Load balancing of dynamical distribution networks with flow constraints and unknown in/outflows

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A B S T R A C T

We consider a basic model of a dynamical distribution network, modeled as a directed graph with storage variables corresponding to every vertex and flow inputs corresponding to every edge, subject to unknown but constant inflows and outflows. As a preparatory result it is shown how a distributed proportional–integral controller structure, associating with every edge of the graph a controller state, will regulate the state variables of the vertices, irrespective of the unknown constant inflows and outflows, in the sense that the storage variables converge to the same value (load balancing or consensus). This will be proved by identifying the closed-loop system as a port-Hamiltonian system, and modifying the Hamiltonian function into a Lyapunov function, dependent on the value of the vector of constant inflows and outflows. In the main part of the paper the same problem will be addressed for the case that the input flow variables are constrained to take value in an arbitrary interval. We will derive sufficient and necessary conditions for load balancing, which only depend on the structure of the network in relation with the flow constraints.

1. Introduction

In this paper, we study a basic model for the dynamics of a distribution network. Identifying the network with a directed graph we associate with every vertex of the graph a state variable corresponding to storage, and with every edge a control input variable corresponding to flow, which is constrained to take value in a given closed interval. Furthermore, some of the vertices serve as terminals where an unknown but constant flow may enter or leave the network in such a way that the total sum of inflows and outflows is equal to zero. The control problem to be studied is to derive necessary and sufficient conditions for a distributed control structure (the control input corresponding to a given edge only depending on the difference of the state variables of the adjacent vertices) which will ensure that the state variables associated to all vertices will converge to the same value equal to the average of the initial condition, irrespective of the values of the constant unknown inflows and outflows.

The structure of the paper is as follows. Some preliminaries and notations will be given in Section 2. In Section 3 we will show how in the absence of constraints on the flow input variables a distributed proportional–integral (PI) controller structure, associating with every edge of the graph a controller state, will solve the problem if and only if the graph is weakly connected. This will be shown by identifying the closed-loop system as a port-Hamiltonian system, with state variables associated both to the vertices and the edges of the graph, in line with the general definition of port-Hamiltonian systems on graphs [1–4]; see also [5,6]. The proof of asymptotic load balancing will be given by modifying, depending on the vector of constant inflows and outflows, the total Hamiltonian function into a Lyapunov function. In the examples the obtained PI-controller often has a clear physical interpretation, emulating the physical action of adding energy storage and damping to the edges.

The main contribution of the paper resides in Sections 4 and 5, where the same problem is addressed for the case of constraints on the flow input variables. In Section 4 it will be shown that in the case of zero inflow and outflow the state variables of the vertices converge to the same value if and only if the network is strongly connected. This will be shown by constructing a C 1 Lyapunov function based on the total Hamiltonian and the constraint values. This same construction will be extended in Section 5 to the case of nonzero inflows and outflows, leading to the result that in this case asymptotic load balancing is reached if and only the graph is not only strongly connected but also balanced. Finally, Section 6 contains the conclusions.

Some preliminary results, in particular concerning Section 3, have been already reported before in [7].

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2. Preliminaries and notations

First we recall some standard definitions regarding directed graphs, as can be found e.g. in [8]. A directed graph $\tilde{G}$ consists of a finite set $V$ of vertices and a finite set $E$ of edges, together with a mapping from $E$ to the set of ordered pairs of $V$, where no self-loops are allowed. Thus to any edge $e \in E$ there corresponds an ordered pair $(v, w) \in V \times V$ (with $v \neq w$), representing the tail vertex $v$ and the head vertex $w$ of this edge.

A directed graph is completely specified by its incidence matrix $B$, which is an $n \times m$ matrix, $n$ being the number of vertices and $m$ being the number of edges, with $(i,j)$th element equal to 1 if the $j$th edge is towards vertex $i$, and equal to $-1$ if the $j$th edge is originating from vertex $i$, and 0 otherwise. Since we will only consider the directed graphs in this paper ‘graph’ will throughout mean ‘directed graph’ in the sequel. A directed graph is strongly connected if it is possible to reach any vertex starting from any other vertex by traversing edges following their directions. A directed graph is called weakly connected if it is possible to reach any vertex from every other vertex using the edges not taking into account their direction. A graph is weakly connected if and only if ker $B^T = \text{span} \{1\}_{m \times 1}$. Here $\text{ker} \{\cdot\}$ denotes the $n$-dimensional vector with all elements equal to 1. A graph that is not weakly connected falls apart into a number of weakly connected subgraphs, called the connected components. The number of connected components is equal to dim ker $B^T$. For each vertex, the number of incoming edges is called the in-degree of the vertex and the number of outgoing edges its out-degree. A graph is called balanced if for every vertex their in-degree and out-degree of every edge are equal. A graph is balanced if and only if $1_m \in \text{ker} B$.

Given a graph, we define its vertex space as the vector space of all functions from $V$ to some linear space $R$. In the rest of this paper we will take for simplicity $R = \mathbb{R}$, in which case the vertex space can be identified with $\mathbb{R}^n$. Similarly, we define its edge space as the vector space of all functions from $E$ to $R = \mathbb{R}$, which can be identified with $\mathbb{R}^m$. In this way, the incidence matrix $B$ of the graph can be also regarded as the matrix representation of a linear map from the edge space $\mathbb{R}^m$ to the vertex space $\mathbb{R}^n$.

