On the behaviour of streams in angle and frequency spaces in different potentials

Abstract

We have studied the behaviour of stellar streams in the Aquarius fully cosmological N-body simulations of the formation of Milky Way haloes. In particular, we have characterised the streams in angle and frequency spaces derived using an approximate but generally well-fitting spherical potential. We have also run several test-particle simulations to understand and guide our interpretation of the different features we see in the Aquarius streams. Our goal is both to establish which deviations of the expected action-angle behaviour of streams exist because of the approximations made on the potential, but also to derive to what degree we can use these coordinates to model streams reliably.

We have found that many of the Aquarius streams wrap in angle space along relatively straight lines, and distribute themselves along linear structures also in frequency space. On the other hand, from our controlled simulations we have been able to establish that deviations from spherical symmetry, the use of incorrect potentials and the inclusion of self-gravity lead to streams in angle space to still be along relatively straight lines but also to depict wiggly behaviour whose amplitude increases as the approximation to the true potential becomes worse. In frequency space streams typically become thicker and somewhat distorted. In all cases, the energy gradient along the stream seems almost intact in frequency space, but this is not the case for angle space. Therefore, our analysis explains most of the features seen in the approximate angle and frequency spaces for the Aquarius streams with the exception of their somewhat ‘noisy’ and ‘patchy’ morphologies. These are likely due to the interactions with the large number of dark matter subhaloes present in the cosmological simulations. Since the measured angle-frequency misalignments of the Aquarius streams can largely be attributed to using the wrong (spherical) potential, the determination of the mass growth history of these halos will only be feasible once (and if) the true potential has been determined robustly.
4.1 Introduction

In the last two decades much progress has been made on the discovery and characterisation of tidal streams around our Milky Way and in other nearby galaxies (see e.g. Koposov et al. 2012; Martin et al. 2014). Tidal streams consist of stars stripped from disrupted satellites (dwarf galaxies and globular clusters) that move on nearly parallel orbits. As such they constitute extremely sensitive probes of the mass distribution in the host system (Johnston et al. 1999; Ibata et al. 2001; Johnston & Bullock 2004; Law et al. 2005; Law & Majewski 2010; Koposov et al. 2010; Vera-Ciro & Helmi 2013; Sanders & Binney 2013b; Sanderson et al. 2014). This is one of the drivers of the observational and theoretical studies of streams, as one of their ultimate goals is to establish the mass distribution in the dark halo of the Milky Way, which in turn will lead to a better understanding of the nature of dark matter (see e.g. Strigari 2013).

The recently launched Gaia satellite (Perryman et al. 2001) will provide the phase-space coordinates of a vast sample of stars in the Milky Way in the next decade. Together with spectroscopic follow-up surveys of the fainter Gaia stars such as 4MOST (de Jong et al. 2012) and WEAVE (Dalton et al. 2012), these datasets will produce an unprecedented detailed view of our Galaxy. This also means that our understanding of the dynamics of the halo and its streams needs to be sharpened as to maximally exploit the wealth of data that will soon become available (see e.g. Johnston et al. 1999; Law & Majewski 2010; Eyre & Binney 2011; Bonaca et al. 2014).

Action-angle coordinates provide an excellent tool to describe the evolution of streams (Tremaine 1999; Helmi & White 1999). For example, the evolution of stars in angle space is linear with time for a static potential while the actions are adiabatic invariants. The difficulty lies in finding the necessary coordinate transformations. Only for spherical and Staeckel potentials can we directly compute the angles and actions because the Hamilton-Jacobi equation is separable (Goldstein 1950; de Zeeuw 1985; Binney & Tremaine 2008). However, in recent years several approximate schemes have been developed to overcome this problem, based on the work of McGill & Binney (1990) and Kaasalainen & Binney (1994). In this case an appropriately chosen toy potential is used to compute the true action-angles (see e.g. McMillan & Binney 2008; Fox 2012; Sanders & Binney 2014; Bovy 2014). An alternative is to approximate the potential locally by a Staeckel potential (axisymmetric or triaxial); this is known as the ‘Staeckel fudge’ (Binney 2012; Sanders & Binney 2015).

The availability of full phase-space information for a large number of stars in the Gaia dataset will assist greatly in exploiting the power of action-angle coordinates for streams (McMillan & Binney 2008). Actions (and integrals of motion) may be used as well to derive the accretion history of the halo of the Milky Way, because even when a stream is fully phase mixed, relics are left behind that are clumped in this space (Helmi & de Zeeuw 2000). This clumpiness in action space can also be employed to determine the gravitational field in which the stars have evolved because the largest degree of clustering occurs when the actions are computed in the true potential (Sanderson et al. 2014; Peñarrubia et al. 2012). The angles and frequencies can also be used to this end, because streams should lie along straight lines that have the same slope in angle and in frequency space for the correct gravitational potential under the condition that this is static (Sanders & Binney 2013a,b). In Buist & Helmi (2015) we argued that an adiabatically growing potential will cause a small difference in these slopes, or an angle-
frequency misalignment. We also pointed out that there are several other indicators in angle and frequency space that the potential used in the computation may be incorrect, and this allows the determination of the characteristic parameters of the true potential to be separated from its time-evolution.

We continue along that line of study in this Chapter, where we explore the behaviour of streams evolved in fully cosmological N-body simulations. In particular, we characterise the streams in angle and frequency spaces that were obtained using an approximate, good fitting spherical potential. We also run several test-particle simulations to understand and guide our interpretation of the different features we see in the N-body simulations. Our goal is both to establish which perturbations of the action-angle behaviour of streams exist because of the approximations made on the potential, but also to derive to which degree we can use these coordinates to study streams.

This Chapter is built up as follows. In Sec. 4.2 we describe the cosmological N-body simulations we study and show the stream catalogue used. In Sec. 4.3 we discuss the behaviour in the action-angle coordinates that we compute using an appropriately chosen spherical potential. In Sec. 4.4 we present a set of test-particle simulations evolved in an axisymmetric potential that are loosely based on the streams found in the cosmological simulations, and show their true action-angle behaviour. In Sec. 4.5 we demonstrate why streams are generally on straight lines in angle and in frequency space. In Sec. 4.6 we determine the impact of computing the action-angles of the test-particle simulations of Sec. 4.4 in several incorrect potentials, and discuss the effect of self-gravity. We end in Sec. 4.7 with a discussion and conclusions.

