Mode-Observability Conditions for Linear and Nonlinear Systems

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Abstract—This paper addresses the problem of mode-observability, namely the problem of determining whether or not system outputs obtained with different modes can be distinguished one from another. We introduce a notion of mode-observability for nonlinear systems and establish a number of related properties. For mode-observable systems we also provide tools for analysing their mode-observability degree. Examples are discussed to substantiate the analysis.

I. INTRODUCTION

Over the last decade, a lot of attention has been devoted to switching or multimode systems from both research and industrial areas, as they allow one to represent and investigate the properties of a large class of plants in numerous applications resulting from the interactions of continuous dynamics, discrete dynamics, and logic decisions [1]. Such systems are described by several dynamics around different operating points or trajectories, possible examples being faulty mode dynamics for systems subjects to failures [2], or multimode systems in servo positioning [3].

Motivated by the wide range of possible applications through the use of switching systems modeling, in the last decade considerable effort has been devoted to the analysis of such systems. Among the properties used to analyse switching systems, the so-called mode-observability plays an important role. By mode-observability one refers to the problem of determining whether or not system outputs obtained with different modes can be distinguished one from another. Mode-observability is therefore a natural question that arises in many situations of practical interest: fault detection schemes where different modes are introduced to model possible faults in the plant; design and analysis of switched systems, where different dynamical systems are connected through a switching signal specifying the current system at each instant of time. In this respect, there has been very recently some interest [4], [5] in finding control actions which can ensure, along with stability, mode-observability properties for switching systems.

In connection with mode-observability problems, while several results have been reported for linear systems [6]-[9], very little is known for nonlinear systems. Moreover, in the literature, most of the attention has been devoted to the only question of whether or not mode-observability

properties hold. In this paper, we introduce a notion of mode-observability for nonlinear systems and establish a number of related properties. While extending mode-observability conditions to a nonlinear setting, we also provide, for mode-observable systems, tools for analysing their mode-observability degree. It will be shown that such an approach can be particularly useful in contexts where the problem is that of determining the unknown current mode from noisy-corrupted observations.

The remainder of the paper is as follows. In Section II, we introduce a mode-observability notion for nonlinear systems and discuss basic concepts. In Section III, we establish a number of properties related to mode-observability and we provide tools for analysing the mode-observability degree of two dynamic systems. In Section IV, we discuss how the proposed results translate into a linear setting. In Section V, we discuss an application of the proposed results to a surveillance-like problem. Section VI ends the paper with conclusive remarks.

II. BASIC CONCEPTS

Consider a family of dynamic systems

\[ x_{t+1} = f_i(x_t), \]
\[ y_t = h_i(x_t) \]

where \( t \in \mathbb{Z}_+ \triangleq \{0, 1, \ldots\}; x_t \in \mathbb{R}^n \) is the state; \( y_t \in \mathbb{R}^p \) is the vector of measurements; the subscript \( i \) specifies the \( i \)-th subsystem of (1), which is assumed to belong to a finite index set \( \mathcal{M} \triangleq \{1, 2, \ldots, M\} \). In this paper, the \( i \)-th element of \( \mathcal{M} \) will be referred to as the \( i \)-th mode. The problem of mode-observability is that of determining whether or not system outputs obtained with different modes can be distinguished one from another.

In order to make this precise, given an \((N+1)\)-length observation window \([t - N, t]\) with \( t \geq N \) we denote by \( F_i(x_{t-N},N) \) the \( N+1 \)-output mapping

\[ F_i(x_{t-N},N) \triangleq \begin{bmatrix} h_i(x_{t-N}) \\ h_i \circ f_i(x_{t-N}) \\ \vdots \\ h_i \circ f_i \circ \cdots \circ f_i(x_{t-N}) \end{bmatrix} \]

where \( \circ \) denotes composition. Then \( y_{t-N,t} = F_i(x_{t-N},N) \). \(^1\) The following definitions can be introduced.