**Notation:** for $a, b \in \mathbb{R}^m$ the notation $a \leq b$ will denote elementwise inequality $a_i \leq b_i$, $i = 1, \ldots, m$. For $a_i < b_i$, $i = 1, \ldots, m$ the multidimensional saturation function sat$(x; a, b) : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is defined as

$$\text{sat}((x; a, b)) = \begin{cases} a_i & \text{if } x_i \leq a_i, \\ x_i & \text{if } a_i < x_i < b_i, \\ b_i & \text{if } x_i \geq b_i, \end{cases}, i = 1, \ldots, m. \hspace{1cm} (1)$$

3. A dynamic network model with PI controller

Let us consider the following dynamical system defined on the graph $\tilde{G}$

$$\dot{x} = Bu, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m$$

$$y = B^T \frac{\partial H}{\partial x}(x), \quad y \in \mathbb{R}^m, \hspace{1cm} (2)$$

where $H : \mathbb{R}^n \rightarrow \mathbb{R}$ is any differentiable function, and $\frac{\partial H}{\partial x}(x)$ denotes the column vector of partial derivatives of $H$. Here the $i$th element $x_i$ of the state vector $x$ is the state variable associated to the $i$th vertex, while $u_i$ is a flow input variable associated to the $j$th edge of the graph. System (2) defines a port-Hamiltonian system [9, 10], satisfying the energy-balance

$$\frac{d}{dt} H = u^T y. \hspace{1cm} (3)$$

Note that geometrically its state space is the vertex space, its input space is the edge space, while its output space is the dual of the edge space.

**Example 3.1 (Hydraulic Network).** Consider a hydraulic network, modeled as a directed graph with vertices (nodes) corresponding to reservoirs, and edges (branches) corresponding to pipes. Let $x_i$ be the amount of water stored at vertex $i$, and $u_j$ the flow through edge $j$. Then the mass-balance of the network is summarized in

$$\dot{x} = Bu, \hspace{1cm} (4)$$

where $B$ is the incidence matrix of the graph. Let furthermore $H(x)$ denote the stored energy in the reservoirs (e.g. gravitational energy). Then $P_i := \frac{\partial H}{\partial x}(x)$, $i = 1, \ldots, n$, are the pressures at the vertices, and the output vector $y = B^T \frac{\partial H}{\partial x}(x)$ is the vector whose $i$th element is the pressure difference $P_i - P_k$ across the $j$th edge linking vertex $k$ to vertex $i$.

As a next step we will extend the dynamical system (2) with a vector $d$ of inflows and outflows

$$\dot{x} = Bu + Ed, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m, \quad d \in \mathbb{R}^k$$

$$y = B^T \frac{\partial H}{\partial x}(x), \quad y \in \mathbb{R}^m, \hspace{1cm} (5)$$

with $E$ an $n \times k$ matrix whose columns consist of exactly one entry equal to 1 (inflow) or $-1$ (outflow), while the rest of the elements is zero. Thus $E$ specifies the $k$ terminal vertices where flows can enter or leave the network.

In this paper we will regard $d$ as a vector of constant disturbances, and we want to investigate control schemes which ensure asymptotic load balancing of the state vector $x$ irrespective of the (unknown) disturbance $d$. The simplest control possibility is to apply a proportional output feedback

$$u = -Ry = -RB^{T} \frac{\partial H}{\partial x}(x), \hspace{1cm} (6)$$

where $R$ is a diagonal matrix with strictly positive diagonal elements $r_1, \ldots, r_m$. Note that this defines a decentralized control scheme if $H$ is of the form $H(x) = H_1(x_1) + \cdots + H_m(x_m)$, in which case the $i$th input as given by (6) equals $r_i$ times the difference of the component of $\frac{\partial H}{\partial x}(x)$ corresponding to the head vertex of the $i$th edge and the component of $\frac{\partial H}{\partial x}(x)$ corresponding to its tail vertex. This control scheme leads to the closed-loop system

$$\dot{x} = -B^{T}R^{T}\frac{\partial H}{\partial x}(x) + Ed. \hspace{1cm} (7)$$

In the case of zero in/outflows $d = 0$ this implies the energy-balance

$$\frac{d}{dt} H = -B^{T} \frac{\partial H}{\partial x}(x)BR^{T} \frac{\partial H}{\partial x}(x) \leq 0. \hspace{1cm} (8)$$

Hence if $H$ is radially unbounded it follows that the system trajectories of the closed-loop system (7) will converge to the set

$$\mathcal{E} := \left\{ x \in \mathbb{R}^n | B^{T} \frac{\partial H}{\partial x}(x) = 0 \right\}$$

and thus to the load balancing set

$$\mathcal{E} = \left\{ x \in \mathbb{R}^n | \frac{\partial H}{\partial x}(x) = \alpha 2 \right\}, \alpha \in \mathbb{R} \hspace{1cm} (9)$$

if and only if ker $B^T = \text{span}\{1\}$, or equivalently [8], if and only if the graph is weakly connected.

In particular, for the standard Hamiltonian $H(x) = \frac{1}{2} \|x\|^2_2$ this means that the state variables $x_i(t), i = 1, \ldots, n$, converge to a common value $a$ as $t \rightarrow \infty$. Since $\frac{d}{dt} H(x)(t) = 0$ it follows that this common value is given as $\alpha = \frac{1}{n} \sum_{i=1}^{n} x_i(0)$.

For $d \neq 0$ proportional control $u = -Ry$ will not be sufficient to reach load balancing, since the disturbance $d$ can only be attenuated at the expense of increasing the gains in the matrix $R$. Hence
we consider proportional–integral (PI) control given by the dynamic output feedback\(^1\)


dot{x}_c = y,
\[ u = -Ry - \frac{\partial H_c}{\partial x_c}(x_c), \]

where \(H_c(x_c)\) denotes the storage function (energy) of the controller. Note that, this PI controller is of the same decentralized nature as the static output feedback \(u = -Ry\).