4.2 Streams in cosmological simulations

4.2.1 Description of the Aquarius project and its stellar haloes

The Aquarius project (Springel et al. 2008; Navarro et al. 2010) consist of a set of six re-simulations of Milky-Way mass (∼ 10^{12} M_⊙) dark matter haloes extracted from a larger cosmological parent simulation (Gao et al. 2008). They were selected to have no close massive neighbours at z = 0, and form late-type galaxies when evolved using semi-analytic galaxy formation models. We use 5 out of the 6 haloes, Aq-A to Aq-E in this work, because halo Aq-F experiences a major merger at recent times and hence does not resemble the Milky Way (Wang et al. 2011).

We extract streams from the accreted component of stellar haloes modelled using the Durham semi-analytic model GALFORM. Cooper et al. (2010) have associated stellar populations with dark matter particles in the simulations via a ‘tagging’ scheme. Lowing et al. (2015) took this a step further and generated individual stars from these populations by re-sampling the dark matter particles and using stellar population synthesis modelling. Another difference between these works is that Cooper et al. used the Bower et al. (2006) version of GALFORM, while Lowing et al. used the Font et al. (2011) version which has improved physics on dwarf galaxy scales that makes model satellites more similar to those observed around the Milky Way. Here we use the public catalogue that Lowing et al. offer online\(^1\), which has all stars with magnitude M_g < 7.

\(^1\) http://galaxy-catalogue.dur.ac.uk:8080/StellarHalo
The Lowing dataset re-samples the dark matter particles’ positions and velocities in a way that aims to preserve their distribution in phase-space. To observe streams in the halo of the Milky Way it is common to select the bright giant stars such as the Red Giant Branch (RGB) or Main Stream Turnoff (MSTO) stars. We noticed that some of the thinner simulated streams look quite clumpy when only using the RGB stars, most likely because of the re-sampling of the dark matter particles. Since our interest is only in the dynamical features of the streams, we instead decided to use the source (‘tagged’) dark matter particles. To this end we have matched the dark matter ID’s to those in the outputs of the Aquarius dark matter simulations and found the positions and velocities of the source particles for the Lowing dataset. Whether we would have used the RGB stars or the dark matter particles does not matter much for the number statistics as both sets have similar sizes.

We aligned the coordinate system to that of the parent dark halo by using the principal axes determined at 100 kpc from the centre by Vera-Ciro et al. (2011). These authors used a method based on the reduced inertia tensor method (Allgood et al. 2006) which closely follows isodensity contours. The z-direction is chosen along the major axis for haloes Aq-A to Aq-D, and along the minor axis for Aq-E. Further details on the shapes are discussed in Sec. 4.4.1. This coordinate system aligns the particles to the planes of symmetry of the parent halo. Depending on the approximations we use for the potential this can give a more satisfactory understanding of the dynamics of these streams.

### 4.2.2 Mass distribution of the Aquarius haloes

The Aquarius haloes can be fit with a spherical mass distribution, although they are not really spherical. Here we use the Navarro-Frenk-White profile (Navarro et al. 1996, 1997, hereafter NFW) that provides a relatively good description of the mass distribution in the regions where we study streams ($r \sim 50 – 100$ kpc, see Springel et al. 2008; Navarro et al. 2010), and we prefer it because of its simplicity and computational efficiency compared to the slightly better fitting Einasto profile (Einasto 1965). We computed the spherically averaged mass profiles for each of the main haloes and fitted these using the parametrisation in scale mass $M_s$ and scale radius $r_s$ as described in Buist & Helmi (2014).

In Fig. 4.1 we show the results obtained when fitting the spherical circular velocity profile for halo Aq-A and Aq-D (see also Navarro et al. 2010). Halo Aq-D is much better fitted by an NFW profile than halo Aq-A, which has two (local) maxima in the circular velocity profile. Halo Aq-A is rather triaxial and perhaps this is due to its shape. For our purposes, the fits in Fig. 4.1 are quite reasonable and we will use these for our analyses throughout this Chapter.

The streams in the Aquarius haloes have evolved in a fully dynamic time-dependent potential. In Buist & Helmi (2014) we fitted the time-evolution of the Aquarius main haloes assuming spherical symmetry, according to a model in which the growth of the scale mass $M_s$ and scale radius $r_s$ with redshift $z$ are given by

\[
M_s(z) = M_{s,0} \exp(-2a_g z),
\]

\[
r_s(z) = r_{s,0} \left( \frac{M_s(t)}{M_{s,0}} \right)^{1/\gamma}.
\]  

(4.1)

Here $a_g$ is the formation epoch: for an earlier formation epoch, the final mass of the halo is sooner in place, and the growth rate is lower at late times. The parameter $\gamma$ is fixed.
Figure 4.1: The enclosed mass (left) and circular velocity (right) for halo Aq-A (top panels) and halo Aq-D (bottom panels) with NFW fits to the circular velocity (blue lines). The red dashed curve shows the profile when we match the peak velocity. For an NFW the position of this maximum depends only on $r_s$ and the magnitude only on $M_s$. The radial range is from the convergence radius (Power et al. 2003; Navarro et al. 2010) up to $r_{200}$ (see Springel et al. 2008).

Figure 4.2: Scale mass $M_s$ as a function of time for the best fitting growth rate $a_g$ and inside-out growth ($\gamma = 2$) for halo Aq-A and halo Aq-D.
at 2, which ensures that the halo grows inside-out. As the scale mass and scale radius are increasing in our model, the particles’ orbits shrink with time (i.e. $r/r_s$ is decreasing). The results for halo Aq-A and Aq-D are shown in Fig. 4.2.

