\dot{z}(t) \in \mathcal{K}(g(z(t)))$ then $\dot{z}(t) \in \times_{i=0}^{N} \mathcal{K}(g_i(z(t)))$ ([29]), where the symbol $\times$ denotes the Cartesian product. Consider the case in which $i = 1$. Since $0 < z_1(t) < d_0$, then $g_1(z(t)) = \kappa + \kappa\,\mathrm{sgn}(z_2(t) - d_1)$ and $\mathcal{K}(g_1(z(t))) = \overline{\mathrm{co}}\{0, 2\kappa\} = \{v_1 \in \mathbb{R} : v_1 = 2\lambda_1\kappa, \; \lambda_1 \in [0,1]\}$. Since any Krasowskii solution $z(t)$ is such that $\dot{z}_1(t) \in \mathcal{K}(g_1(z(t))) \subseteq [0, +\infty)$, this contradicts $\dot{z}_1(t) < 0$. Similarly, if $i = 2, \ldots, N$, since $0 < z_i(t) < d_{i-1}$, then

$$\begin{array}{rl} \dot{z}_i(t) \in \mathcal{K}(g_i(z(t))) = & \!\!\overline{\mathrm{co}}\{0, 4\kappa\} \\ = & \!\!\{v_i \in \mathbb{R} : v_i = 4\kappa\lambda_i, \; \lambda_i \in [0,1]\} \subseteq [0, +\infty) \,, \end{array}$$

which is again a contradiction. Finally, if $i = N+1$, then

$$\begin{array}{rl} \dot{z}_{N+1}(t) \in \mathcal{K}(g_{N+1}(z(t))) = & \!\!\overline{\mathrm{co}}\{0, 2\kappa\} \\ = & \!\!\{v_{N+1} \in \mathbb{R} : v_{N+1} = 2\lambda_{N+1}\kappa, \; \lambda_{N+1} \in [0,1]\} \\ \subseteq & \!\![0, +\infty) \,, \end{array}$$

and the conclusion is the same as before. This concludes the proof. ☐

It is now straightforward to prove the following:

Theorem 3 *The problem of deploying kinematic agents by binary information is solvable.*

Proof: Let $\overline{x} \in \mathbb{R}^N$ be such that $x_0 < \overline{x}_1 < \overline{x}_2 < \ldots < \overline{x}_N < x_{N+1}$ and consider any Krasowskii solution $x(t)$ to (3) which starts from $\overline{x}$. Consider the function $z(t) = Sx(t)$ and observe that by Lemma 2 $z(t)$ is a Krasowskii solution to (7). By construction, $z_i(0) > 0$ for all $i = 1, 2, \ldots, N+1$ and therefore $z_i(t) > 0$ for all $i = 1, 2, \ldots, N+1$ and for all $t \geq 0$ by Lemma 3. The definition of $z(t)$ then implies that $x_0 < x_1(t) < x_2(t) < \ldots < x_N(t) < x_{N+1}$ for all $t \geq 0$, i.e. the solution $x(t)$ is feasible. Finally, for any initial condition $\overline{x} \in \mathbb{R}^N$, any Krasowskii solution to (3) converges to x_*, as proven by Theorem 2. In particular this is true for any Krasowskii solution to (3) which starts from $\overline{x} \in \mathbb{R}^N$ such that $x_0 < \overline{x}_1 < \overline{x}_2 < \ldots < \overline{x}_N < x_{N+1}$. This ends the proof. $\square$

V. Numerical results

We have run simulations of the system (3) in the case $x_0 = -x_{N+1} = -50$, $d_0 = d_N = \frac{x_{N+1}-x_0}{2N}$, $d_i = \frac{x_{N+1}-x_0}{N}$ for $i = 1, 2, \ldots, N-1$, which correspond to the configuration where the agents should deploy uniformly within the segment $[x_0, x_{N+1}]$. We have considered a formation with $N = 20$ agents and with gain $\kappa = 2$. In the first simulation (Fig. 1, top left), the agents start very close to each other and nearby a point chosen randomly in the segment. In the second simulation (Fig. 1, top right), the agents start from initial positions which are drawn from a uniform distribution. In the figures, the dotted lines represent the desired final positions of the agents. It is seen that the agents converge to the desired configuration in finite time. In both simulations, each agent tries to move towards its final position if no other agent prevents it to do so. Otherwise, it stays still, until the distance from the neighbor which lies between the agent and its final position becomes equal to the prescribed inter-agent (safety) distance. At this time, the agent starts moving while trying to keep the distance from its neighbors constant. During the motion, the agent can stop momentarily even though it has not reached its desired final position. This can happen if the distance from both its neighbors becomes too large (or too small). The switching between positive (or negative) and zero velocities gives rise to the curved lines seen in the graphs.
The system is then simulated with a gain $\kappa = 5$ and the result is illustrated in the bottom graphs of Fig. 1. It is seen that a larger gain yields a faster response of the system. In fact, a higher gain κ implies that the function V decreases towards its minimum with an increased speed, as it is deduced from the proof of Theorem 2.