The jth element of the controller state \(x_c\) can be regarded as an additional state variable corresponding to the jth edge. Thus \(x_c \in \mathbb{R}^m\), the edge space of the network. The closed-loop system resulting from the PI control (10) is given as

\[
\begin{bmatrix}
\dot{x}_c \\
\dot{\tilde{x}}_c
\end{bmatrix} =
\begin{bmatrix}
-BRB^T & -B^T \\
B & 0
\end{bmatrix}
\begin{bmatrix}
\frac{\partial H}{\partial x}(x) \\
\frac{\partial H_c}{\partial x_c}(x_c)
\end{bmatrix}
+ \begin{bmatrix} E \end{bmatrix} d_c.
\]

(11)

This is again a port–Hamiltonian system,\(^2\) with total Hamiltonian \(H_{tot}(x, x_c) := H(x) + H_c(x_c)\), and satisfying the energy-balance

\[
\frac{d}{dt} H_{tot} = -\frac{\partial^2 H}{\partial x^2}(x)BRB^T \frac{\partial H}{\partial x}(x) + \frac{\partial^2 H}{\partial x^2}(x)Ed
\]

(12)

Consider now a constant disturbance \(\tilde{d}\) for which there exists a matching controller state \(\tilde{x}_c\), i.e.,

\[
\tilde{E}\tilde{d} = B \frac{\partial H_c}{\partial x_c}(\tilde{x}_c).
\]

(13)

This allows us to total the Hamiltonian \(H_{tot}(x, x_c)\) into\(^3\)

\[
V_{\tilde{d}}(x, x_c) := H(x) + H_c(x_c) - \frac{\partial^2 H}{\partial x^2}(\tilde{x}_c)(x_c - \tilde{x}_c) - H_c(\tilde{x}_c),
\]

(14)

which will serve as a candidate Lyapunov function; leading to the following theorem.

**Theorem 1.** Consider the system (5) on the graph \(\mathcal{G}\) in closed loop with the PI-controller (10). Let the constant disturbance \(d\) be such that there exists a \(\tilde{x}_c\) satisfying the matching equation (13). Assume that \(V_{\tilde{d}}(x, x_c)\) is radially unbounded. Then the trajectories of the closed-loop system (11) will converge to an element of the load balancing set

\[
\mathcal{E}_{tot} = \left\{ (x, x_c) \mid \frac{\partial H}{\partial x}(x) = \alpha \tilde{x}, \alpha \in \mathbb{R}, B \frac{\partial H_c}{\partial x_c}(x_c) = \tilde{E}d \right\}
\]

(15)

if and only if \(\mathcal{G}\) is weakly connected.

**Proof.** Suppose that \(\mathcal{G}\) is weakly connected. By (12) for \(d = \tilde{d}\) we make use of (13),

\[
\frac{d}{dt} V_{\tilde{d}} = -\frac{\partial^2 H}{\partial x^2}(x)BRB^T \frac{\partial H}{\partial x}(x) + \frac{\partial^2 H}{\partial x^2}(x)Ed
\]

\[
-\frac{\partial^2 H_c}{\partial x_c}(\tilde{x}_c)B \frac{\partial H}{\partial x_c}(x)
\]

\[
= -\frac{\partial^2 H}{\partial x^2}(x)BRB^T \frac{\partial H}{\partial x}(x) \leq 0.
\]

Hence by LaSalle’s invariance principle the system trajectories converge to the largest invariant set contained in

\[
\left\{ (x, x_c) \mid B \frac{\partial H}{\partial x}(x) = 0 \right\}.
\]

Substitution of \(B \frac{\partial H}{\partial x}(x) = 0\) in the closed-loop system equations (11) yields \(x_c\) constant and \(-B \frac{\partial H_c}{\partial x_c}(x_c) + \tilde{E}d = 0\). Since the graph is weakly connected \(B \frac{\partial H_c}{\partial x_c}(x_c) = 0\) implies \(\tilde{E}d = 0\). If the graph is not weakly connected then the above analysis will hold on every connected component, and the common value \(\alpha\) will be different for different components. □

**Corollary 2.** If \(ker B = 0\), which is equivalent [8] to the graph having no cycles, then for every \(d\) there exists a unique \(x_c\) satisfying (13), and convergence is towards the set \(\mathcal{E}_{tot} = \left\{ (x, x_c) \mid \frac{\partial H}{\partial x}(x) = \alpha \tilde{x}, \alpha \in \mathbb{R}, x_c = \tilde{x}_c \right\}\)

**Corollary 3.** In the case of the standard quadratic Hamiltonians \(H(x) = \frac{1}{2} \|x\|^2, H_c(x_c) = \frac{1}{2} \|x_c\|^2\) there exists for every \(\tilde{d}\) a controller state \(\tilde{x}_c\) such that (13) holds if and only if

\[
imE \subset imB.
\]

Furthermore, in this case \(V_{\tilde{d}}\) equals the radially unbounded function \(\frac{1}{2} \|x\|^2 + \frac{1}{2} \|x_c - \tilde{x}_c\|^2\), while convergence will be towards the load balancing set \(\mathcal{E}_{tot} = \left\{ (x, x_c) \mid x = \alpha \tilde{x}, \alpha \in \mathbb{R}, Bx_c = \tilde{E}d \right\}\).

A necessary (and in the case the graph is weakly connected necessary and sufficient) condition for the inclusion \(imE \subset imB\) is that \(\Pi^T E = 0\). In its turn \(\Pi^T E = 0\) is equivalent to the fact that for every \(d\) the total inflow into the network equals to the total outflow. The condition \(\Pi^T E = 0\) also implies

\[
\Pi^T x = -\Pi^T BRB^T \frac{\partial H}{\partial x}(x) + \Pi^T \tilde{E}d = 0,
\]

yielding (as in the case \(d = 0\)) that \(\Pi^T x\) is a conserved quantity for the closed-loop system (11). In particular it follows that the limit value \(\lim_{t \to \infty} x(t) \in \text{span}\{1\}\) is determined by the initial condition \(x(0)\).

**Example 3.2** (Hydraulic Network Continued). The proportional part \(u = -Ry\) of the controller corresponds to adding damping to the dynamics (proportional to the pressure differences along the edges). The integral part of the controller has the interpretation of adding compressibility to the hydraulic network dynamics. Using this emulated compressibility, the PI-controller is able to regulate the hydraulic network to a load balancing situation where all pressures \(P_i\) are equal, irrespective of the constant inflow and outflow \(d\) satisfying the matching condition (13). Note that for the Hamiltonian \(H(x) = \frac{1}{2} \|x\|^2\) the pressures \(P_i\) are equal to each other if and only if the water levels \(x_i\) are equal.