### 4.2.3 Selection of streams

Not all objects in the Aquarius stellar haloes are apparent as stream-like structures, for example this may be the case if the object is too small or if it has not been significantly disrupted. In each of the 5 Aquarius haloes there are about 100-200 individual progenitors, of which about 20% have produced visible stream-like features by $z = 0$. We selected these by eye and verified this selection by checking that they also appeared streamy in their approximate angle coordinates, as explained in the next sections. Fig. 4.3 shows that halo Aq-A, Aq-C and Aq-D are hosting large streamy structures out to very large radii, while halo Aq-B and Aq-E are typically more compact (see also Cooper et al. 2010). To avoid flooding this Chapter with figures we focus next on the streams from halo Aq-A and Aq-D as these show many interesting streams up to large radii. We also impose a lower limit of at least 500 dark matter ‘tagged’ particles for each progenitor to exclude the really small streams. This limit corresponds to a ‘tagged’ dark matter mass of $\sim 7 \times 10^6 \, M_\odot$. Further, we selected only some of the shell-like structures that appear near the long axis (here the $z$-axis) because they are often phase mixed, and these structures correspond to more radial orbits that contribute a lot to the inner regions of, for example, halo Aq-D.

### 4.2.4 The morphology of selected streams

Streams consist of groups of stars that have similar orbits, and the relatively small variance in their orbits creates structures whose trajectory follows closely the orbit of the progenitor (Jin & Lynden-Bell 2007; Binney 2008), although not exactly (Choi et al. 2007; Eyre & Binney 2009; Sanders & Binney 2013b). We can therefore analyse the streams in terms of the orbits permitted by the potential.

The individual streams for halo Aq-A and Aq-D with more than 500 dark matter particles are shown in Fig. 4.4. The IDs of the streams given in this figure correspond one-to-one with tree-IDs in the Lowing catalogue. The colours in the figure represent the binding energy computed in the best fitting spherical NFW potential (yellow is the most bound, blue is the least bound). Each of the streams was selected to have at least one stream-like feature or loop.

Typically the streams consist of one or more petals, but they are not the clean rosette-like figures seen in the case of spherical potentials (see e.g. Buist & Helmi 2015), because in a triaxial potential many more orbit families exist. The base class of orbits in a triaxial potential are the box and tube orbits around the major, intermediate and minor axes. Box orbits get arbitrarily close to the centre of the potential, a property they share with purely radial orbits in a spherical potential and do not have a sense of rotation. Tube orbits circulate about one of the axes of the potential and never get to the centre of the potential, which they have in common with loop orbits in a spherical potential. For example, stream S56 in halo Aq-D and stream S98 in halo Aq-A are distributed on a structure similar to that defined by a box orbit, while stream S108 in halo Aq-A seems to be similar to a ‘fish’-orbit (3:2 resonance, see e.g. Miralda-Escudé & Schwarzschild 1989; Merritt & Valluri 1999; Binney & Tremaine 2008).
As a first step in our characterisation of streams we now focus on how close the streams follow orbits in the best fitting spherical potential. The initial conditions of the orbit are determined from a particle located in the highest density portion of the stream, which typically means we take a bound particle in the progenitor. We integrate this particle forwards and backwards for 4 Gyr in our best fitting NFW potential to approximately match the streams’ length.

Fig. 4.5 shows the results in the $y$-$z$ plane. For the thinner streams with only one or two wraps the orbits seem to be able to follow the stream, such as for halo Aq-A S158 and S151, but also the much thicker S164 has at least one loop reasonably matched. Many of the heavier streams that have more wraps show a big difference in radial extent when compared to the orbits, with halo Aq-A S104 giving one of the least satisfactory results. This is not too surprising given that the spherical potential only supports a very limited range of orbit families. Also, a single orbit cannot fit the stream’s radial extent because this depends on the range of energies of the particles, and this is particularly large for some of our objects.

In Fig. 4.6 we show the streams in the $r$-$v_r$ plane with the corresponding orbits overlaid. In this figure we see again that some streams have many more wraps, and that our orbits integrated in a spherical potential are not able to match their radial extent. But in
CHAPTER 4: STREAMS IN ANGLES AND FREQUENCIES IN DIFFERENT POTENTIALS

Figure 4.4: The stream-like objects from halo Aq-A (top) and halo Aq-D (bottom) in the $y$-$z$ projection as in the top panel of Fig. 4.3. Shown here are the dark matter source particles of the streams, and the numbers indicate the object IDs used throughout this Chapter. The colours represent the energy gradient computed using the best-fitting spherical NFW potential, with yellow the most bound particles, and blue those least bound. The streams have been sorted by dark matter mass, with the lightest stream on the top-left and the most massive on the bottom-right.
Figure 4.5: The y-z projection for spherical orbits in the best fitting spherical potential to halo Aq-A (top panels) and halo Aq-D (bottom panels) overplotted on the streams from Fig. 4.4. The orbit is taken from a particle (indicated with a red cross) near or in the progenitor and has been evolved 4 Gyr forward and backwards in time assuming a static potential. The colours of the stream particles indicate the energy gradient as in Fig. 4.4.
Figure 4.6: The $r$-$v_r$ projection for spherical orbits in the best fitting spherical potential to halo Aq-A (top panels) and halo Aq-D (bottom panels). The colour and labelling schemes are the same as in Fig. 4.5.
all cases, the orbit seems to match reasonably at least a single wrap. Often the progenitor
appears as a vertical diamond shaped object in the stream. Its location coincides quite
well with the location of the highest density of the stream, which was chosen to define
the initial conditions for the orbital integration and is indicated by the red cross. Near
the progenitors of halo Aq-D S72 and S73 we see very long arms that at the ends seem to
dissolve in the stream. Most likely these particles are not yet completely unbound from
the progenitor (see e.g. Gibbons et al. 2014)

We conclude that our orbits do reproduce some of the wraps of the streams, but
typically they lack the complexity that is seen in the Aquarius simulations’ streams. This
is partially related to the size of the systems, but for a major part also to the kind of orbits
the actual streams are on.

4.3 Action-angle behaviour of Aquarius streams in spherical potentials

We now investigate the properties of our streams in action-angle space, since in this
space the behaviour of streams is expected to be particularly simple. In this section we
assume the underlying potential is spherical and represented by our best fitting NFW
potential. Staeckel potentials are another example of separable potentials that allow
the direct computation of the angles, frequencies and actions associated with a stream
(Helmi & White 1999; de Zeeuw 1985), and these are used in Sec. 4.4. For other more
complex, non-separable potentials approximations for the computation of e.g. the actions
are necessary (see Kaasalainen & Binney 1994; McMillan & Binney 2008; Binney 2012;
Sanders 2012; Sanders & Binney 2014, 2015; Bovy 2014) but their application to our
streams is beyond the scope of this Chapter.

