Effort- and flow-constraint reduction methods for structure preserving model reduction of port-Hamiltonian systems

Rostyslav V. Polyuga a,∗, Arjan J. van der Schaft b,1

a ABN AMRO N.V. Bank, Gustav Mahlerlaan 10, 1082PP (PAC HQ2015), Amsterdam, The Netherlands
b Johann Bernoulli Institute for Mathematics and Computer Science, University of Groningen, P.O.Box 407, 9700 AK Groningen, The Netherlands

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A B S T R A C T
The geometric formulation of general port-Hamiltonian systems is used in order to obtain two structure preserving reduction methods. The main idea is to construct a reduced-order Dirac structure corresponding to zero power flow in some of the energy-storage ports. This can be performed in two canonical ways, called the effort- and the flow-constraint methods. We show how the effort-constraint method can be regarded as a projection-based model reduction method. Both the effort- and flow-constraint reduction methods preserve the stability and passivity properties of the original system, as a consequence of preserving the port-Hamiltonian structure.

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1. Introduction

A standard way to model large-scale physical systems is network modeling. In this approach, the overall system is decomposed into (possibly many) interconnected subsystems. Network modeling has many advantages, such as reusability of subsystem models (libraries), flexibility (coarse models of subsystems may be replaced by more refined ones, leaving the rest of the system modeling untouched), hierarchical modeling, and control (by adding new subsystems as control components). In port-based network modeling (e.g., bond graph modeling), the overall system is decomposed into subsystems which are interconnected to each other through (vector) pairs of variables, whose product is the power exchanged among the subsystems. This approach is especially useful for the systematic modeling of multi-physics systems, where the subsystems belong to different physical domains (mechanical, electrical, hydraulic, etc.).

Since the beginning of the nineties of the previous century, it has been realized [1–5] that the mathematical models arising from port-based network modeling have an insightful geometric structure, which can be regarded as a generalization of the geometric formulation of analytical mechanics into its Hamiltonian form. These geometric dynamical system models that follow directly from port-based network modeling have been called port-Hamiltonian systems [1,3,6].

The state-space dimensions of mathematical models arising from network modeling easily become very large; think, for example, of electrical circuits, multi-body systems, or spatial discretization of distributed-parameter systems. Thus, there is an immediate need for model reduction methods. However, since we want the reduced order models again to be interconnectable to other (sub-) systems, we want to retain the port-Hamiltonian structure of the reduced order systems. Furthermore, we want to preserve structural properties, such as energy conservation, passivity and existence of conservation laws as implied by the port-Hamiltonian structure. Thus, the problem arises of structure preserving model reduction of port-Hamiltonian systems.

The geometric formulation of port-Hamiltonian systems motivates a model reduction approach for general port-Hamiltonian systems (possibly also including the algebraic constraints), which involves the construction of a reduced order Dirac structure, and subsequently the construction of a reduced Hamiltonian. This approach is directly based on port-based modeling by replacing interconnections with almost zero energy flow by zero-power constraints. In this paper we treat two canonical structure preserving model reduction methods, called the effort-constraint reduction method and the flow-constraint reduction method. We show how the effort-constraint method in suitable coordinates can be regarded as a projection-based model reduction method. We
suggest these coordinates for the effort-constraint method and balanced coordinates for both the effort- and flow-constraint methods as a possible choice of the coordinate system in order to obtain the reduced order models.

Structure preserving model reduction of port-Hamiltonian systems was also studied in [7–9]. The perturbation approach is considered in [10,11]. The use of the (rational) Krylov methods is addressed in [12–16], see also [17]. A recent overview of port-Hamiltonian model reduction methods can be found in [18]. Model reduction of nonlinear port-Hamiltonian systems is discussed in [19,20]. Model reduction of general nonlinear systems is gaining more attention in recent years, see for example [21–24] and references therein. For a general overview of model reduction methods, we refer the reader to [25–27].

Preliminary results of this work are presented in [28].

The paper is organized as follows. The general definition of port-Hamiltonian systems using the notion of a Dirac structure is given in Section 2. In Section 3, we explain the idea behind structure preserving model reduction based on zero-power constraints. Equational representations of the reduced order models are given in Section 4. These equational representations give rise to the reducedordermodels. In Section 5, we explain the idea behind structure methods, we refer the reader to [25–27].

of the resistive elements, while \( \mathcal{F}_p \times \mathcal{E}_p \) is the space of flow and effort variables corresponding to the external ports (or sources). The property \( \langle e \mid f \rangle = 0 \) for all \( (f,e) \in \mathcal{D} \) implies that the power supplied through the external port is distributed between the energy-storing port and the resistive port.

The vector of all the flow and effort variables of a port-Hamiltonian system

\[
\begin{align*}
&f_x \in \mathcal{F}_x, \\ &e_x \in \mathcal{E}_x, \\ &f_R \in \mathcal{F}_R, \\ &e_R \in \mathcal{E}_R
\end{align*}
\]

and \( f_p \in \mathcal{F}_p, \ e_p \in \mathcal{E}_p \) is required to be in the Dirac structure

\[
(f_x, e_x, f_R, e_R, f_p, e_p) \in \mathcal{D}.
\]

The constitutive relations for the energy-storing elements are defined as follows. Let the Hamiltonian \( H : \mathcal{X} \rightarrow \mathbb{R} \) denote the total energy of the energy-storing elements with state variables \( x = (x_1, x_2, \ldots, x_n)^T \); i.e., the total energy is given as \( H(x) \). In the sequel, we will take \( \mathcal{X} = \mathcal{F}_x \). Then the energy-storage constitutive relations are given as\(^3\)

\[
\dot{x} = -f_x, \quad e_x = \frac{\partial H}{\partial x}(x).
\]

This immediately implies the following energy balance

\[
\frac{d}{dt} H = -e^T f_x.
\]

that is, the increase in total energy \( H(x) \) is equal to the power \( -e^T f_x \) provided to the energy-storing elements.