### 4. Constrained flows: the case without in/outflows

In many cases of interest, the elements of the vector of flow inputs \(u \in \mathbb{R}^m\) corresponding to the edges of the graph will be constrained, that is

\[
u \in \mathcal{U} := \{u \in \mathbb{R}^m \mid u^- \leq u \leq u^+\}\]

(19)

for certain vectors \(u^-\) and \(u^+\) satisfying \(u^-_i \leq u^+_i, i = 1, \ldots, m\) (throughout \(\leq\) denotes element-wise inequality). This leads to the
following constrained version of the PI controller (10) given in the previous section
\[
\dot{x}_c = y, \\
u = \text{sat} \left( -Ry - \frac{\partial H}{\partial x_c}(x_c); u^-, u^+ \right).
\] (20)
Throughout this paper we make the following assumption on the flow constraints.

**Assumption 4.**

\[u_i^- < 0, \quad u_i^+ > 0, \quad u_i^- < u_i^+, \quad i = 1, \ldots, m.\] (21)

It is important to note that we may change the orientation of some of the edges of the graph at will; replacing the corresponding columns \(b_i\) of the incidence matrix \(B\) by \(-b_i\). Noting the identity \(\text{sat}(x; u_i^-, u_i^+) = -\text{sat}(x; -u_i^-, -u_i^+\rangle\) this implies that we may assume without loss of generality that the orientation of the graph is chosen such that
\[u_i^- < 0 < u_i^+, \quad i = 1, \ldots, m.\] (22)

*This will be assumed throughout the rest of the paper.* In general, we will say that any orientation of the graph is compatible with the flow constraints if (22) holds. If the \(j\)-th edge is such that \(u_j^- = 0\) then we will call this edge an uni-directional edge, while if \(u_j^+ < 0\) then the edge is called a bi-directional edge.

In this section, we will first analyze the closed-loop system for the constrained PI-controller under the simplifying assumption of zero inflow and outflow \((d = 0)\). In the next section, we will deal with the general case. Furthermore, for the simplicity of exposition we consider throughout the rest of this paper the standard Hamiltonian
\[H_c(x_c) = \frac{1}{2}\|x_c\|^2\]
for the constrained PI controller and the identity gain matrix \(R = I\), while we also throughout assume that the Hessian matrix of Hamiltonian \(H(x)\) is positive definite for any \(x\). Thus we consider in the rest of this section the closed-loop system
\[
\dot{x} = B\text{sat} \left( -B^T \frac{\partial H}{\partial x}(x) - x_c; u^-, u^+ \right),
\]
\[
\dot{x}_c = B^T \frac{\partial H}{\partial x}(x).
\] (23)

In order to state the main theorem of this section we need one more definition.

**Definition 5.** Consider the directed graph \(\mathcal{G}\) together with the constraint values \(u^-, u^+\) satisfying (22). Then we will call the graph strongly connected with respect to the flow constraints \(u^- \leq u \leq u^+\) if the following holds: for every two vertices \(v_1, v_2\) there exists an orientation of the graph compatible with the flow constraints\(^4\) and a directed path (directed with respect to this orientation) from \(v_1\) to \(v_2\).\(^5\)

**Theorem 6.** Consider the closed-loop system (23) on a graph \(\mathcal{G}\) with flow constraints \(u^- \leq u \leq u^+\) satisfying (22). Then its solutions converge to the load balancing set
\[
\mathcal{E}_{\text{tot}} = \left\{ (x, x_c) \mid \frac{\partial H}{\partial x}(x) = \alpha \mathbb{1}_n, \text{ sat}(x_c; u^-, u^+) = 0 \right\}
\] (24)
if and only if the graph is strongly connected with respect to the flow constraints.

**Proof.** Sufficiency: consider the Lyapunov function given by
\[
V(x, x_c) = \mathbb{E}_m S \left( -B^T \frac{\partial H}{\partial x}(x) - x_c; u^-, u^+ \right) + H(x),
\] (25)
with
\[
S(x; u^-, u^+) := \int_0^1 \text{sat}(y; u^-_i, u^+_i) \, dy.
\] (26)
It can be easily verified that \(V\) is positive definitive, radially unbounded and \(C^1\). Its time-derivative is given as
\[
\dot{V} = -\text{sat}^T \left( -B^T \frac{\partial H}{\partial x}(x) - x_c; u^-, u^+ \right) B^T
\]
\[
\times \text{sat}^T \left( -B^T \frac{\partial H}{\partial x}(x) - x_c; u^-, u^+ \right)
\]
\[
\leq 0.
\] (27)
By LaSalle’s invariance principle, all trajectories will converge to the largest invariant set, denoted as \(L\), contained in \(\mathcal{K} = \{ (x_c, x_i) \mid \text{ sat} \left( -B^T \frac{\partial H}{\partial x}(x) - x_c; u^-, u^+ \right) = 0 \}\). Whenever \(x \in \mathcal{K}\) it follows that \(\dot{x} = 0\) and thus \(x(t) = x\) for some constant vector \(x\). Hence, since \(\dot{x} = B^T \frac{\partial H}{\partial x}(x)\), it follows that \(x(t) = B^T \frac{\partial H}{\partial x}(x) t + x(0)\).