\(^1\)In general, given a generic sequence of vectors \( \{w_t\} \) and \( t_1, t_2 \in \mathbb{Z}_+ \), let \( w_{t_1,t_1+t_2} \triangleq [w'_{t_1}, \ldots, w'_{t_1+t_2}]' \).
**Definition 1**: Given a set $\mathcal{X} \subseteq \mathbb{R}^n$, two different modes $i, j$ are said to be $(N + 1)$-steps distinguishable on $\mathcal{X}$ if $F_i(x, N) \neq F_j(v, N)$ for any $x, v \in \mathcal{X}$.

**Definition 2**: The family of systems in (1) is said to be $(N + 1)$-steps mode-observable on $\mathcal{X}$ if any two different modes are $(N + 1)$-steps distinguishable on $\mathcal{X}$.

In words, two modes are said to be $(N + 1)$-steps distinguishable on $\mathcal{X}$ when, over an observation window composed by $N + 1$ measurements, they lead to different output data for all initial states belonging to $\mathcal{X}$. In the relevant literature for linear systems [6]-[9], the analysis is typically devoted to provide answers to the binary question of whether or not mode-observability holds.

The developments of this paper, while extending mode-observability conditions to a nonlinear setting, also aim at providing tools for characterising how the mode-observability properties of a family of systems vary over a certain set $\mathcal{X}$. To substantiate our discussion, consider the simple case where the family of systems in (1) is composed by linear systems,

\[
\begin{align*}
    x_{t+1} &= A_t x_t, \quad (2a) \\
    y_t &= C_t x_t \quad (2b)
\end{align*}
\]

where, for every $i \in \mathcal{M}$, $A_i$ and $C_i$ are matrices of suitable dimensions. According to Definition 1, two modes turn out to be $(N + 1)$-steps distinguishable on $\mathcal{X}$ if and only if

\[
[O_i(N) \quad -O_j(N)] \begin{bmatrix} x \\ v \end{bmatrix} \neq 0 \quad (3)
\]

for all $x, v \in \mathcal{X}$, where $O_i(N)$ denotes the $(N + 1)$-th order observability matrix of the $i$-th subsystem. It is immediate to see that if the joint observability matrix

\[
O_{i,j}(N) \triangleq [O_i(N) \quad -O_j(N)] \quad (4)
\]
is full rank, then modes $i$ and $j$ turns out to be distinguishable on every set $\mathcal{X}$ such that $0 \notin \mathcal{X}$. Moreover, the larger the distance of $x$ or $v$ from the origin, the more distinguishable the modes $i$ and $j$ will be. In fact, provided that $O_{i,j}(N)$ is full rank, we get

\[
\|O_{i,j}(N) \begin{bmatrix} x \\ v \end{bmatrix}\| \geq \sqrt{\lambda_{\text{min}}(O_{i,j}(N))} \| \begin{bmatrix} x \\ v \end{bmatrix}\| \quad (5)
\]

for all $x, v \in \mathcal{X}$, where $\lambda_{\text{min}}$ denotes minimum eigenvalue. These simple observations suggest that in general the distinguishability property has more structure to reveal than the one entailed by Definition 1. As we will see, providing a characterization of how the mode-observability properties of a family of dynamic systems vary over $\mathcal{X}$ can be useful not only for design and analysis of switched systems, but also in mode-identification contexts, where the aim is to reconstruct the current mode of the system from observations.

### III. Mode-Observability Conditions

In this section, we establish a number of properties related to mode-observability. To this end, it is first convenient to derive an equivalent condition for mode-observability which is better suited for analysis purposes.