The constitutive relations for the resistive elements are given as\(^5\)

\[
f_R = -\varphi(e_R),
\]

for some function \( \varphi \) satisfying

\[
e^T \varphi(e_R) > 0 \quad \text{for all } e_R \neq 0.
\]

Linear resistive elements are given as

\[
f_R = -R e_R, \quad R = R^T \geq 0.
\]

The interpretation is that power is always dissipated by the resistive elements.

\begin{figure}
\centering
\includegraphics[width=1.0\textwidth]{Fig1.png}
\caption{Geometric definition of a port-Hamiltonian system.}
\end{figure}

\footnotesize
\begin{enumerate}
\item This can be immediately generalized to taking \( \mathcal{X} \) to be an \( n \)-dimensional manifold with tangent space being \( \mathcal{F}_x \).
\item The vector \( \frac{\partial H}{\partial x}(x) \) of partial derivatives of \( H \) will throughout be denoted as a column vector.
\item This can be immediately generalized to a nonlinear resistive relation \( R(f_x, e_R) = 0 \) having the property that \( e_R f_x \leq 0 \) for all \( f_x, e_R \) satisfying this relation.
\end{enumerate}
It follows \cite{3,6} from the power-conservation property of Dirac structures, and (4) and (6) that

\[
\frac{d}{dt} H = -e_k^T \varphi(e_k) + e_f^T f_p \leq e_i^T f_p.
\]

(9)

thus showing passivity if the Hamiltonian \(H\) is bounded from below.

3. Structure preserving model reduction based on power conservation

Consider a general port-Hamiltonian system \((8)\), with state variables \(x\) and total stored energy \(H(x)\). Let us assume that we have been able to find (e.g., by some balancing technique) a splitting of the state-space variables \(x = (x_1^T, x_2^T)^T\), \(x_1 \in \mathbb{R}^r\), \(x_2 \in \mathbb{R}^{n-r}\), having the property that the \(x_2\) coordinates hardly contribute to the input–output behavior of the system, and thus could be omitted from the state-space description. It is seen easily that the usual truncation method for obtaining a reduced order model in the reduced state \(x_1\) in general does not preserve the port-Hamiltonian structure, like it does not preserve the passivity property, see e.g., \cite{28,18}. The same holds for the so-called singular perturbation reduction method, as was mentioned in \cite{18, Remark 2.14}; see also \cite{30,31}.

In which way is it possible to retain the port-Hamiltonian structure in model reduction? Recall that in the definition of a port-Hamiltonian system, the vector of flow and effort variables \((2)\) is required to be in the Dirac structure

\[
(\mathbf{f}_x^1, \mathbf{f}_e^1, \mathbf{e}_x^1, \mathbf{e}_e^1, \mathbf{f}_x^2, \mathbf{f}_e^2, \mathbf{e}_x^2, \mathbf{e}_e^2) \in \mathcal{D},
\]

while the flow and effort variables \(\mathbf{f}_x, \mathbf{e}_e\) are linked to the constitutive relations of the energy-storage by

\[
\dot{x}_1 = -\mathbf{f}_x^1, \quad \frac{\partial H}{\partial x_1}(x_1, x_2) = \mathbf{e}_e^1,
\]

\[
\dot{x}_2 = -\mathbf{f}_x^2, \quad \frac{\partial H}{\partial x_2}(x_1, x_2) = \mathbf{e}_e^2,
\]

which is shown in Fig. 2. This figure is a zoomed-in version of Fig. 1. The basic idea of structure preserving model reduction considered in this paper is to “cut” the interconnection

\[
\dot{x}_2 = -\mathbf{f}_x, \quad \frac{\partial H}{\partial x_2}(x_1, x_2) = \mathbf{e}_e,
\]

(11)

between the energy storage corresponding to \(x_2\) and the Dirac structure, in such a way that no energy is transferred. Hence the exchange of energy between the energy storage and the other system elements through the Dirac structure happens only via the port associated to \(x_1\), with \(x_1\) being the reduced order state vector.

The energy flow through the interconnection (11) is set equal to zero by making both power products

\[
\left(\frac{\partial H}{\partial x_2}\right)^T \mathbf{e}_e \quad \text{and} \quad (\mathbf{e}_e)^T \mathbf{f}_e
\]
equal to zero.

This can be done in the two following canonical ways (see also \cite{28})

(i): Set

\[
\frac{\partial H}{\partial x_2}(x_1, x_2) = 0, \quad \mathbf{e}_e = 0.
\]

(12)

The first equation imposes an algebraic constraint on the space variables \(x = (x_1^T, x_2^T)^T\). Under general conditions on the Hamiltonian \(H\), this constraint allows one to solve for \(x_2\) as a function of \(x_1\): \(x_2 = x_2(x_1)\), leading to a reduced Hamiltonian

\[
H_{\text{red}}^R(x_1) := H(x_1, x_2(x_1)).
\]

Furthermore, the second equation defines the reduced Dirac structure\(^6\)

\[
\mathcal{D}_{\text{red}}^R := \{(f_x^1, e_x^1, f_x^2, f_e^2, e_x^2, e_e^2, f_p, e_p) \mid \exists x_2 \text{ such that } (f_x^1, e_x^1, 0, e_x^2, f_e^2, f_p, e_p) \in \mathcal{D}\},
\]

leading to the reduced port-Hamiltonian system

\[
\left(\dot{x}_1, \frac{\partial H_{\text{red}}^R}{\partial x_1}(x_1), -\varphi(e_k), e_k, f_p, e_p\right) \in \mathcal{D}_{\text{red}}^R.
\]

We will call this reduction method the effort-constraint reduction method, since it constrains the efforts \(e_2\) and \(\frac{\partial H}{\partial x_2}\) to zero.