Suppose now that \(B^T \frac{\partial H}{\partial x}(x) \neq 0\). Then for \(t\) large enough
\[
0 = \frac{\partial H}{\partial x} (x) B \text{sat} \left( -B^T \frac{\partial H}{\partial x}(x) - B^T \frac{\partial H}{\partial x}(x) t - x_c(0); u^-, u^+ \right)
\]
\[
= \sum_{i=1}^{m} \left( B^T \frac{\partial H}{\partial x}(x) \right)_i c_i,
\] (28)
where
\[
c_i = \begin{cases} 
    u_i^- & \text{if } \left( B^T \frac{\partial H}{\partial x}(x) \right)_i > 0, \\
    u_i^+ & \text{if } \left( B^T \frac{\partial H}{\partial x}(x) \right)_i < 0.
\end{cases}
\] (29)
Hence, in view of \(u_i^- < 0 < u_i^+\), we have \(\left( B^T \frac{\partial H}{\partial x}(x) \right)_i \neq 0\), for \(i = 1, \ldots, m\). However since the graph is strongly connected with respect to the flow constraints, if \(\left( B^T \frac{\partial H}{\partial x}(x) \right)_i > 0\), then there exists \(j\) such that \(\left( B^T \frac{\partial H}{\partial x}(x) \right)_j < 0\). This yields a contradiction. We conclude that \(B^T \frac{\partial H}{\partial x}(x) = 0\), which implies \(\frac{\partial H}{\partial x}(x) = \alpha \mathbb{1}_n\), and thus all trajectories converge to \(\mathcal{E}_{\text{tot}}\).

Necessity: assume without loss of generality that the graph is weakly connected. (Otherwise the same analysis can be performed on every connected component.) If the graph is not strongly connected with respect to the flow constraints then there is a pair of vertices \(v_1, v_2\) for which there exist a compatible orientation and a directed path from \(v_1\) to \(v_2\), but not a compatible orientation and directed path from \(v_2\) to \(v_1\). In other words, there can be positive flow from \(v_1\) to \(v_2\), but not vice versa. Then for suitable initial condition, \(\frac{\partial H}{\partial x}(x(t)) < \frac{\partial H}{\partial x}(x(t)) \) for all \(t \geq 0\), and thus there is no convergence to \(\mathcal{E}_{\text{tot}}\).\(\square\)

**Remark 7.** Note that for \(u_i^- \rightarrow -\infty\), \(u_i^+ \rightarrow \infty\), the Lyapunov function (25) tends to the function \(H(x) + \frac{1}{2} ||B^T \frac{\partial H}{\partial x}(x) + x_c||^2\), which is different from the Lyapunov function \(H(x) + \frac{1}{2} ||x_c||^2\) used in the previous section.

In the special case that the flow constraints are such that all the flows \(u_i\) can follow both directions, we obtain the following corollary.
Theorem 11. Consider a graph \( G \) with dynamics (30) with flow constraints such that \( u_i^- = 0, i = 1, \ldots, m \) (uni-directional flow). Then for any \( u^+ \in \mathbb{R}^m \) and any in/outflow \( \delta \) satisfying the matching condition for the constrained case, the trajectories of (30) converge to
\[
\mathcal{E}_{\text{tot}} = \left\{ (x, x_c^+) | \frac{\partial H}{\partial x}(x) = \alpha \mathbb{1}, \alpha \in \mathbb{R}, \right\}
\]
if and only if the graph in the (only) orientation compatible with the flow constraints is strongly connected and balanced.

In order to prove Theorem 12 we need the following two lemmas. Recall that a directed graph is balanced if every vertex has indegree (number of incoming edges) equal to outdegree (number of outgoing edges). Furthermore, we will say that two cycles of a graph are non-overlapping if they do not have any edges in common.

Lemma 13. A strongly connected graph is balanced if and only if it can be covered by non-overlapping cycles.

Proof. Sufficiency: if a graph can be covered by non-overlapping cycles, then every vertex necessarily has the same in-degree and out-degree; so this graph is balanced.

Necessity: since the graph is strongly connected, every two vertices can be connected by a directed path, and the graph can be covered by cycles. Now suppose that the graph cannot be covered by non-overlapping cycles. We will show that this implies that the graph is not balanced.

Let \( k \) be the smallest number of cycles needed to cover the graph, and let \( \mathcal{T} = (C_1, C_2, \ldots, C_k) \) be a covering set of cycles. According to our assumption, at least one edge of the graph is shared by two or more cycles in \( \mathcal{T} \). We claim that the set of shared edges cannot contain any cycles. Indeed, suppose that there is one cycle, denoted as \( \mathcal{D} \) (depicted in Fig. 1(a)), whose edges are all shared by elements of \( \mathcal{T} \). If \( \mathcal{D} \in \mathcal{T} \), then obviously \( \mathcal{T} \) is not a minimal covering set, since by deleting the cycle \( \mathcal{D} \) from \( \mathcal{T} \) we have a covering set of \( k - 1 \) elements.

Thus \( \mathcal{D} \notin \mathcal{T} \). It can be seen that the minimal number \( c \) of cycles in \( \mathcal{T} \) which cover \( \mathcal{D} \) twice is at least 4. Denote such a minimal set of cycles in \( \mathcal{T} \) which cover \( \mathcal{D} \) by \( \mathcal{T}_2 \). We will now show that by combining these \( c \) cycles with the cycle \( \mathcal{D} \) there exist 3 cycles in the original graph \( G \) which cover the subgraph given by \( \mathcal{T}_2 \) thus reaching a contradiction with the minimality of \( \mathcal{T} \). The construction of these 3 cycles is indicated in Fig. 1. Consider for simplicity the case that 4 cycles in \( \mathcal{T} \), denoted by \( C_1, C_2, C_3, C_4 \), cover \( \mathcal{D} \) twice. Combining (depending on the orientation of the cycles) part of \( C_1 \) with part of \( C_3 \) and part of \( C_2 \) with part of \( C_4 \) (see Fig. 1), we can define 2 cycles which together with the cycle \( \mathcal{D} \) yield a set of 3 cycles which cover the subgraph spanned by \( C_1, C_2, C_3, C_4 \).

In conclusion, there must exist at least one shared edge, say \((v_i, v_j)\), such that all edges with tail-vertex \( v_i \) are used only once in \( \mathcal{T} \). But this implies that \( v_j \) has larger out-degree than in-degree, i.e., the graph is not balanced.