Let

\[
\omega_{t+1} \triangleq \begin{bmatrix} x_{t+1} \\ v_{t+1} \end{bmatrix} = \begin{bmatrix} f_i(x_t) \\ f_j(v_t) \end{bmatrix}, \quad (6a)
\]

\[
\zeta_t = h_i(x_t) - h_j(v_t) \quad (6b)
\]
denote the composite system made up by the $i$-th and the $j$-th subsystem. Let

\[
F_{i,j}(x_{t-N}, N) \triangleq F_i(x_{t-N}, N) - F_j(v_{t-N}, N). \quad (7)
\]

Then, we have $\zeta_{t-N,t} = F_{i,j}(x_{t-N}, N)$. Finally, let

\[
\Omega \triangleq \left\{ \omega \in \mathbb{R}^{2n} : \omega = \begin{bmatrix} x \\ v \end{bmatrix}, x, v \in \mathcal{X} \right\}. \quad (8)
\]

The next proposition follows at once.

**Proposition 1**: For the family of systems in (1), two different modes $i, j$ are $(N + 1)$-steps distinguishable on $\mathcal{X}$ if and only if $F_{i,j}(\omega, N) \neq 0$ for any $\omega \in \Omega$.

As can be seen, Proposition 1 provides a simple and natural nonlinear counterpart of (3). Hence, following the arguments made after (4), a natural way to determine whether, and how, two dynamic systems are distinguishable over a certain set $\mathcal{X}$ is in terms of the distance of $\omega$ from the subset of $\mathcal{X}$ where $F_{i,j}(\omega, N)$ is zero, or, more generally from the subset of $\mathcal{X}$ where $F_{i,j}(\omega, N)$ attains its minimum. As shown below, under some mild regularity conditions, such a characterization is always possible. To this end, we avail of the following result which can be easily proved by resorting to standard topological arguments.

**Proposition 2**: Consider a compact set $\mathcal{Q}$ and assume that, for every $i \in \mathcal{M}$, $f_i$ and $h_i$ are continuous on $\mathcal{Q}$. Then, the subset $\mathcal{Q}_s \triangleq \text{arg min}_{\omega \in \mathcal{Q}} \|F_{i,j}(\omega, N)\|$ is closed.

Given a bounded set $\mathcal{X}$, let $\overline{\mathcal{X}}$ denote the closure of $\mathcal{X}$, and assume that for every $i \in \mathcal{M}$, the functions $f_i$ and $h_i$ are continuous on $\overline{\mathcal{X}}$. Let

\[
\Xi \triangleq \text{arg min}_{\omega \in \Omega} \left\{ \min_{i,j \in \mathcal{M}} \|F_{i,j}(\omega, N)\| \right\} \quad (9)
\]

where $\overline{\Omega}$ denotes the closure of $\Omega$. In order to provide a measure of the mode-observability degree, let us consider the point-to-set distance

\[
d(\omega, \Xi) \triangleq \min_{\xi \in \Xi} \|\omega - \xi\|
\]

which, based on Proposition 2, is always well-defined. Next result holds (the proof is omitted here for the sake of brevity).
Theorem 1: Let \( \mathcal{X} \) be a bounded set. Further assume that for every \( i \in \mathcal{M} \) the functions \( f_i \) and \( h_i \) are continuous on \( \mathcal{X} \). Then, the following statements are equivalent:

1) System (1) is \((N + 1)\)-steps mode-observable on \( \mathcal{X} \);

2) There exists a function \( \varphi \) continuous, strictly increasing, with \( \varphi(0) \geq 0 \), such that

\[
\| F_{i,j}(\omega, N) \| > \varphi(d(\omega, \Xi)) \tag{10}
\]

for all \( \omega \in \Omega \) and all \( i,j \in \mathcal{M} \).

In words, Theorem 1 states that for a pair of states \( x, v \in \mathcal{X} \), the value \( \varphi(d(\omega, \Xi)) \), with \( \omega \triangleq [x' v'] \), can be considered as a measure of the distinguishability of modes \( i \) and \( j \) with respect to \( x, v \in \mathcal{X} \).

A. Discussion

The main implication of Theorem 1 is that, under certain conditions, it is possible to characterise how the mode-observability properties of (1) vary over the observation set \( \mathcal{X} \).