(ii): Set

\[
\dot{x}_2 = 0, \quad f_2 = 0.
\]

(13)

The first equation imposes the constraint

\[
x_2 = c.
\]

where the constant \(c\) can be taken to be zero, and thus defines the reduced Hamiltonian

\[
H_{\text{red}}^R(x_1) := H(x_1, c),
\]

(14)

while the second equation leads to the reduced Dirac structure

\[
\mathcal{D}_{\text{red}}^R := \{(f_x^1, e_x^1, f_x^2, f_e^2, e_x^2, e_e^1, f_p, e_p) \mid \exists x_2 \text{ such that } (f_x^1, e_x^1, 0, e_x^2, f_e^2, f_p, e_p) \in \mathcal{D}\},
\]

(15)

and the corresponding reduced port-Hamiltonian system

\[
\left(\dot{x}_1, \frac{\partial H_{\text{red}}^R}{\partial x_1}(x_1), -\varphi(e_k), e_k, f_p, e_p\right) \in \mathcal{D}_{\text{red}}^R.
\]

(16)

We call this approach the flow-constraint reduction method, because it constrains the flows \(-\dot{x}_2, f_2\).

An important open question, which will not be answered in this paper, is how to choose the coordinates \(x_1, x_2\) in such a way that the energy flow between the energy storage corresponding to \(x_2\) and the rest of the system through the Dirac structure is very small (negligible) at all time instants. Then the approximations (12) and (13) are at least from an energy transfer point of view well justified.

In Section 5.5 we will briefly discuss the closely related question of how to choose the coordinates in such a manner that the reduced model is close to the full order model from an input–output point of view.

\(^6\) \(\mathcal{D}_{\text{red}}^R\) is the composition of the full order Dirac structure \(\mathcal{D}\) with the Dirac structure on the space of flow and effort variables \(f_x, e_x\) defined by \(e_x = 0\). It is proven in \cite{12} that \(\mathcal{D}_{\text{red}}^R\) is indeed a Dirac structure.
4. Equational representations of the reduced order models

We will now provide explicit equational representations of the above two methods for structure preserving model reduction starting from the general representation by DAEs of the full order model (for details see [1–3,6,18]):

\[ F_0 \dot{x} = E_0 \frac{\partial H}{\partial x}(x) - F_0 \psi(e_R) + E_0 e_R + F_0 f_R + E_0 e_R. \]

(17)

where the matrices \( F_0, E_0, F_R, E_R, F_R \) and \( E_R \) satisfy [3,6]

\[ E_0 F_0^T + F_0 E_0^T + E_0 F_R^T + F_R E_0^T + E_0 F_R^T + F_R E_R^T = 0, \]

\[ \text{rank } [F_0, F_R, F_P, E_0, E_R, E_P] = n_x + n_R + n_R. \]

(18)

with \( n_x = \text{dim } \mathcal{X}, \ n_R = \text{dim } \mathcal{X}_R, \ n_R = \text{dim } \mathcal{X}_R. \)

Corresponding to the splitting of the state vector \( x \) into \( (x_1^T, x_2^T) \), \( x_1 \in \mathbb{R}^{n_x}, \ x_2 \in \mathbb{R}^{n_R} \), where \( r \) is the dimension chosen for the reduced order model, and the respective splitting of the flow and effort vectors \( f_x, e_x \), into \( f_x, e_x \), we write

\[ F_x = \begin{bmatrix} F_x^1 & F_x^2 \end{bmatrix}, \quad E_x = \begin{bmatrix} E_x^1 & E_x^2 \end{bmatrix}. \]

(19)

**Proposition 1.** The reduced Dirac structure \( \mathcal{D}_{\text{red}}^{\text{eff}} \) corresponding to the effort-constraint \( e_{\text{red}}^2 = 0 \) is given by the explicit equations

\[ L^{e_2} F_{x,1} + L^{e_2} E_{x,1} + L^{e_2} F_{x,2} + L^{e_2} E_{x,2} = 0, \]

(20)

where \( L^{e_2} \) is any matrix of maximal rank satisfying

\[ L^{e_2} F_x^2 = 0. \]

(21)

**Proof.** For the proof of the statement, we refer the reader to [32]. □

Similarly, the following result holds true.

**Proposition 2.** The reduced Dirac structure \( \mathcal{D}_{\text{red}}^{\text{eff}} \) corresponding to the flow-constraint \( f_{\text{red}}^2 = 0 \) is given by the equations

\[ L^{f_2} F_{x,1} + L^{f_2} E_{x,1} + L^{f_2} F_{x,2} + L^{f_2} E_{x,2} = 0, \]

(22)

where \( L^{f_2} \) is any matrix of maximal rank satisfying

\[ L^{f_2} E_x^2 = 0. \]

(23)

**Proof.** The proof can be found again in [32]. □

It follows that the reduced order model resulting from applying the effort-constraint method is given by

\[ L^{e_2} F_{x,1} \dot{x}_1 = L^{e_2} E_{x,1} \frac{\partial H_{\text{red}}}{\partial x_1}(x_1) - L^{e_2} F_{x,2} \phi(e_R) + L^{e_2} E_{x,2} e_R + L^{e_2} F_{x,2} f_R + L^{e_2} E_{x,2} e_R, \]

(24)

whereas the reduced order model resulting from applying the flow-constraint method is given by

\[ L^{f_2} F_{x,1} \dot{x}_1 = L^{f_2} E_{x,1} \frac{\partial H_{\text{red}}}{\partial x_1}(x_1) - L^{f_2} F_{x,2} \phi(e_R) + L^{f_2} E_{x,2} e_R + L^{f_2} F_{x,2} f_R + L^{f_2} E_{x,2} e_R. \]

(25)

The steps of model reduction leading to the reduced order models (24), (25) are depicted in Fig. 3. First, we consider a full order port-Hamiltonian system with the corresponding full order Dirac structure. Second, we reduce the full order Dirac structure to obtain the reduced order Dirac structure. Finally, given the reduced order Dirac structure, we obtain the reduced order system. At the same time, we are approximating the full order Hamiltonian of the full order model in order to obtain the reduced order Hamiltonian of the reduced order model. Note that the reduced order models obtained in this way are port-Hamiltonian by construction.