Lemma 14. Consider a strongly connected and balanced graph with dynamics (30) with flow constraints and disturbance as given in Theorem 12. Then the following statements hold:
Fig. 1. (a) The cycle $\mathcal{D}$. We divide the edges of $\mathcal{D}$ into two disjoint sets: $\mathcal{D}_1$ contains the left part of $\mathcal{D}$ and $\mathcal{D}_2$ contains the rest. (b) The subgraph given by $T_D$. Without $\mathcal{D}$, we need at least 3 cycles to cover $\mathcal{D}$ twice; these cycles are given as $C_1 = T_{D_1} \cup \mathcal{D}_1$, $C_2 = T_{D_2} \cup \mathcal{D}_1$, $C_3 = T_{D_2} \cup \mathcal{D}_2$ and $C_4 = T_{D_2} \cup \mathcal{D}_1$. It follows that $T_D$ is also covered by the 3 cycles: $\mathcal{D}_2$, $T_{D_2}$ and $\mathcal{D}_1$ (clockwise) $\mathcal{D}_1$, $T_{D_2}$ and $\mathcal{D}_2$ (counterclockwise).

(i) along every trajectory $(x(t), \dot{x}(t))$, $t \geq 0$, of (30), the function
$$V(x, \dot{x}) = 1^T S \left(-B^T \frac{\partial H}{\partial x}(x) - \dot{x}; \dot{x}, u^+ + \dot{x}_c\right) + H(x)$$
\text{is bounded from below,}
(ii) the trajectory $(x(t), \dot{x}(t))$, $t \geq 0$, is bounded,
(iii) $\lim_{t \to \infty} V(x(t), \dot{x}(t)) = 0$,
where $\dot{x}_c = x_c - \dot{x}_c$.

\textbf{Proof.} (i) By using (33) and the matching condition for the constrained case, we rewrite the system as
$$\begin{align*}
\dot{x} &= B \text{sat} \left(-B^T \frac{\partial H}{\partial x}(x) - \dot{x}; \dot{x}_c, u^+ + \dot{x}_c\right), \\
\dot{x}_c &= B^T \frac{\partial H}{\partial x}(x),
\end{align*}$$
where $\dot{x}_c = x_c - \dot{x}_c$. Since for a balanced network $B_{11} = 0$, we obtain the following implications
$$\begin{align*}
0 &= 1^T B^T \frac{\partial H}{\partial x}(x) \\
&= \sum_{i=1}^m \left( B^T \frac{\partial H}{\partial x}(x) \right)_i = 0 \\
&= \sum_{i=1}^m \dot{x}_c(t) = \sum_{i=1}^m \dot{x}_c(0) \\
&= \sum_{i=1}^m \left(-B^T \frac{\partial H}{\partial x}(x(t)) - \dot{x}_c(t) \right)_i \\
&= \sum_{i=1}^m -\dot{x}_c(0), \quad \forall t > 0.
\end{align*}$$

Next, we prove that $1^T S \left(-B^T \frac{\partial H}{\partial x}(x) - \dot{x}; \dot{x}_c, u^+ + \dot{x}_c\right)$ is bounded from below. Indeed, suppose that $1^T S \left(-B^T \frac{\partial H}{\partial x}(x(t)) - \dot{x}_c(t); \dot{x}_c, u^+ + \dot{x}_c\right)$ has an increasing sequence $\{t_k\}$ with $t_k \geq 0$ converging to $-\infty$, i.e.,
$$\lim_{k \to \infty} 1^T S \left(-B^T \frac{\partial H}{\partial x}(x(t_k)) - \dot{x}_c(t_k); \dot{x}_c, u^+ + \dot{x}_c\right) = -\infty.$$  

Notice that
$$\int_0^{\text{sat}(y; \dot{x}_c, u^+ + \dot{x}_c)} \text{d}y (40)$$
can have a negative value only when $\left(-B^T \frac{\partial H}{\partial x}(x) - \dot{x}_c\right) < 0$. Therefore we may assume without loss of generality that
$$\left(-B^T \frac{\partial H}{\partial x}(x(t_k)) - \dot{x}_c(t_k) \right)_i < 0, \quad i \in \mathcal{E}_1 \subset \mathcal{E},$$
$$\lim_{k \to \infty} \sum_{i \in \mathcal{E}_1} \left(-B^T \frac{\partial H}{\partial x}(x(t_k)) - \dot{x}_c(t_k) \right)_i = -\infty.$$  

By Eq. (38), we have
$$\lim_{k \to \infty} \sum_{i \in \mathcal{E}_2} \left(-B^T \frac{\partial H}{\partial x}(x(t_k)) - \dot{x}_c(t_k) \right)_j = +\infty$$
where $\mathcal{E}_2 = \mathcal{E} \setminus \mathcal{E}_1$. Then Definition 9 implies that $(\dot{x}_c + u^+)_c = \dot{x}_c$, $V_s$, $s = 1, 2, \ldots, m$, which leads to $\lim_{k \to \infty} 1^T S \left(-B^T \frac{\partial H}{\partial x}(x(t_k)) - \dot{x}_c(t_k); \dot{x}_c, u^+ + \dot{x}_c\right) = +\infty$. This is a contradiction.

Moreover, since $H(x)$ has a lower bound, then $V(x, \dot{x}_c)$ is bounded from below for any given initial condition $(x(0), \dot{x}_c(0))$.

(ii) Notice that $V = -\dot{x}_c^2 < 0$.

Suppose that $x(t), \dot{x}(t) > 0$, then there exists a sequence $\{t_k\}$, $t_k > 0$ such that $\lim_{t \to \infty} \|x(t_k)\| = \infty$. Since $H(x)$ is unbounded, this implies
$$\lim_{k \to \infty} V(x(t_k), \dot{x}_c(t_k)) = +\infty,$$
This is a contradiction with $V < 0$.