The procedure to compute the actions, angles and frequencies in a spherical poten-
tial has been worked out before (see Goldstein 1950; Binney & Tremaine 2008). When
applied to our streams, we lose a small fraction of the particles in this procedure, for ex-
ample because some of the numerical integrals do not converge well if the particles are
almost unbound in the approximated spherical potential. Note that in a spherical poten-
tial, $\Omega_\phi$ and $\Omega_\theta$ are equal (apart from a possible sign difference, see Binney & Tremaine
2008). However the corresponding angles derived for the particles in our streams are
different because they depend on the positions of the particles in configuration space and
are computed using a spherical approximation to the true potential.

In Figs. 4.7 and 4.8 we show the distribution of particles in the spaces $\theta_r-\theta_\phi$ and $\theta_r-
\theta_\theta$ respectively. The most striking feature in these figures is that each of the streams is
distributed along more or less straight lines as we had seen in Buist & Helmi (2015), even
though the host potential is really not spherical. For all streams, the behaviour in $\theta_r-\theta_\phi$
space is generally cleaner and the individual streams are seen more clearly than in $\theta_r-\theta_\theta$
space. This difference in behaviour is due of course, to the haloes being non-spherical.
Interestingly the range in the slopes of the streams in $\theta_r-\theta_\phi$ for Aq-D is remarkably small,
which may be related to the fact that halo Aq-D is more symmetric than halo Aq-A.

We fit straight lines to the streams, in analogy to what we did in Buist & Helmi
(2015). Our method to fit lines to the distributions of streams in angle space is discussed
in Appendix 4.A. We fit the distributions in $\theta_r-\theta_\phi$ and $\theta_r-\theta_\theta$ separately, because we know
the potential is not spherical and, as we just saw, the behaviour in both spaces is very
Figure 4.7: The $\theta_r-\theta_\phi$ space for the streams selected from halo Aq-A (top panel) and Aq-D (bottom panel), computed with the best fitting spherical potentials. The angles were centred around the most bound particle in the progenitor, or around a particle closest to the highest density in $\theta_r-\theta_\phi$ space. The colours indicate the energy gradient assuming the spherical potential. The distributions were fitted with straight lines after removing generously particles still bound to the progenitor. The insets show the fitted slope and its error as estimated from bootstrapping 20 times.
Figure 4.8: As Fig. 4.7, but now the $\theta_r-\theta_\theta$ angle space for the streams selected from halo Aq-A (top panel) and Aq-D (bottom panel).
different. Particles that still form a bound progenitor are removed in the fitting procedure because they are not expected to follow the general action-angle behaviour of the stream. The result of this removal is clearly visible for example, for Aq-D S73 and Aq-A S164 where we see a gap in the middle of the distribution in angle space.

The fitting procedure puts into evidence several noticeable distortions in angle space because sometimes the different wraps of a stream are not on parallel lines, such as for halo Aq-A S108 and S116. This clearly poses a challenge to the determination of the slope. For this reason we have to take the slopes determined with our method generally with some care (especially in the case of S116) as the quoted errors do not account for such systematic uncertainties. A quick visual inspection usually helps to evaluate the outcome and the reliability of the fit. Much more subtle is the deviation seen in the bottom-left of Aq-A stream S158, which shows a straight structure that suddenly deviates and spreads out.

The slopes in \( \theta_r - \theta_\phi \) space vary much more than those in \( \theta_r - \theta_\vartheta \) space, which is especially seen in halo Aq-D, where the slopes in \( \theta_r - \theta_\phi \) are all very close to 0.5. In a spherical potential, this slope would be the same as in \( \theta_r - \theta_\vartheta \) space. Generally, we also notice that the streams in halo Aq-D seem less perturbed than the streams in halo Aq-A.

In our selection of streams we also included several objects on very radial orbits, and whose debris is distributed in an hour-glass shape in configuration space, such as Aq-A S98, and Aq-D S56 and S98. In \( \theta_r - \theta_\vartheta \) space they seem very mixed and do not show very distinct structures, but there still is some structure present and this is fitted by our algorithm. These objects are typically more massive and close to the centre of the halo and therefore much phase mixed, especially in \( \theta_r - \theta_\vartheta \).

In Fig. 4.9 we show the frequency distributions of the streams. These follow closely a straight line, although sometimes they are quite thick. For Aq-A S108 and S116 the width in frequency space is not everywhere the same. This is one of the signatures expected when using the wrong potential to compute the frequencies, although it is possible that the features were caused by the process of disruption of the progenitor or interactions with dark matter subhaloes.

The determination of the fitted slopes is more robust in frequency space, and these show a considerably smaller range than the slopes in angle space. Furthermore, the slopes in angle space and frequency space differ, especially when comparing to the \( \theta_r - \theta_\vartheta \) space to frequency space. They are expected to be equal in the true (static) potential (Sanders & Binney 2013b,a). In Buist & Helmi (2015) we found that in a time-dependent potential, the slope in angle space is steeper than in frequency space (\( S(\Delta \theta) > S(\Delta \Omega) \)). Therefore, since the potentials in our N-body simulations have grown in time we might expect the angle space slope to be larger. For some of the streams in our sample this is indeed the case, but there are quite a few streams for which this does not hold, such as Aq-D S116 and Aq-A S112, for which the magnitude of the angle-frequency difference is far greater than what can be expected for (adiabatic) evolution of the halo. Also in \( \theta_r - \theta_\vartheta \) space many streams are not in line with our predictions. Overall we find that in \( \theta_r - \theta_\vartheta \), 27 of the 45 streams have a larger slope in frequency than in angle space, i.e. they do not follow the expected behaviour, and for \( \theta_r - \theta_\vartheta \), 17 out of the 45 streams. We therefore attribute these differences to the incorrect potential, and not to time-dependence.