Fig. 3. Steps of model reduction of a full order port-Hamiltonian system (PHS) with the Hamiltonian \( H \) and the Dirac structure \( \mathcal{D} \).

5. Reduced models for linear input-state-output port-Hamiltonian systems

In this section we specialize the results of the previous section to the case of linear input-state-output port-Hamiltonian systems [3,6]

\[ \begin{align*}
    \dot{x} &= (J - R)Qx + Gu, \\
    y &= G^T Qx.
\end{align*} \]

(26)

The model (26) is obtained after the termination of the resistive port. In order to use the Dirac structure representation (17) of this model, we rewrite (26) in the form

\[ \begin{align*}
    \dot{x} &= fQx + G_R u, \\
    y &= G^T Qx, \\
    e_R &= G_R^T Qx, \\
    f_R &= -\bar{R} e_R.
\end{align*} \]

(27)

where the matrix \( \bar{R} \) is such that

\[ G_R \bar{R} G_R^T = R. \]

(28)

Splitting of the state vector into \( x = [x_1^T, x_2^T] \), \( x_1 \in \mathbb{R}^{n_x}, \ x_2 \in \mathbb{R}^{n_R} \), for \( r \) being the dimension of the reduced order model, then leads to the following partitioned system description

\[ \begin{bmatrix}
    \dot{x}_1 \\
    \dot{x}_2
\end{bmatrix} =
\begin{bmatrix}
    J_{11} & J_{12} \\
    J_{21} & J_{22}
\end{bmatrix}
\begin{bmatrix}
    Q_{11} & Q_{12} \\
    Q_{21} & Q_{22}
\end{bmatrix}
\begin{bmatrix}
    x_1 \\
    x_2
\end{bmatrix}
+ \begin{bmatrix}
    G_{R_1} \\
    G_{R_2}
\end{bmatrix}
\begin{bmatrix}
    u \\
    y
\end{bmatrix},
\]

(29)

\[ e_R = \begin{bmatrix}
    G_{R_1}^T & G_{R_2}^T
\end{bmatrix}
\begin{bmatrix}
    Q_{11} & Q_{12} \\
    Q_{21} & Q_{22}
\end{bmatrix}
\begin{bmatrix}
    x_1 \\
    x_2
\end{bmatrix},
\]

(30)

\( f_R = -\bar{R} e_R. \)

5.1. Effort-constraint method

Rewriting these equations into the form (17), and applying the general effort-constraint reduction method (20) from the previous section, yields (assuming that \( Q_{22} \) is invertible) the reduced order port-Hamiltonian model (30).

**Proposition 3.** The effort-constraint reduction method (20) results in the following reduced order port-Hamiltonian model

\[ \begin{align*}
    \dot{x}_1 &= (J_{11} - R_{11})(Q_{11} - Q_{12} Q_{22}^{-1} Q_{21}) x_1 + G_{1} u, \\
    y_{ec} &= G_{1}^T (Q_{11} - Q_{12} Q_{22}^{-1} Q_{21}) x_1.
\end{align*} \]

(30)
for the full order model (26).

**Proof.** Full details for the derivation of the reduced order model (30) are relegated to Appendix A, see also Proposition 1. □

The reduced model (30) was already obtained by direct methods in [8], as well as in scattering coordinates in [9].

5.2. Flow-constraint method

The application of the flow-constraint method (22) to (29) (rewritten in the DAE-form (17)) is more involved. Assuming invertibility of $J_{22}$, the flow-constraint method is seen to lead to the reduced order port-Hamiltonian model (31).

**Proposition 4.** The flow-constraint reduction method (22) results in the following reduced order port-Hamiltonian model

\[
\begin{align*}
\dot{x}_1 &= [(J_r - \beta^T Z_{sk} \beta) - \beta^T Z_{sym} \beta] Q_{11} x_1 \\
&\quad + [(\alpha^T - \beta^T Z_{sk} \gamma^T) - \gamma^T Z_{sym} \gamma] u, \\
y_{fc} &= [(\alpha - \gamma Z_{sk} \beta) - \gamma Z_{sym} \beta] Q_{11} x_1 \\
&\quad + [(\eta + \gamma Z_{sk} \gamma^T) + \gamma^T Z_{sym} \gamma] u,
\end{align*}
\]

where we have adopted the notation

\[
\alpha := G_{J_{12} J_{21}}^{-1} J_{21} - G_{r}, \quad \beta := G_{J_{12} J_{21}}^{-1} J_{21} - G_{r}, \\
\gamma := G_{J_{12} J_{21}}^{-1} G_{r}, \quad \delta := G_{J_{12} J_{21}}^{-1} G_{r}, \\
\eta := G_{J_{12} J_{21}}^{-1} \beta, \quad Z := R(I - \delta R)^{-1}, \\
Z_{sym} := \begin{cases} 
\frac{1}{2}(Z + Z^T), & \text{if } e \in \mathbb{D}_1 \\
\frac{1}{2}(Z - Z^T), & \text{if } e \in \mathbb{D}_2 
\end{cases}, \\
J_s := J_{11} - J_{12} J_{21}.
\]