As simple illustrative examples, consider a family of scalar systems as in (1) composed by two candidate modes \( i \) and \( j \). Let \( \mathcal{X} = \{ x | 0 < x < 1 \} \). As for the output equations consider the cases where

\[
h_i(x) = +0.25, \quad h_j(x) = -0.25 \tag{11}
\]

and

\[
h_i(x) = +\arctan(x), \quad h_j(x) = -\arctan(x) \tag{12}
\]

In both cases, system (1) turns out to be 1-step mode-observable, \( h_i \) and \( h_j \) always being of opposite sign. \(^2\) However, the mode-observability properties of (1) in these two cases are quite different. When output functions (11) are considered, the mode-observability properties of the system do not vary over \( \mathcal{X} \). In fact, we have \( \| F_{i,j}(\omega, 0) \| = 0.5 \) and \( \Xi = \mathcal{X} \). Moreover, \( d(\omega, \Xi) = 0 \) for all \( \omega \in \Omega \). Accordingly, (10) is satisfied by \( \varphi(d(\omega, \Xi)) = 0.5(1 - \epsilon) \) for any \( 0 < \epsilon < 1 \). On the contrary, the same conclusion does not hold when output functions (12) are considered, since the modes are less distinguishable as \( \omega \) approaches the origin (the function \( \phi \) going to zero). In particular, we have \( \Xi = \{0\} \) and \( d(\omega, \Xi) > 0 \) for all \( \omega \in \Omega \). Moreover, by some trigonometry,

\[
\| F_{i,j}(\omega, 0) \| = \arctan \left( \frac{x + v}{1 - x v} \right) > \arctan(x + v)
\]

since \( x v < 1 \). Since \( x + v > \sqrt{x^2 + v^2} = d(\omega, \Xi) \), we conclude that (10) is satisfied by \( \varphi(d(\omega, \Xi)) = \arctan(d(\omega, \Xi)) \).

This example shows that, while both Proposition 1 and Theorem 1 result in the same binary answer in terms of mode-observability, Theorem 1 provides much more insight of this property (see Fig. 1).

![Fig. 1. Behavior of the function \( \varphi(d(\omega, \Xi)) \) over \( \Omega \) in the first example with \( \epsilon = 10^{-4} \) (cyan) and second example (magenta)](image)

From a practical point of view, such an additional information can be useful in mode-identification contexts, where the aim is to reconstruct the current mode of the system from observations when they are corrupted by disturbances.

A possible application of the mode estimation will be to switching systems, i.e., systems of type (1) where the index \( i \) can vary over time in an unpredictable way. This makes the use of a fixed-dimension observation windows suitable and suggests a moving-horizon mode estimation strategy [10]. To be more precise, consider the situation where the vector \( y_{t - N, t} \) composed by \( N \) measurements has been generated by an unknown mode \( \sigma \in \mathcal{M} \). Accordingly, the mode-identification problem can be approached by asking whether or not \( \sigma \) can be identified from \( y_{t - N, t} \), As for the present case, assume that \( y_{t - N, t} \) has been generated according to the equations

\[
x_{t+1} = f_\sigma(x_t), \tag{13a}
\]

\[
y_t = h_\sigma(x_t) + \eta_t \tag{13b}
\]

where \( \eta_t \in \mathbb{R}^p \) is a measurement noise vector. Accordingly, one can write

\[
y_{t - N, t} = F_\sigma(x_{t - N}, N) + \eta_{t - N, t}. \tag{14}
\]

From the foregoing, by resorting to a least-square approach, one can consider the following moving-horizon strategy for estimating at time \( t \) the mode \( \sigma \) for system (13) from the most recent \( N + 1 \) measurements:

\[
\hat{\sigma}_t \in \arg \min_{\sigma \in \mathcal{M}} \left\{ \min_{x} \| y_{t - N, t} - F_i(x, N) \|^2 \right\}. \tag{15}
\]