We also expect an energy gradient to be present along a stream (Buist & Helmi 2015), and although this is visible in frequency space it is less clear in angle space. In the \( \theta_r - \theta_\vartheta \) space this is more difficult to see because the streams tend to overlap, for example for
Figure 4.9: Frequencies computed with the best fitting spherical potentials for our selection of streams from halo Aq-A (top panels) and Aq-D (bottom panels). We only show one frequency space, because in a spherical potential $\Omega_\phi = \Omega_\theta$. The frequencies are centred around the same particle as in Figs. 4.7 and 4.8. The distributions were fitted with straight lines after removing generously the bound particles from the progenitor. The colours indicate the energy gradient for the spherical potential. The insets show the fitted slope and estimated errors.
Aq-D S56 and S98. This is not unexpected, as even small (of order 10\%) differences in the characteristic parameters of the potential can lead to the energy gradient being lost, even when the stream itself has a normal appearance in angle space. An exception is Aq-A S164 in $\theta_r-\theta$-$\phi$-space, which does seem to have a continuous energy gradient when following the stream along its various wraps. We experimented with the enclosed mass for the particular case of Aq-A S158 to see if the behaviour in angle space and the energy could be improved. However we were unable to remove the bend at the bottom-left in $\theta_r-\theta$-$\phi$ space, a behaviour that in the spherical case is indicative of wrong values of the characteristic parameters of the potential. It is clear from the analysis presented in this section that it is the shape of the potential that is wrong rather than the value of the enclosed mass.

We conclude that streams can still look rather regular in angle and frequency space when using the incorrect shape of the potential, even if they evolved in a potential that grew via accretion and merging. In later sections of this Chapter we will investigate what are the conditions for streams to be distributed along such straight lines.

4.4 Test-particle simulations of streams in axisymmetric potentials

We will now study the behaviour in action-angle space of streams evolved in an axisymmetric Staeckel potential. To this end we ran several test-particle simulations. The initial orbital conditions for the streams are the same as for the orbits in the spherical potential in Sec. 4.2.4 and the potential is based on the best fitting spherical NFW potential that we derived in Sec. 4.2.2. In axisymmetric potentials there is a richer variety of orbits than in a spherical potential, and in the case of a Staeckel potential, the true action-angle coordinates can be quickly found.

4.4.1 Potential set-up

For the orbit integrations we use the Kuzmin-Kutuzov axisymmetric Staeckel potential (Dejonghe & de Zeeuw 1987). In cylindrical coordinates this potential has the following form

$$\Phi_K(R,z) = -\frac{GM_K}{\sqrt{a_K^2 + c_K^2 + R^2 + z^2 + 2\sqrt{a_K^2 c_K^2 + R^2 c_K^2 + a_K^2 z^2}}}$$

and in prolate ellipsoidal coordinates it takes simple expression

$$\Phi_K(\lambda, \nu) = -\frac{GM_K}{\sqrt{\lambda + \sqrt{\nu}}}$$

where the relations between $\lambda$, $\nu$ and $R$, $z$ can be expressed as

$$\lambda \nu = c_K^2 R^2 + a_K^2 z^2 + a_K^2 c_K^2,$$
$$\lambda + \nu = R^2 + z^2 + a_K^2 + c_K^2.$$

This potential reduces in the spherical limit ($a_K = c_K$) to Henon's isochrone potential (Dejonghe & de Zeeuw 1987; Henon 1959). For convenience we link the parameters $M_K$, $a_K$
and $c_K$ to the isochrone scale radius $r_{iso}$ and scale mass $M_{iso}$, and introduce the flattening parameter $q_K \equiv c_K/a_K$. We define the isochrone scale mass as $M_{iso} = M_{iso}(r_{iso})$, such that

$$M_K = M_{iso} \left( \frac{3}{\sqrt{2}} - 2 \right)^{-1},$$

$$a_K = r_{iso} \frac{2}{1 + q_K},$$

$$c_K = r_{iso} \frac{2q_K}{1 + q_K}.$$  (4.5)

These relations satisfy $a_K + c_K = 2r_{iso}$, i.e. the isochrone scale radius is the average of the Kuzmin-Kutuzov axis lengths.

To set the numerical values of the parameters we proceed as follows. We first set up the isochrone potential by choosing a radius $r_{fix}$ at which the enclosed mass and the slope of an isochrone potential equals that of the best-fitting NFW potential of the Aquarius haloes. This ensures that around $r_{fix}$ the (spherically averaged) mass distributions are similar. We choose $r_{fix} = 50$ kpc because most of the streams are located around 50-100 kpc from the halo centre, and so we will roughly match the more robust parameter in mass modelling, i.e. the enclosed mass (Breddels & Helmi 2013; Sanderson et al. 2014).

Having chosen $r_{fix}$, we first match the slope of the mass profile and then proceed to set the enclosed mass. The logarithmic slope $\kappa$ is

$$\kappa_{NFW}(x) \equiv \frac{\partial \log M_{NFW}(x)}{\partial \log x}$$

with $x = r/r_s$ and $r_s$ the scale radius of the NFW potential such that $M_{NFW}(r_s) = M_s$. The condition of equal slopes at $r_{fix}$ is

$$\kappa_{NFW}(r_{fix}/r_s) = \kappa_{iso}(r_{fix}/r_{iso}),$$

which we can invert to find $r_{iso}$. The scale mass $M_{iso}$ ensures the enclosed mass is equal at $r_{fix}$

$$M_s \frac{A_{NFW}(r_{fix}/r_s)}{A_{NFW}(1)} = M_{iso} \frac{A_{iso}(r_{fix}/r_{iso})}{A_{iso}(1)},$$

where $A_i$ are the normalised radial mass profiles for the used potentials.

In Fig. 4.10 we show the result of our fitting procedure for halo Aq-D and Table 4.1 summarises the parameters obtained for all Aquarius haloes. The overall mass and velocity profiles differ significantly outside $r \sim r_{fix}$. We note that the correspondence of $r_{fix}$ with the location of the maximum circular velocity for the best fitting NFW potential to halo Aq-D is a coincidence.