**Proof.** Full details can be found in Appendix B, see also Proposition 2. □

Note that even though we started with a full order port-Hamiltonian system (26) without feed-through terms, the flow-constraint method, in contrast to the effort-constraint method, results in the reduced order model (31), which is a linear input-state-output port-Hamiltonian system with feedthrough terms [6]:

\[
\begin{align*}
\dot{x}_1 &= (J_r - R_e) Q_{11} x_1 + (G_r - P_r) u, \\
y_{fc} &= (G_r^T + P_r^T) Q_{11} x_1 + (M_r + S_r) u,
\end{align*}
\]

where the reduced order matrices are

\[
J_r = J_{11} - \beta^T Z_{sk} \beta, \quad R_r = \beta^T Z_{sym} \beta, \\
Q_r = Q_{11}, \quad G_r = -\alpha^T + \beta^T Z_{sk} \gamma^T, \\
P_r = -\beta^T Z_{sym} \gamma^T, \quad M_r = \eta + \gamma Z_{sk} \gamma^T, \\
S_r = \gamma Z_{sym} \gamma^T.
\]

One can easily verify that $J_r$, $M_r$, $S_r$ are skew-symmetric, $R_r$, $S_r$ are positive semi-definite symmetric, $Q_r$ is positive definite symmetric, while $R_r$, $P_r$ and $S_r$ satisfy

\[
\begin{bmatrix}
R_r & P_r \\
\eta^T & S_r
\end{bmatrix} \geq 0
\]

(Lemma 1 in Appendix B demonstrates that $Z_{sym}$ in (32) is positive definite).

\[\text{Fig. 4. 0-junction (left) and 1-junction (right).}\]

**Remark 2.** Whenever $G_r = 0$, then the reduced order port-Hamiltonian system (31) specializes to the reduced order system without feed-through terms

\[
\begin{align*}
\dot{x}_1 &= J_{11} Q_{11} x_1 + (G_r - J_{12} J_{21}^{-1} G_r) u, \\
y_{fc} &= (G_r^T + P_r^T) Q_{11} x_1 + (M_r + S_r) u.
\end{align*}
\]

**Remark 3.** In the case of a lossless full order port-Hamiltonian system (26), that is $R = 0$ and $R = 0$, the reduced order port-Hamiltonian system (31) is also lossless and is given as

\[
\begin{align*}
\dot{x}_1 &= J_{11} Q_{11} x_1 + (G_r - J_{12} J_{21}^{-1} G_r) u, \\
y_{fc} &= (G_r^T + P_r^T) Q_{11} x_1 + (M_r + S_r) u.
\end{align*}
\]

5.3. Effort- and flow-constraint methods in the bond-graph modeling framework

Effort- and flow-constraint methods have a direct interpretation from the bond-graph modeling point of view. Constraining the efforts

\[
e_{x_1}^2 = \frac{\partial H}{\partial x_2}(x_1, x_2) = 0, \quad e_2 = 0,
\]

in the lower part of Fig. 2, which results in the effort-constraint method, corresponds to the so-called 0-junction with constraint $e = 0$, shown in Fig. 4 (without orientations). On the other hand, constraining the flows

\[
f_x^2 = -\dot{x}_2 = 0, \quad f_2 = 0,
\]

as in the flow-constraint method, corresponds to the 1-junction with constraint $f = 0$ (see again Fig. 4). The 0- and 1-junctions represent generalized, i.e., domain independent, Kirchhoff current and voltage laws, respectively, and are the common ways to model physical constraints in bond-graph modeling. For details see e.g., [6].

5.4. The effort-constraint method and moment matching

Consider a single-input single-output port-Hamiltonian system (26)

\[
\begin{align*}
\dot{x} &= (J - R) Q x + g u, \\
y &= g^T Q x,
\end{align*}
\]

with an input matrix $g \in \mathbb{R}^{n \times 1}$. The effort-constraint method from Proposition 3, which leads in this case to the following reduced order model

\[
\begin{align*}
\dot{x}_1 &= f_{11}(Q_{11} - Q_{12} Q_{21}^{-1} Q_{21}) x_1 + g u, \\
y_{fc} &= f_{11}^T (Q_{11} - Q_{12} Q_{21}^{-1} Q_{21}) x_1,
\end{align*}
\]

turns out to have a relation to the projection based methods matching moments of the full order system at certain points in the complex plane. The moment-matching approach, discussed in [25].
Using the well-known matrix inversion formula, we get
\[
\begin{split}
Q^{-1}e &= (J - R)e + gu, \\
y &= g^T e.
\end{split}
\]

Recall from the literature on moment matching (see again [25]) that a map \( V_i \in \mathbb{R}^{n \times n} \) matches the first \( r \) moments of (37) at infinity or at \( s_0 \in \mathbb{C} \) if (for \( A := (J - R) \))
\[
\begin{split}
\text{im}V_e &= \text{im}(Q \cdot AQ \cdots : A^{-1}Qg), \quad \text{or} \\
\text{im}V_e &= \text{im}((A - s_0I)^{-1}Qg : \cdots : (A - s_0I)^{-1}Qg),
\end{split}
\]
respectively. Then the following result holds true.

**Theorem 1.** Suppose that the energy coordinates \( x \) of (35) are such that the projection map
\[
V_e = \begin{bmatrix} V_1 \\ 0 \end{bmatrix}, \quad \text{with } V_1 \in \mathbb{R}^{r \times r} \text{ invertible,}
\]
matches the first \( r \) moments at \( s_0 \in \mathbb{C} \) or at infinity of the full order system in co-energy coordinates (37). Then the reduced order port-Hamiltonian model obtained by the effort-constraint method
\[
\begin{split}
\dot{x}_1 &= F_{11}(Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})x_1 + g_1u, \\
y_{ec} &= g_{11}^T(Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})x_1,
\end{split}
\]
matches the first \( r \) moments of the full order system (35) at \( s_0 \in \mathbb{C} \) or at infinity.