Suppose that $\dot{x}_c(t)$ is not bounded, then as follows from the proof of (i), there exists a sequence $\{t_k\}$ such that
$$\lim_{k \to \infty} 1^T S \left(-B^T \frac{\partial H}{\partial x}(x(t_k)) - \dot{x}_c(t_k); \dot{x}_c, u^+ + \dot{x}_c\right) = +\infty,$$
which implies $\lim_{t \to \infty} V(x(t_k), \dot{x}_c(t_k)) = +\infty$. Again this is a contradiction with $V < 0$.

In conclusion, $(x(t), \dot{x}_c(t))$ is bounded.

(iii) From the dynamics (30) and (ii), it can be shown that $\frac{d}{dt} \left(-B^T \frac{\partial H}{\partial x}(x) - \dot{x}_c\right)$ is bounded. Combining the facts that $V(x, \dot{x}_c)$ is bounded from below with $V < 0$, we have that $\lim_{t \to \infty} V(x(t), \dot{x}_c(t)) = 0$.

Indeed, suppose $\dot{V}(x(t), \dot{x}_c(t))$ does not converge to zero. In other words, there exist a real $\delta > 0$ and a sequence $\{t_k\}$, satisfying $\lim_{k \to \infty} t_k = +\infty$, such that $\dot{V}(x(t_k), \dot{x}_c(t_k)) < -\delta$. Since $\frac{d}{dt} \left(-B^T \frac{\partial H}{\partial x}(x) - \dot{x}_c\right)$ is bounded, then for each $k = 1, 2, \ldots$, there exists a time interval $I_k$ and an $\epsilon > 0$ such that $|I_k| > \epsilon$, $t_k \in I_k$, and $\forall t \in I_k$, $V(x(t), \dot{x}_c(t)) < -\frac{\epsilon}{2}$. This implies that
$$\lim_{t \to \infty} V(x(t), \dot{x}_c(t)) = -\infty,$$
which is a contradiction with (i). In conclusion, \( \lim_{t \to \infty} \tilde{V}(x(t), \tilde{x}_c(t)) = 0. \) □

**Proof of Theorem 12.** Sufﬁciency: consider now the following function

\[
V(x, x_c) = 1^T S \left( -B^T \frac{\partial H}{\partial x}(x) - \tilde{x}_c; x_c, u^+ + \tilde{x}_c \right) + H(x).
\] (43)

Using Lemma 14 and LaSalle’s principle, it can be shown that \((x(t), \tilde{x}_c(t))\) converges to the largest invariant set \( I \) contained in \( \{(x, \tilde{x}_c) \mid V = 0\} \). Similar to the proof of Theorem 6, if a solution \((x(t), \tilde{x}_c(t))\) \( \in I \), then \( x \) is a constant vector, denoted as \( v \). Furthermore, \( I \) is given as

\[
I = \left\{ (v, \tilde{x}_c) \mid \tilde{x}_c = B^T \frac{\partial H}{\partial x}(v) t + \tilde{x}_c(0), B \text{sat}( -B^T \frac{\partial H}{\partial x}(v) ) 
- B^T \frac{\partial H}{\partial x}(v) t - \tilde{x}_c(0); \tilde{x}_c, u^+ + \tilde{x}_c \right\}, \quad \forall t \geq 0 \right\}. \quad (44)
\]

Suppose now that \( B^T \frac{\partial H}{\partial x}(v) \neq 0 \). Then for \( t \) large enough, we have

\[
0 = \frac{\partial T}{\partial x} (v) B \text{sat}( -B^T \frac{\partial H}{\partial x}(v) - B^T \frac{\partial H}{\partial x}(v) t - \tilde{x}_c(0); \tilde{x}_c, u^+ + \tilde{x}_c )
= \sum_{i=1}^{m} \left( B^T \frac{\partial H}{\partial x}(v) \right) c_i, \quad (45)
\]

where

\[
c_i = \begin{cases} \tilde{x}_c, & \text{if} \left( B^T \frac{\partial H}{\partial x}(v) \right) > 0, \\ u^+ + \tilde{x}_c, & \text{if} \left( B^T \frac{\partial H}{\partial x}(v) \right) < 0. 
\end{cases} \quad (46)
\]

Since the graph is balanced we have \( B \Sigma_m = 0 \), and thus

\[
\sum_{i=1}^{m} \left( B^T \frac{\partial H}{\partial x}(v) \right) c_i = 0. \quad (47)
\]

By the definition of the permission set \( \mathcal{P}(0_m, u^+) \), \( 0 < \tilde{x}_c < u^+_i + \tilde{x}_c_j \) for any \( i, j = 1, 2, \ldots, m \), so

\[
\sum_{i=1}^{m} \left( B^T \frac{\partial H}{\partial x}(v) \right) c_i < 0. \quad (48)
\]

This yields a contradiction. Hence \( B^T \frac{\partial H}{\partial x}(v) = 0 \) and therefore

\[
I = \left\{ (v, \tilde{x}_c) \mid \frac{\partial H}{\partial x}(v) = c \Sigma_m, B \text{sat}( -\tilde{x}_c; \tilde{x}_c, u^+ + \tilde{x}_c ) = 0 \right\}
\]

**Necessity:** first, if the graph \( \tilde{\gamma} \) is not strongly connected, then by the same argument as in Theorem 6, it can be easily seen that \( \frac{\partial H}{\partial x}(x) \) will not reach consensus.

Now we will show that if the strongly connected network is unbalanced, then there exist a constraint interval \([0_m, u^+]\) and an in/out ﬂow d for which there exists \( x_c \in \mathcal{P}(0_m, u^+) \) such that \( Ed = B \Sigma_m \) while \( \frac{\partial H}{\partial x}(x) \) is not converging to consensus.

For the simplicity of exposition we shall take the set of constraint intervals as \([0_m, \Sigma_m]\).