The next step in the potential set-up is to determine the characteristic parameters of the Kuzmin-Kutuzov potential. For this we may use Eq. (4.5). This requires us to find the axis ratios of the density distribution, and we use those derived by Vera-Ciro et al. (2011) for the Aquarius haloes at $r_{fix} = 50$ kpc. All main haloes from the Aquarius simulations are triaxial, and therefore have two axis ratios: $q = c/a$ and $s = b/a$ (where $a \geq b \geq c$). A useful quantity is the triaxiality parameter (Franx et al. 1991)

$$T = \frac{a^2 - b^2}{a^2 - c^2} = \frac{1 - s^2}{1 - q^2},$$

(4.9)
Table 4.1: Potential parameters

<table>
<thead>
<tr>
<th>Halo</th>
<th>$M_s$ $(10^{11} M_\odot)$</th>
<th>$r_s$ (kpc)</th>
<th>$M_{iso}$ $(10^{11} M_\odot)$</th>
<th>$r_{iso}$ (kpc)</th>
<th>$q(50)$</th>
<th>$q_K$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1.55</td>
<td>15.33</td>
<td>1.53</td>
<td>19.44</td>
<td>1.67</td>
<td>2.27</td>
</tr>
<tr>
<td>B</td>
<td>1.07</td>
<td>20.34</td>
<td>0.90</td>
<td>21.67</td>
<td>1.63</td>
<td>2.12</td>
</tr>
<tr>
<td>C</td>
<td>1.74</td>
<td>15.72</td>
<td>1.69</td>
<td>19.63</td>
<td>1.93</td>
<td>3.01</td>
</tr>
<tr>
<td>D</td>
<td>2.04</td>
<td>23.37</td>
<td>1.58</td>
<td>22.85</td>
<td>1.53</td>
<td>1.87</td>
</tr>
<tr>
<td>E</td>
<td>1.20</td>
<td>16.41</td>
<td>1.14</td>
<td>19.96</td>
<td>0.56</td>
<td>0.48</td>
</tr>
</tbody>
</table>

Figure 4.10: Enclosed mass and circular velocity curves for the best fitting NFW profile to halo Aq-D, and the isochrone potential that matches the enclosed mass and slope at $r_{fix} = 50$ kpc.

Figure 4.11: Triaxiality parameter $T$ as a function of the axis ratios $q = c/a$ and $s = b/a$, where $a \geq b \geq c$. The red crosses indicate the values for the 5 Aquarius haloes at 50 kpc from Vera-Ciro et al. (2011). For $T < 1/3$ the halo is nearly oblate (halo Aq-E), for $T > 2/3$ the halo is nearly prolate (halo Aq-A to Aq-D).
which is zero for a strictly oblate halo and equals unity for a strictly prolate halo. In Fig. 4.11 we show the triaxiality parameter as a function of $s$ and $q$, and the red crosses show the axis ratios at $r_{\text{fix}}$ from Vera-Ciro et al. (2011). Haloes Aq-A to Aq-D have triaxiality parameters between $T = 2/3$ and 1 at this radius, making them more prolate, and halo Aq-E falls between $T = 0$ and 1/3 and is more oblate. None of these haloes falls in the most triaxial range ($T = 1/3$ to $2/3$, see also Warren et al. 1992), although at larger radii the haloes do become more triaxial (Vera-Ciro et al. 2011). After distinguishing between prolate and oblate haloes, we define the axisymmetric equivalent axis ratio of the density $q_\rho$ as

\[
q_\rho = \begin{cases} 
  \frac{2a}{b+c} & \text{if } T \geq 0.5, \\
  \frac{2c}{a+b} & \text{if } T < 0.5.
\end{cases}
\] (4.10)

We list the axis ratios for all haloes in the fifth column of Table 4.1. Halo Aq-D has $q_\rho = 1.53$ ($q_\rho = 1.39$ when the axis ratios $b/a$ and $c/a$ are determined at 100 kpc). To find $q_K$ we need the density profile of the Kuzmin-Kutuzov potential (Dejonghe & de Zeeuw 1987), which is given by

\[
\rho_K(\lambda, v) = \frac{M_K c_K^2}{4\pi} \frac{\lambda v + a_K^2 (\lambda + 3\sqrt{\lambda} v + v)}{(\lambda v)^{3/2} (\sqrt{\lambda} + \sqrt{v})^3},
\] (4.11)

here expressed in prolate spheroidal coordinates $\lambda$, $v$. We then solve numerically for the value of $q_K$ such that the isodensity-contour at $R = 50$ kpc has the desired axis ratio $q_\rho$. For halo Aq-D this results in $q_K = 1.87$, and the values for the other haloes are given in the last column of Table 4.1. The shape of the isodensity surfaces for halo Aq-D assuming the Kuzmin-Kutuzov form are shown in Fig. 4.12, and it can be seen that the distribution is quite prolate.

In Fig. 4.13 we show the axis ratios of iso-contours of the potential $q_\Phi$ and density $q_\rho$ as a function of radius for $q_K = 1.87$. As expected, the shape of the potential is much rounder than the density distribution. Note that $q_\rho(100) = 1.44$ is not to far off from the Aq-D value of 1.39 at 100 kpc.

### 4.4.2 Streams set-up

As mentioned earlier we start our test-particle simulations using a subset of the initial positions and velocities of the spherical orbits shown in Fig. 4.5, i.e. effectively extracted from the Aquarius simulations. The stream progenitor is formed by 10,000 particles that follow an isotropic Gaussian distribution in position and velocity, characterised by dispersions $\sigma_{\text{pos}} = 0.3$ kpc and $\sigma_{\text{vel}} = 10$ kpc/Gyr respectively (the same as the ‘Sculptor’ progenitor in Buist & Helmi 2015). There is no self-gravity, and the particles are released at once, implying there is no energy gap between the leading and trailing arms. This simplification is justified because the dynamics of the individual particles are essentially the same after they are released (see also the discussions in Sanders & Binney 2013a; Gibbons et al. 2014; Buist & Helmi 2015, and Sec. 4.6.4 of this Chapter). The orbits are integrated for 10 Gyr in the Kuzmin-Kutuzov potential with the parameters derived from Aquarius halo D.
Figure 4.12: Density contours of the Kuzmin-Kutuzov potential for $q_K = 1.87$, meant to represent halo Aq-D in the axisymmetric limit.