**Proof.** The moment matching projection of the rewritten port-Hamiltonian system (37)
\[
\begin{split}
Q^{-1}e &= (J - R)e + gu, \\
y &= g^T e,
\end{split}
\]
is given by
\[
\begin{split}
V_e^T Q^{-1}V_1 e &= V_e^T (J - R) V_1 e + V_e^T gu, \\
\dot{y} &= g^T V_1 e.
\end{split}
\]
Using the well-known matrix inversion formula, we get
\[
V_e^T Q^{-1}V_1 e = [V_1^T \\ 0]\begin{bmatrix} Q^{-1} \\ * \end{bmatrix}\begin{bmatrix} V_1 \\ 0 \end{bmatrix} = V_1^T Q^{-1}V_1,
\]
where \( Q_0 = Q_{11} - Q_{12}Q_{22}^{-1}Q_{21} \) is the Schur complement of \( Q \). Therefore the reduced order system becomes
\[
\begin{split}
V_e^T Q^{-1}V_1 e = V_1^T (J_{11} - R_{11}) V_1 e + V_1^T g_1 u, \\
\dot{y} &= g_{11}^T V_1 e.
\end{split}
\]
Since \( e = V_1 e \) implies that \( e_1 = V_1 e \), and since \( V_1^T \) is invertible, the reduced order model transforms to
\[
\begin{split}
Q^{-1}e_1 &= (J_{11} - R_{11}) e_1 + g_1 u, \\
\dot{y} &= g_{11}^T e_1,
\end{split}
\]
which is, after the transformation from co-energy to energy coordinates \( y_1 = Q_1 x_1 \), nothing but the reduced order system (40) obtained by the effort-constraint method
\[
\begin{split}
\dot{x}_1 &= F_{11}(Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})x_1 + g_1u, \\
y_{ec} &= g_{11}^T(Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})x_1,
\end{split}
\]
Since there are only linear coordinate transformations involved, the moments of (42) and (41), and thus (42) and (35) are the same, which completes the proof. \( \square \)

5.5. The choice of the coordinate system for model reduction

As already indicated before, we do not address in this paper the question of how to choose the coordinate system in which we apply either the effort- or the flow-constraint method. One possible choice of coordinates is balanced coordinates using Lyapunov balancing, positive real (Chapter 4 of [18]) or some other type of balancing. Another choice for the flow-constraint method would be to choose the coordinates where \( G_2 = 0 \), which would significantly simplify the expression of the reduced order model (31), see (33). The effort-constraint method for the SISO port-Hamiltonian systems naturally suggests coordinates \( x \) as in Theorem 1, in order to match moments at specific points in the complex plane, which would pose a question of how to find such coordinates in a numerically efficient way.

6. Numerical example

Consider an \( n \)-dimensional full order port-Hamiltonian mass-spring-damper system as shown in Fig. 5, with masses \( m_i \), spring constants \( k_i \), and damping constants \( c_i \geq 0 \), for \( i = 1, \ldots, n/2 \), \( p_i \) and \( q_i \) are the momentum and displacement of the mass \( m_i \), respectively. The external force acting on the first mass, \( m_1 \), is the input \( u \), while its velocity is the output \( y \). State variables are defined in the following way: for \( i = 1, \ldots, n/2 \), \( x_{2i-1} = q_i \) and \( x_{2i} = p_i \). A detailed port-Hamiltonian description of this system is given in [13].

We considered a 100-dimensional mass-spring-damper system with \( m_1 = 1 \), \( k_1 = 2 \), and \( c_1 = 3.6 \), and applied the effort-constraint method from (30), the flow-constraint method as in (31) and the regular balanced truncation. The coordinates chosen for reduction are (Lyapunov) balanced coordinates.

The reduced order systems are constructed for the orders \( r = 2 \) to \( r = 30 \) with increments of 2. Evolution of the relative \( H_\infty \)- and \( H_\infty \)-norms is shown in Fig. 6. The figure demonstrates that both relative norms for the effort-constraint method consistently decay as the dimension of the reduced order models increases, perhaps apart from the orders \( r = 28 \) and \( r = 30 \). The relative \( H_\infty \)-norm for the flow-constraint method surprisingly does not show similar decaying behavior. Therefore the effort-constraint method outperforms the flow-constraint method for the considered mass-spring-damper system for all dimension of the reduced order models except for \( r = 6 \). The performance of the effort-constraint method was also studied in [8,13,18]. Note that a feed-through term is present in the flow-constraint method (31). Thus, the \( H_2 \)-norms of the flow-constraint method are unbounded and are not shown in the figure.

The regular balanced truncation method, as seen from Fig. 6, outperforms the presented effort- and flow-constraint methods for all dimensions of the reduced order models. Yet we want to underline that the balanced truncation method does not preserve the port-Hamiltonian structure (as explained in [18, Remark 2.12]).

The amplitude Bode plots of the full, reduced and error systems for \( r = 10 \) are shown in Fig. 7. The figure exhibits that the approximation by the flow-constraint method is better for low
frequencies, while the approximation by the effort-constraint method does a better job for high frequencies. The error plot illustrates that the $H_\infty$-norm is larger for the reduced order model by the flow-constraint method. This is consistent with the information from Fig. 6.

Naturally, the considered reduced order models, produced by the effort- and flow-constraint methods, inherit the port-Hamiltonian structure, are asymptotically stable and passive.

7. Conclusions

In this paper, we considered two port-Hamiltonian structure preserving model reduction methods: the effort-constraint method and the flow-constraint method. Both reduction methods preserve the stability and passivity properties of the original system, as a consequence of preserving the port-Hamiltonian structure. These methods arise from the geometric description of
general port-Hamiltonian systems, and are based on the idea of replacing the interconnections to the energy-storage, which carry little power, by zero-power constraints. These constraints can be interpreted within the bond-graph modeling framework as effort- or flow-constraints. We showed that the effort-constraint method, applied in particular coordinates, matches the first moments of the SISO full order port-Hamiltonian system at specific points in the complex plane. A numerical example illustrates the performance of the effort- and flow-constraint methods. A systematic way of choosing the coordinates for the full order port-Hamiltonian system in order to obtain the most accurate approximation from the input–output point of view is an important question left for future research. Relation of the flow-constraint method to moment matching methods, as well as possible error bounds for the effort- and flow-constraint methods, are additional open questions.