As in the proof of Lemma 13 we let \( k \) be the minimal number of cycles to cover \( \tilde{\gamma} \), and we let \( \mathcal{C} = (C_1, \ldots, C_k) \) be a minimal covering set for \( \tilde{\gamma} \). With some abuse of notation

\[
BC_i = 0, \quad i = 1, \ldots, k \quad (49)
\]

where \( C_i \) is the \( m \)-dimensional vector whose \( j \)-th component is equal to the number of times the \( j \)-th edge appears in the cycle \( C_i \).

In the following, we will prove that there exist \( B^T \frac{\partial H}{\partial x}(v) \neq 0, \tilde{x}_c \in \mathcal{P}(0_m, \Sigma_m), \tilde{x}_c(0) \) and \( \lambda \in \mathbb{R} \), such that

\[
\text{sat} \left( -B^T \frac{\partial H}{\partial x}(v) - B^T \frac{\partial H}{\partial x} (v) t - \tilde{x}_c(0); \tilde{x}_c, \Sigma_m + \tilde{x}_c \right) = \lambda T,
\]

\( \forall t \geq 0, \quad (50) \)

where \( v \) is the equilibrium value of \( x \) as above, and \( T \) is the \( m \)-dimensional vector whose \( i \)-th component is the number of cycles in \( \mathcal{T} \) which contain the \( i \)-th edge. This implies that the system has an equilibrium \((v, \tilde{x}_c)\) which satisfies \( \frac{\partial H}{\partial x}(v) \) span \([\Sigma_m]\).

Consider as above a minimal covering set \( \mathcal{T} = (C_1, \ldots, C_k) \) for \( \tilde{\gamma} \). Let \( T_{max} := \max(\{T_i \mid i = 1, 2, \ldots, m\} \), and denote \( \tilde{E}_i = \{\text{the } i\text{-th edge } | T_i = T_{max}\} \). Every cycle in \( \mathcal{T} \) has at least one non-overlapped edge (see the proof of Lemma 13), and we denote by \( \tilde{E}_2 \) the set of all the non-overlapped edges in the cycles in \( \mathcal{T} \) which contain at least one edge which is overlapped \( \tilde{E}_1 \) times.

In the last step, we will make the flows through the edges in \( \tilde{E}_1 \) reach the upper bounds of the constraints intervals, and the flows through the edges in \( \tilde{E}_2 \) reach their lower bounds. By taking

\[
\text{sat} \left( \frac{\partial H}{\partial x}(v) - \frac{\partial H}{\partial x}(v) t - \tilde{x}_c(0); \tilde{x}_c, \Sigma_m + \tilde{x}_c \right) = \lambda T,
\]

\( \forall t \geq 0, \quad (50) \)

for suitable \( \tilde{x}_c \) and \( \tilde{x}_c(0) \), it follows that \( (50) \) holds. Indeed, in the set \( \tilde{E}_1 \cup \tilde{E}_2 \), the Eq. \( (50) \) takes the form

\[
1 + \tilde{x}_c = \lambda T_{p}, \quad q\text{-th edge belongs to } \tilde{E}_1
\]

\[
\tilde{x}_c = \lambda T_{p}, \quad p\text{-th edge belongs to } \tilde{E}_2.
\]

Now take \( \lambda \) be such that \( \frac{1}{T_{max}} < \lambda < 1 \). Then \( (52) \) contains \(|\tilde{E}_1| + |\tilde{E}_2|\) equations and the same number of variables, and has a unique solution \( \tilde{x}_c, \tilde{x}_c(0) \) such that

\[
0 < \tilde{x}_c < 1 \quad (53)
\]

\[
0 < \tilde{x}_c(0) < 1 \quad (54)
\]

Furthermore, pick \( \tilde{x}_c(0) \) in the third equation of \((51)\) as

\[
\tilde{x}_c(0) = -\lambda T_{r}, \quad r\text{-th edge belongs to } \tilde{E} \setminus (\tilde{E}_1 \cup \tilde{E}_2).
\]

Obviously, there exists \( 0 < \tilde{x}_c < 1 \) such that

\[
\tilde{x}_c(0) < 1 + \tilde{x}_c(0) \quad (55)
\]

In conclusion, there exists an equilibrium \((v, \tilde{x}_c)\) that does not satisfy \( B^T \frac{\partial H}{\partial x}(v) = 0 \), and thus \( \frac{\partial H}{\partial x}(x) \) cannot reach the consensus. □

The above constructive proof is illustrated by the following example.

**Example 5.1.** Consider a directed graph in Fig. 2 with dynamics given by system \((30)\) where \( H(x) = \frac{1}{2} |x|^2 \) and \([u^-, u^+] = [0_8, \Sigma_8]\), that is

\[
x = B \text{sat}(-B^T x - x_c; 0_7, \Sigma_7) + Ed
\]

\[
x_c = B^T x.
\]

The purpose of this example is to show that there exist in/out ﬂows \( d \) satisfying the matching condition for which \( x \) does not converge to consensus. By taking \( Ed = B \Sigma_8 \), where \( x_c = \frac{1}{2} x^T x(0) = (3, 75, 1, 4)^T \) and \( \tilde{x}_c(0) = (1, -1, -1, 1, 1, 1)^T \), the state \( x \) in system \((56)\) will converge to \( v \) with \( v_2 = v_3 > v_5 > v_4 > v_1 \) and
In this analysis is the construction of a C\(^1\) Lyapunov function. We distinguish between the case that the flow constraints corresponding to all the edges allow for bi-directional flow and the case that all the edges only allow for uni-directional flow. For both cases we have derived necessary and sufficient conditions for asymptotic load balancing based on the structure of the graph.

An obvious open problem is the extension of our results to the general case where some of the edges allow for bi-directional flow and others only for uni-directional flow. This is currently under investigation. Many other questions can be addressed in this framework. For example, what is happening if the in/outflows are not assumed to be constant, but are e.g. periodic functions of time; see already [16]. Furthermore, the use of constrained PI-controllers may suggest a fruitful connection to anti-windup control ideas.

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