Figure 4.13: Axis ratios of the potential $q_\phi$ and the density $q_\rho$ as a function of cylindrical radius $R$ for the Kuzmin-Kutuzov axisymmetric potential used to represent halo Aq-D. The axis ratios were derived from the isocontours shown in Fig. 4.12.
Figure 4.14: Spatial distribution of our selection of test-particle streams after 10 Gyr of integration in the axisymmetric Kuzmin-Kutuzov potential. The colours represent the energy gradient with the most bound particles in yellow and those least bound in blue. The z-axis is aligned with the major axis of the potential.

We note that our goal is not to reproduce the original streams from the Aquarius haloes in the axisymmetric limit. In that case it would be better to use a fuller representation of the stream following the streakline method (Küpper et al. 2012; Bonaca et al. 2014) or possibly even including the gravity of the progenitor (Gibbons et al. 2014). Instead our interest is on the streams well after they have formed and to ensure this we focus on long streams. Therefore here we discuss only a selection of streams that feature several wraps at the final time.

The selection is shown in Fig. 4.14, with the inset labels corresponding to the original Aq-D streams IDs they are based on. The streams mean trajectories often appear similar to tube orbits around the major axis (here the z-axis), as expected for a prolate axisymmetric potential.

4.4.3 Action-angles in the true potential

For each of the streams shown in Fig. 4.14 we computed the angles and frequencies in the true Kuzmin-Kutuzov potential. For a discussion how to compute the actions in an
Figure 4.15: Test-particle streams in the true $\lambda - \phi$ angle and frequency spaces after 10 Gyr of evolution. The colours represent the energy gradient with the most bound particles in yellow and those least bound in blue. The panels have been centred on the current position of the centre of mass of the progenitor system. The insets give the best-fitting slopes to the distributions as well as the angle-frequency misalignment $\Delta S$.

axisymmetric potential we refer to Helmi & White (1999); Sanders (2012). The results are shown in Fig. 4.15 and Fig. 4.16 for both projections of frequency space (top) and angle space (bottom). In angle space, the streams are distributed along straight lines just as they are in frequency space, but they are much longer as they have spread out in the 10 Gyr of evolution.

The insets in the figures show the slopes obtained from fitting straight lines to the distributions, where the error is estimated from bootstrapping the fits 20 times. We also give in the top panels the angle-frequency misalignment $\Delta S = S(\Delta \theta) - S(\Delta \Omega)$, with the corresponding total error. Notice that typically $\Delta S \sim 0.01 - 0.02$, which is larger than found for the streams evolved in static spherical potentials in Buist & Helmi (2015). The reason for this difference lies mostly in the fact that the progenitor of the streams shown in Figs. 4.15 and Fig. 4.16 is larger, which results in systematically thicker streams. As a consequence the determination of the slope is less precise. Furthermore, the additional width causes streams to overlap more often, which makes it harder to group the
individual particles in the corresponding wraps, which especially happens in the $\theta_\lambda - \theta_\nu$ space. Finally, shorter streams do not have the same slope in frequency and angle space because their initial spread in angle space is still imprinted in the trajectories followed by the particles in angle space (see Fig. 15 of Buist & Helmi 2015).

To make more evident the dependence on the properties of the progenitor, in Fig. 4.17 we show a stream on the same orbit as simulation S140 but now for the Carina-like progenitor ($\sigma_{\text{pos}} = 0.1$ kpc and $\sigma_{\text{vel}} = 5$ kpc/Gyr) used in Buist & Helmi (2015) evolved for 10 Gyr. We see that the angle-frequency misalignment has been reduced significantly, which can be attributed mostly to the improvement in the fitting, although the smaller initial angle spreads also matter.

The values of the slopes in $\theta_\lambda - \theta_\phi$ space for most of the streams are $\approx 0.6$, similar to those found in Buist & Helmi (2015) for a spherical NFW potential. There are two exceptions, S103 and S118, for which the slope is $\sim 0.3$. These are the innermost streams in our sample (see Fig. 4.14), and they experience a more prolate halo because of that, which in turn leads to a shallower slope. In a spherical potential the corresponding slope in $\theta_r - \theta_\phi$ space would not be expected to be smaller than 0.5 because for individual particles $0.5 \leq \Omega_\phi / \Omega_r \leq 1$ always.
Figure 4.17: Stream S140 but now with a Carina-like progenitor, after 10 Gyr of evolution in the Kuzmin-Kutuzov potential. The colours and insets are the same as in Fig. 4.15. With the smaller progenitor the stream is much thinner and the error on the fitting is significantly reduced, which basically removes the angle-frequency misalignment.

In the $\lambda - \nu$ angle and frequency spaces, quite a few of the streams have slopes larger than 1. In the limiting case of a spherical potential $\Omega_\nu \sim 2 \Omega_\theta$, because $J_\nu$ does not reduce to $J_\theta$ in the spherical case. We also note that the energy gradient is smooth along the stream, but this is not always apparent in cases where the different wraps of the stream overlap, such as the $\theta_\lambda - \theta_\nu$ space for streams S138 and S56. In those cases, the fitting becomes also more difficult because it is harder to group the particles in the different wraps as required by our fitting method.

4.5 Why do streams follow straight lines?

In the previous sections we saw that streams appear as elongated structures in angle and frequency space for all cases explored. In a static potential, the angles and frequencies are related as $\theta_i = \theta_i(0) + \Omega_i t$. For a stream (i.e. an ensemble of particles) we are more interested in the spread, which evolves as $\Delta \theta_i \sim \Delta \Omega_i t$ for $t \gg 0$. The frequencies are constant with time, and therefore a spread in angles is a direct consequence of the initial distribution of the particles in the progenitor, provided that the potential has not evolved in time and one can neglect self-gravity (and the initial spread in angles). In this section we try to build some intuition on why streams appear on straight lines in frequency and angle space.

In the computations of Sanders (2012) this was changed by making the $\Omega_\nu$ angle correspond directly to the $z$-motions, but we chose to keep with the original definitions of de Zeeuw (1985), although later in the Chapter we show $2 \Delta \theta_\nu$ for spherical potentials to make comparison between streams evolved in different potentials more direct.
4.5: Why do streams follow straight lines?