\(^2\)Note that as we are considering 1-step mode-observability, the functions \( f_i \) and \( f_j \) play no role.
Whenever $\mathcal{X}$ is compact, minimization over $\mathcal{X}$ for the (correct) index $\sigma$ yields
\[
\min_{x \in \mathcal{X}} \| y_{t-N,t} - F_{\sigma}(x, N) \| \leq \| \eta_{t-N,t} \|.
\]
Conversely, by exploiting (14), for any other candidate mode one obtains
\[
\| F_{\sigma}(x_{t-N}, N) - F_{\delta}(v, N) \|^2 \leq 2 \| y_{t-N,t} - F_{\delta}(v, N) \|^2
\]
which yields
\[
\| y_{t-N,t} - F_{\delta}(v, N) \|^2 \geq \frac{1}{2} \| F_{\sigma}(x_{t-N}, N) - F_{\delta}(v, N) \|^2
\]
\[
- \| \eta_{t-N,t} \|^2
\]
\[
= \frac{1}{2} \| F_{\sigma,\delta}(\omega_{t-N,t}, N) \|^2 - \| \eta_{t-N,t} \|^2
\]
for all
\[
\omega_{t-N} \triangleq \begin{bmatrix} x_{t-N} \\ u \end{bmatrix} \text{ with } v \in \mathcal{X}.
\]
Hence, by exploiting (10), we conclude that a sufficient condition for reconstructing the system mode is that
\[
\min_{v \in \mathcal{X}} \varphi(d(\omega_{t-N}, \Xi)) > 2 \| \eta_{t-N,t} \|.
\]
(16)

The latter inequality basically says that the larger is the distance of $\omega_{t-N}$ from $\Xi$ (because of $x_{t-N}$), the smaller the effect of noises on the mode reconstruction.

More generally, Theorem 1 can be used to determine whether or not the mode-observability properties of (1) are independent of some noise level or, in the negative, how they vary over $\mathcal{X}$.

In connection with the previous example, by exploiting (16), we have that in the case of output functions (11) $\sigma$ can be identified provided that the disturbance upper bound $\overline{\eta}$ is such that
\[
\overline{\eta} < \frac{(1 - \epsilon)}{4},
\]
which is consistent with the fact that any disturbance level below 0.25 does not destroy the mode-observability properties of (1). As for output functions in (12) we have that $\sigma$ can be identified provided that
\[
\overline{\eta} < \frac{1}{2} \arctan(x_t)
\]
In contrast with the previous case, here, there is no positive $\overline{\eta}$ ensuring that mode-identification occurs over $\mathcal{X}$. Obviously, in this simple cases, one could have been arrived at the same conclusions by direct inspection.

IV. LINEAR CASE AND FINITE SENSITIVITY PROPERTIES

In this section we discuss how some of the conditions given above can be translated in a linear setting. As we will see, this is also convenient in order to discuss the concept of mode-observability with finite sensitivity.

Consider the linear counterpart of (6),
\[
\omega_{t+1} = \begin{bmatrix} A_i & 0 \\ 0 & A_j \end{bmatrix} \omega_t =: A_{i,j} \omega_t,
\]
\[
\zeta_t = \begin{bmatrix} C_i & -C_j \end{bmatrix} \omega_t =: C_{i,j} \omega_t.
\]
In this case, the conditions of Proposition 1 amount to requiring that $F_{i,j}(\omega, N) = \mathcal{O}_{i,j}(N) \omega \neq 0$, where $\mathcal{O}_{i,j}$ is as in (4). Thus, next proposition follows at once.

Proposition 3: For the family of systems in (2), two different modes $i, j$ are $(N + 1)$-steps distinguishable on $\mathcal{X}$ if and only if
\[
\Omega \cap \ker \{ \mathcal{O}_{i,j}(N) \} = \emptyset.
\]
(18)

The result is in agreement with previous results on mode-observability for linear systems, where two different modes are said to be distinguishable if their outputs are different for any nonzero initial conditions. Indeed, in the standard literature on the topic, the set $\mathcal{X}$ is chosen equal to $\mathbb{R}^n \setminus \{0\}$ so that (18) reduces to the requirement that $\mathcal{O}_{i,j}(N)$ is full-rank [6]. Moreover, since in the linear case the domain of definition is $\mathbb{R}^n$, it is possible to obtain a more simple counterpart of Theorem 1.