Appendix A. Effort-constraint reduction

Consider the full order port-Hamiltonian system (29) with a splitting of the state according to the dimension \( r \) chosen for the reduced order model:

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{y}_1 \\
\dot{y}_2 \\
\end{bmatrix} =
\begin{bmatrix}
J_{11} & J_{12} & 0 & 0 \\
J_{21} & J_{22} & 0 & 0 \\
G_{11} & G_{12} & 0 & 0 \\
G_{21} & G_{22} & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
y_1 \\
y_2 \\
\end{bmatrix} +
\begin{bmatrix}
0 & 0 & I_{m,r} & 0 \\
0 & 0 & 0 & I_{m,r} \\
G_{R1} & 0 & 0 & 0 \\
0 & G_{R2} & 0 & 0 \\
\end{bmatrix}
f_R +
\begin{bmatrix}
0 & 0 & m_{x,m} & 0 \\
0 & 0 & 0 & m_{x,m} \\
0 & 0 & m_{x,m} & 0 \\
0 & 0 & 0 & m_{x,m} \\
\end{bmatrix}
e_R +
\begin{bmatrix}
0 & 0 & m_{x,m} & 0 \\
0 & 0 & 0 & m_{x,m} \\
0 & 0 & m_{x,m} & 0 \\
0 & 0 & 0 & m_{x,m} \\
\end{bmatrix}
u +
\begin{bmatrix}
0 & 0 & m_{x,m} & 0 \\
0 & 0 & 0 & m_{x,m} \\
0 & 0 & m_{x,m} & 0 \\
0 & 0 & 0 & m_{x,m} \\
\end{bmatrix}\tilde{y},
\]  
(43)

which is the equilibrium representation (20)

\[
L^eF_x^f \mathbf{F}_x^f + L^eG_x^f \mathbf{G}_x^f + L^eE_x^f \mathbf{E}_x^f = 0
\]

of the reduced order Dirac structure (note that \( f^f_1 = -\dot{x}_1 \)).

Recall from [18, Section 2.6.1] that setting \( e^2_1 = 0 \) implies that \( e^1_1 = Q_1 x_1 \), where \( Q_1 = Q_{11} - Q_{12} Q_{22}^{-1} Q_{21} \) is the Schur complement of the energy matrix \( Q \). The equilibrium representation (47) is hence equivalent to

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{y}_1 \\
e_R \\
\end{bmatrix} =
\begin{bmatrix}
J_{11} & J_{12} & 0 & 0 \\
G_{11} & G_{12} & 0 & 0 \\
G_{R1} & 0 & 0 & 0 \\
G_{R2} & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
y_1 \\
y_2 \\
\end{bmatrix} +
\begin{bmatrix}
0 & 0 & m_{x,m} & 0 \\
0 & 0 & 0 & m_{x,m} \\
0 & 0 & m_{x,m} & 0 \\
0 & 0 & 0 & m_{x,m} \\
\end{bmatrix}
e_R
\]

This is the reduced order port-Hamiltonian model by the effort-constraint method with the open resistive port. Termination of the resistive port employing the original linear relation \( f_R = -R e_R \) (while using \( R_{11} = G_{R1} R_{E1} \) after the corresponding splitting of \( R \) from (28)) leads to the reduced order port-Hamiltonian model by the effort-constraint method (30)

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{y}_1 \\
e_R \\
\end{bmatrix} =
\begin{bmatrix}
J_{11} - R_{11} \quad Q_{12} - Q_{11} Q_{22}^{-1} Q_{21} \\
0 & 1 \\
G_{R1} & 0 \\
G_{R2} & 0 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
y_1 \\
y_2 \\
\end{bmatrix} +
\begin{bmatrix}
J_{11} & J_{12} & 0 & 0 \\
G_{11} & G_{12} & 0 & 0 \\
G_{R1} & 0 & 0 & 0 \\
G_{R2} & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
y_1 \\
y_2 \\
\end{bmatrix} +
\begin{bmatrix}
0 & 0 & m_{x,m} & 0 \\
0 & 0 & 0 & m_{x,m} \\
0 & 0 & m_{x,m} & 0 \\
0 & 0 & 0 & m_{x,m} \\
\end{bmatrix}
e_R
\]

\( \mathbf{G} \)

(48)

Appendix B. Flow-constraint reduction

We start with the equilibrium representation of the full order Dirac structure (45). A maximal rank matrix \( L^e \) satisfying

\[
L^eF_x^f = 0
\]

is

\[
L^e =
\begin{bmatrix}
I_r & -J_{12}^{-1} & 0 & 0 \\
J_{21} & J_{22}^{-1} & 0 & 0 \\
0 & 0 & I_m & 0 \\
0 & 0 & 0 & I_{m,r} \\
\end{bmatrix}
\]