VI. Conclusion

The problem of formation control under very coarse information is receiving increasing attention in the literature. In this paper we have presented a deployment control law for continuous-time kinematic agents which uses binary information. Despite the coarse information, the control law is able to achieve and keep the formation.
Another advantage of the use of binary information is that the sensed quantities can be transmitted via a digital channel. Consider for instance the control law for agent i. To implement u_i, the agent i needs the measurements $\mathrm{sgn}(x_i - x_{i-1} - d_{i-1})$, $\mathrm{sgn}(x_{i+1} - x_i - d_i)$. They can either be measured by sensors installed on agent i or measured by the agents $i-1$ or $i+1$ and then transmitted to agent i. Hence, a model like (3) allows to reduce the number of sensors installed on each single agent.
The possible occurrence of chattering in practical implementation due to the use of the binary information may be overcome with the introduction of hysteresis. A formal analysis was presented in [8] for the case of consensus with hysteretic quantizers. For a related deployment problem, the use of sign functions with hysteresis has been studied in [12]. A general setting to study deployment problems is available in [11] (see also [2] for deployment problems on a grid). The tools adopted there are the same as the ones used in this note. A natural question arises: how the results of [11] can be adapted to deal with the scenario in which only coarse information is available?

Acknowledgements The authors would like to thank Paolo Frasca for insightful comments on an earlier version of the paper.

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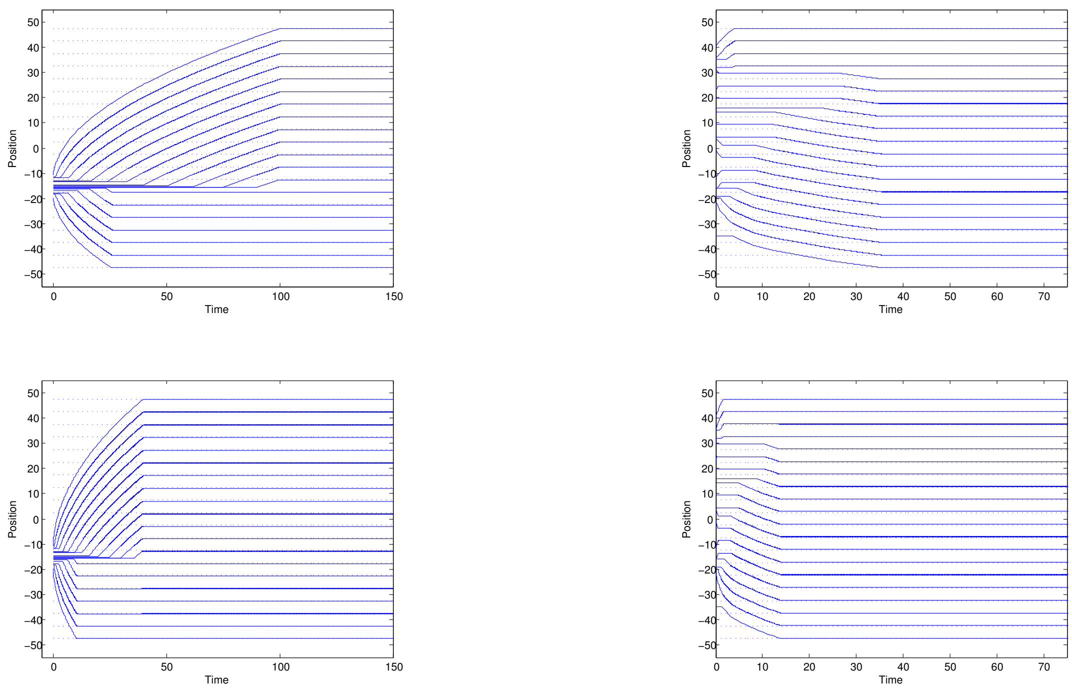

Fig. 1. The evolution of the position of 20 agents which are deploying within the segment $[-50, 50]$. In the top row, the gain κ is set equal to 2. The initial positions are chosen nearby a randomly picked point on the segment (left) or chosen within the segment according to a uniform distribution (right). The dotted lines represent the desired final positions for the agents. The graphs in the bottom row depict the outcome of the simulations with the same initial conditions but with an increased gain $\kappa = 5$.

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