To see this, consider the observability gramian of (17)
\[
G_{i,j}(N) \triangleq (\mathcal{O}_{i,j}(N))' \mathcal{O}_{i,j}(N),
\]
and assume that $G_{i,j}(N)$ is not full-rank, i.e. rank($G_{i,j}$) = $r < 2n$ (the other case being obvious). By standard results of linear systems theory, the observability gramian can be decomposed as $G_{i,j}(N) = UTU'$, where
\[
T \triangleq \begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda \end{bmatrix}
\]
with $\Lambda = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_r\}$ and $U$ orthogonal. Here, for notational simplicity, we omitted the dependence of $U, T$ and $\Lambda$ on $i,j$ and $N$. Therefore, by letting $\hat{C}_{i,j} \triangleq C_{i,j}U$ and $\hat{A}_{i,j} \triangleq U^{-1}A_{i,j}U$, and recalling that $U' = U^{-1}$ since $U$ is orthogonal, we get
\[
T = U' G_{i,j}(N) U = (\hat{\mathcal{O}}_{i,j}(N))' \hat{\mathcal{O}}_{i,j}(N)
\]
where $\hat{\mathcal{O}}_{i,j}(N)$ stands for the $(N + 1)$-order observability matrix of $(\hat{C}_{i,j}, \hat{A}_{i,j})$ which represents a canonical observability decomposition of (17). In particular,
\[
U = \begin{bmatrix} H & L \end{bmatrix}
\]
where $L$ is a basis for $\ker\{\mathcal{O}_{i,j}(N)\}$, and $H$ a completion in $\mathbb{R}^{2n}$. Since $U$ is orthogonal we have that $L'L = I$ and the orthogonal projection $\Pi$ on the subspace orthogonal to $L$ is given by
\[
\Pi \triangleq I - LL' = HH'.
\]
Hence,
\[
\|O_{i,j}(N)\omega\|^2 = \omega'(O_{i,j}(N))'O_{i,j}(N)\omega
= \omega'U^TU'\omega = \omega'HH'\omega
\geq \lambda_r\|\Pi\omega\|^2
\]
from which we conclude that
\[
\|O_{i,j}(N)\omega\| \geq \sqrt{\rho} d(\omega, \ker\{O_{i,j}(N)\}).
\] (19)

As (19) indicates, in the linear case, whenever two modes are distinguishable, they are distinguishable in a very special form. In fact, comparing (19) with (10) one can observe that in the linear case one can always replace the function \( \varphi \) with a norm function. In the nonlinear case, a property of this type is usually known as finite sensitivity [11], [12], and amounts to requiring that there exists a positive real \( \delta \) such that
\[
\varphi(d(\omega, \Xi)) \geq \delta d(\omega, \Xi).
\] (20)

In connection with the results of Theorem 1, two possible cases arise: if \( \varphi(0) > 0 \) it is immediate to see that (20) is always satisfied, e.g., by
\[
\delta \triangleq \gamma^{-1}_0 \varphi(0),
\]
where
\[
\gamma_0 \triangleq \max_{\omega \in \overline{\Omega}} d(\omega, \Xi).
\]