(50)

assuming that \( J_{22} \) is invertible (for even dimensions \( J_{22} \) is necessarily invertible). For details, see again Section 4. Premultiplication of the Eqs. (45) with \( L^e \) and setting \( f^f_2 = -\dot{x}_2 = 0 \) leads to the following equilibrium representation of the reduced order Dirac structure

\[
\begin{bmatrix}
I_r & -J_{12}^{-1} & 0 & 0 \\
J_{21} & J_{22}^{-1} & 0 & 0 \\
0 & 0 & I_m & 0 \\
0 & 0 & 0 & I_{m,r} \\
\end{bmatrix}
\begin{bmatrix}
\dot{x}_1 \\
\dot{y}_1 \\
e_R \\
\end{bmatrix} =
\begin{bmatrix}
J_{11} & J_{12} & 0 & 0 \\
G_{11} & G_{12} & 0 & 0 \\
G_{R1} & 0 & 0 & 0 \\
G_{R2} & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
y_1 \\
y_2 \\
\end{bmatrix} +
\begin{bmatrix}
0 & 0 & m_{x,m} & 0 \\
0 & 0 & 0 & m_{x,m} \\
0 & 0 & m_{x,m} & 0 \\
0 & 0 & 0 & m_{x,m} \\
\end{bmatrix}
e_R
\]

(49)

Using the notation as in (32)

\[\alpha := G_{R2} J_{12} - C_{R2}, \quad \beta := C_{R2} J_{12}^{-1} - G_{R1}, \quad \gamma := G_{R2} J_{22}^{-1} C_{R2}, \quad \delta := C_{R2} J_{22}^{-1} G_{R2}, \quad \eta := G_{R2} J_{22}^{-1} G_{R2}, \]

\[L^eF_x^f = 0,
\]

as well as setting \( e^2_1 = 0 \). The simplest choice for \( L^e \) is

\[
L^e =
\begin{bmatrix}
I_r & 0 & 0 & 0 \\
0 & I_m & 0 & 0 \\
0 & 0 & I_{m,r} & 0 \\
0 & 0 & 0 & I_{m,r} \\
\end{bmatrix}
\]

(46)
the above equation takes the form

\[
\begin{bmatrix}
I_r \\
0_{m \times r} \\
0_{m \times r}
\end{bmatrix}
\dot{x}_1 = \begin{bmatrix}
J_r \\
\alpha e_i \\
\beta e_i
\end{bmatrix} e_k + \begin{bmatrix}
0_{r \times m_r} \\
0_{m \times m_r} \\
I_{m_r}
\end{bmatrix} e_R \\
+ \begin{bmatrix}
-\alpha^T \\
\eta \\
-\gamma^T
\end{bmatrix} u + \begin{bmatrix}
0_{r \times m} \\
0_{m \times m} \\
I_{m}
\end{bmatrix} \dot{y},
\]

(51)

The equivalent representation (51) of the reduced order Dirac structure implies the reduced order port-Hamiltonian model

\[
\begin{align*}
\dot{x}_1 &= (J_r - \beta^T Z \beta) Q_1 x_1 + (\alpha - \alpha^T + \beta^T Z \gamma^T) u, \\
\dot{y}_c &= (\alpha - \gamma Z \beta) Q_1 x_1 + (\eta + \gamma Z \delta)^T u, \\
0 &= \beta e_i + \delta e_k + e_k - \gamma^T u,
\end{align*}
\]

(52)

where \(Z = R(1 - \delta R)^{-1}\).

Note that the matrix \(L^T\) from (50) is non-unique, as discussed before. Another possible choice for maximal rank \(L^T\), which leads to the same result (53), is given in Appendix B.2 of [18, p. 143].

Next, we prove that the symmetric part of the matrix \(Z\) is positive-definite, showing that the reduced order model obtained by the flow-constraint method is indeed port-Hamiltonian.

**Lemma 1.** Consider the matrix \(Z\) from (32) given as

\[Z := R(1 - \delta R)^{-1}\]

for a skew-symmetric matrix \(\delta = -\delta^T = G_2^T G_2\), and a symmetric positive definite matrix \(R = R^T > 0\). Then the matrix \(Z\) can be decomposed into its symmetric \(Z_{sym}\) and skew-symmetric \(Z_{sk}\) parts as follows:

\[Z_{sym} = (R^{-1} - \delta R^{-1})^{-1}, \quad Z_{sk} = (R^{-1} - \delta R^{-1})^{-1} - (R^{-1} - \delta R^{-1}).\]

Furthermore, the symmetric part of the matrix \(Z\) is positive definite:

\[Z_{sym} = (R^{-1} - \delta R^{-1})^{-1} > 0.\]

**Proof.** The matrix \(Z\) can be rewritten as \(Z = (R^{-1} - \delta)^{-1}\). Then straightforward calculations show that

\[
\begin{align*}
Z_{sym} &= \frac{1}{2} (Z + Z^T) \\
&= \frac{1}{2} [(R^{-1} - \delta)^{-1} + (R^{-1} - \delta^{-1})] \\
&= \frac{1}{2} (R^{-1} - \delta)^{-1} [(R^{-1} + \delta) + (R^{-1} - \delta)] (R^{-1} + \delta)^{-1} \\
&= (R^{-1} - \delta)^{-1} (R^{-1} + \delta)^{-1} \\
&= (R^{-1} - \delta)^{-1} (I + \delta R)^{-1} \\
&= [(I + \delta R) (R^{-1} - \delta)^{-1}]^{-1} \\
&= (R^{-1} - \delta R)^{-1}.
\end{align*}
\]

Similarly

\[
Z_{sk} = \frac{1}{2} (Z - Z^T) = (R^{-1} - \delta R^{-1} - \delta - \delta^{-1})^{-1}
\]

Moreover, \(Z = (R^{-1} - \delta)^{-1}\) implies that \(Z^{-1} = R^{-1} - \delta\). Hence, the symmetric part of \(Z^{-1}\), which is \(R^{-1}\), is necessarily positive definite.

Since any real vector \(w\) can be written as \(w = Z^{-1} v\) for a certain \(v\), it follows that

\[w^T Z w = v^T Z^{-1} Z^{-1} v = v^T v > 0.\]

This proves that the symmetric part of \(Z\) is positive definite. \(\square\)

Finally, note that in the case of a lossless full order port-Hamiltonian system \(R = 0\) and, consequently, \(Z = 0\).

**References**