If instead \( \varphi(0) = 0 \) (\( \varphi \) is a \( K \)-class function), then \( F_{i,j}(\omega, N) \) attains a minimum on \( \overline{\Omega} \). Since the set \( \overline{\Omega} \) is compact, condition (20) turns out to be equivalent to
\[
\lim_{z \to 0^+} \varphi(z) / z > 0.
\]
Sufficient conditions for the latter inequality to hold are discussed in [12]. While at first glance (20) seems to provide few additional information on the mode-observability problem, one has to observe that the fulfillment of condition (20) ensures that small variations in \( F_{i,j}(\omega, N), i, j \in \mathcal{M}, \) always correspond to small variations of \( d(\omega, \Xi) \). In connection with the developments of Section III-A, this ensures that small variations in the observation vector \( I_t = F_{t}(x, N), \sigma \in \mathcal{M}, \) always correspond to small variations of \( x. \) Such a requirement is quite typical in the nonlinear programming literature, where a nonlinear least-squares problem of the form
\[
\min_{x} \|I_t - F_t(x, N)\|
\] (21)
is said to be stable if the mapping from the observations vector \( I_t \) to the global minimum of the cost function is sufficiently smooth [13], [14].

V. SIMULATION RESULTS

In the present section, we show how the mode estimation method (15) performs on a numerical example where a switching system is considered. A point-mass object is considered free to move in a planar surveillance region with unknown linear and angular velocity. Its position is measured by three angular sensors as shown in Fig. 2. The same problem has been recently used as a benchmark for nonlinear state estimation algorithms [15], [16]. The system will be here defined as a three-modes switching autonomous system with noise corrupting the nonlinear measurement equations. The three modes correspond to (1) rectilinear movement, (2) clockwise and (3) anticlockwise turns.

The system has the form
\[
x_{t+1} = f_{\sigma_t}(x_t)
\] (22a)
\[
y_t = h_{\sigma_t}(x_t) + \eta_t
\] (22b)
where \( \sigma_t \in \{1, 2, 3\}. \) Let \( \omega_2 = -W \) and \( \omega_3 = W, 0 < W < \pi/2, \) be parameters representing the angular velocity of the point-mass object for modes 2 and 3, respectively. We have
\[
f_i(x_t) = \begin{bmatrix}
x_t^{(1)} + t_s x_t^{(2)} \\
x_t^{(2)} \\
x_t^{(3)} + t_s x_t^{(4)} \\
x_t^{(4)}
\end{bmatrix},
\]
for \( i = 2, 3. \) Here \( t_s \) is the sampling time, \( x_t^{(1)} \) and \( x_t^{(3)} \) represent the Cartesian coordinates on the plane, whereas \( x_t^{(2)} \) and \( x_t^{(4)} \) represent the corresponding velocities. Note that \( f_2 \) can be obtained by the continuous prolongation of \( f_2 \) and \( f_3 \) for \( \omega_t \rightarrow 0. \)

Three sensors are used to derive information about the position of the object. Each of these sensors provide information regarding the angular measurements of the object in the planar surface. The location of the sensors in the Cartesian space are denoted as \( (s_x^{(k)}, s_y^{(k)}), k = 1, 2, 3. \) Therefore, for all the three modes \( i = 1, 2, 3, \) the measurement equations of the system are:
\[
h_t^{(k)} = \text{atan2}\left((x_t^{(3)} - s_y^{(k)})/(x_t^{(1)} - s_x^{(k)}), 1\right), k = 1, 2, 3
\]
where \( \text{atan2} \) is the four-quadrant inverse tangent operation. By geometric inspection, one can see that the system results to be \( N + 1 \)-steps mode-observable for \( N + 1 \geq 3. \)

The moving-horizon strategy (15) is adopted, and an estimated mode \( \hat{\sigma}_t \) is obtained at each iteration by solving numerically a minimization problem using the standard least squares optimization function in MATLAB.

The following parameters were used in the simulations. A square with a side length of 900 m was chosen as surveillance region and three angular sensors were located at

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3The \( n \)-th component of a generic vector \( v \) is denoted as \( v^{(n)}. \)
positions \((s^{(1)}_x, s^{(1)}_y) = (0, 0), (s^{(2)}_x, s^{(2)}_y) = (1000, 0),\) and \((s^{(3)}_x, s^{(3)}_y) = (0, 1000)\). The initial position of the object \((x^{(1)}_0, x^{(2)}_0)\) is randomly generated according to a uniform distribution inside the surveillance region and the initial velocity \((x^{(2)}_0, x^{(4)}_0)\) was randomly chosen with direction toward the interior of the region and modulus according to a uniform distribution in the range [10,12]m/s. The sequence of modes \(\sigma_t\) was generated as a piecewise-constant quantity so that the rectilinear motion is randomly alternated with clockwise and anticlockwise turns with \(W = 10 \frac{\pi}{180}\) rad/s. The sampling time is set to \(t_s = 1\) s and the simulations’ length is set to \(T = 120\) s – trajectories for which the object exits the surveillance region before the final time are disregarded. The measurement noise vector \(\eta_t\) was randomly generated with zero-mean normal distributions with different values for the standard deviation: \(\{0,10^{-3},3.10^{-3},5.10^{-3},7.10^{-3},10^{-2}\}\). Simulations were performed for a window length \(N + 1\) equal to 3 and 5.

Though the switching times are supposed to be unknown to the mode estimation algorithm, in each trajectory switches are imposed every 20 s. This makes easier to see their effects on the mode estimation (which assumes constant mode in the observation window) and, conversely, on the capacity of the algorithm to correctly estimate the mode \(\sigma_t\) when it is constant.

For each level of the noise affecting the measurement equations, \(K = 100\) simulation runs were performed.

In Fig 3a, the noise-free case is considered and the percentage (considering all the simulation runs) of correct estimation of the modes for each time \(t\) is shown. As expected, the mode estimation has a 100% of success when the mode is constant in the observation window \([t - N, t]\), while errors in the mode estimate are suffered when the observation window includes non-constant modes (around switching times). One randomly chosen trajectory for this case is shown in Fig. 3b.

The percentage of correct estimation of the modes for each
The problem of mode-observability (i.e., the problem of determining whether or not system outputs obtained with different modes can be distinguished one from another) has been addressed for nonlinear systems. Motivated by some property related to linear systems, we have introduced a notion of mode-observability for the nonlinear case and established a number of related properties. In particular, for mode-observable systems we have also provided tools for analyzing their mode-observability degree.

VI. CONCLUSIONS

The problem of mode-observability (i.e., the problem of determining whether or not system outputs obtained with

TABLE I

<table>
<thead>
<tr>
<th>noises st. dev. =</th>
<th>0 0.001 0.003 0.005 0.007 0.01</th>
</tr>
</thead>
<tbody>
<tr>
<td>N+1=3</td>
<td>96.61 67.37 39.83 33.90 30.51 28.67</td>
</tr>
<tr>
<td>N+1=5</td>
<td>94.49 94.31 79.74 63.36 53.97 45.34</td>
</tr>
</tbody>
</table>

simulation run was computed and the respective boxplots are shown in Fig. 3c and Fig. 3d for N + 1 = 3 and N + 1 = 5, respectively. Note that in our scenario, also for the noise-free case one has a percentage of correct mode reconstruction smaller than 100%. This is due to the presence of switches as already remarked for Fig. 3a.

As expected, the performance of the method decays as disturbances increase. For the case N + 1 = 3, standard deviations of the components of the noises larger than 0.001 result in a quasi-random mode reconstruction. Though such a level of noise is not so high, one should note that with an observation window of length N + 1 = 3 (the minimum required for guaranteeing mode-observability for the system), even small disturbances could compromise the possibility of correctly determine the mode of the system. In this respect, a larger observation window should be preferred as it would benefit of a more strong filtering effect on noises. This is shown clearly while comparing Fig. 3d (where N + 1 = 5) with Fig. 3c (where N + 1 = 3). The values of the medians of the percentage of correct estimation of the modes are summarized in Table I. It is in order to remark that for N + 1 = 5 one has better performances but in the noise-free case. This is due, again, to the presence of switches. In fact, for a larger observation window, they involve in general longer transients before the mode is correctly estimated.

REFERENCES