4.5.1 Spherical potentials

Figure 4.18: Example of the action distributions (top panels) for three streams from the ‘Carina’-like progenitor from Buist & Helmi (2015) in a spherical potential and the resulting frequency distributions (bottom panels). The colours indicate the energy gradient. For comparison we show the direction of the largest eigenvalue of the Hessian, \( \hat{e}_1 \) with a black dashed line and the direction of the progenitor orbit in angle space with a red dashed line.

In a spherical potential, two independent frequencies exist, \( \Omega_r \) and \( \Omega_\phi \), since \( 3\Omega_\theta = \Omega_\phi \), which depend on the actions \( J_r \) and \( L = J_\theta + |J_\phi| \). For a static potential, an orbit in angle space follows \( [\theta_r, \theta_\phi] = [\theta_r(0) + \Omega_r t, \theta_\phi(0) + \Omega_\phi t] \), which results in a straight line in angle space with slope \( S(\theta) = \Omega_\phi / \Omega_r \). Generally particles on a stream have similar frequencies implying that \( S(\theta) \) is also similar, and this results in an almost linear appearance in angle space. Depending on the orbit, the progenitor and the specific form of the potential, the stream follows a line close to that defined by the slope of the progenitor \( S(\theta_{CM}) \). The difference between the stream and the orbit slope is known as the stream-orbit misalignment (Eyre & Binney 2011).

We can understand this more quantitatively using a Taylor expansion of the frequencies \( \Omega_i \) with respect to the actions \( J_j \) (Helmi & White 1999)

\[
\Delta \Omega_i \approx H_{ij} \Delta J_j, \tag{4.12}
\]

with the spreads measured with respect to the centre of mass of the progenitor, and the Hessian of the Hamiltonian \( H_{ij} \) also evaluated at this point. If we diagonalise the Hessian (and assuming the Einstein notation convention)

\[
H_{ij} \Delta J_j = V^T_{ik} D_{kl} V_{lj} \Delta J_j = \hat{\epsilon}_i (\lambda_l \Delta \tilde{I}_l), \tag{4.13}
\]

3 Apart from a possible sign difference.
with $V_{ik}$ the matrix of eigenvectors $\hat{e}_i$ of $H_{ij}$ (as its columns), $D_{kl}$ the diagonal matrix with the eigenvalues $\lambda_i$, and $\Delta \hat{J}_l = V_{lj}^T \Delta J_j$, the action spreads in the eigenspace. It turns out that generally one of the eigenvalues is much larger than the others, and this implies that the stream forms a thin 1-D structure in frequency space (and therefore also in angle space) that is elongated most in the direction associated to the corresponding eigenvector (Tremaine 1999).

In Fig. 4.18 we show three examples of the action and the frequency distributions of streams presented in Buist & Helmi (2015). We show the projected direction of the largest eigenvalue with a black dashed line, and the projected direction of the progenitor orbit in angle space $S(\theta_{\text{CM}})$ with the red dashed line. These dashed lines are generally closely aligned in spherical potentials, and indicate approximately the degree of the stream-orbit misalignment. The stream does not exactly point in the direction of $\hat{e}_1$ because the action distribution is not isotropic. Only in the specific case of a Kepler potential the stream precisely follows the progenitor orbit independent of the action distribution, and there is no stream-orbit misalignment (Eyre & Binney 2011).

Therefore we can conclude that streams in a spherical potential are generally linear structures in angle and frequency spaces, except for systems with a relatively large spread in actions, i.e. when $\Delta J_i \sim J_i$ (Sanders & Binney 2013b).

### 4.5.2 Streams on nearly circular orbits in the symmetry plane of an axisymmetric potential

To develop some intuition on the behaviour of streams in flattened potentials we focus on a stream on a nearly circular orbit in the symmetry plane of an axisymmetric potential, which allows us to use the epicyclic approximation. The 3-D motion in axisymmetric potentials can be reduced to 2-D motion in the meridional plane (the $R$-$z$ plane) where the (conserved) $z$-component of angular momentum $L_z$ is a parameter, and the $\phi$ motion is ‘simple’. The epicyclic approximation admits the decoupling of the motion in $z$ and $R$, and the potential can be approximated as $\Phi(R,z) \approx \Phi_R(R) + \Phi_z(z)$. The resulting motion can be described with 3 frequencies associated to the radial, vertical and azimuthal oscillations.

For a given value of $L_z$ a circular orbit at radius $R_g$ on the symmetry plane $z = 0$ has circular frequency

$$\Omega^2 = \frac{1}{R_g} \left. \frac{\partial \Phi(R,z)}{\partial R} \right|_{(R_g,0)} = \frac{L_z^2}{R_g^4}. \quad (4.14)$$

The epicyclic or radial frequency ($\kappa$), and vertical frequency ($\nu$) are

$$\kappa^2(R_g) = \left. \frac{\partial^2 \Phi}{\partial R^2} \right|_{(R_g,0)} + 3 \Omega^2(R_g),$$

$$\nu^2(R_g) = \left. \frac{\partial^2 \Phi}{\partial z^2} \right|_{(R_g,0)}, \quad (4.15)$$

and characterize harmonic motions in $x = R - R_g$ and $z$ respectively. As an example we will work out these frequencies for the Kuzmin-Kutuzov potential. The three orbital
4.5: Why do streams follow straight lines?

**Figure 4.19:** A quite circular stream in the epicyclic approximation. Left panel: stream in the $x$-$y$ plane, where the colour represents the energy gradient along the stream and the red line shows the progenitor orbit. Middle panel: radial and azimuthal frequency curves for the stream, and the red line indicates the direction of the progenitor orbit and the blue dashed line shows where the stream would be when $J_R = 0$. Right panel: vertical and azimuthal frequency curves for the stream, otherwise like the middle panel.

Frequencies in the epicyclic approximation are

\[
\begin{align*}
\Omega_g^2 &= \frac{GM_K}{(g_K + c_K)^2 g_K}, \\
\nu^2 &= \frac{GM_K}{(g_K + c_K)^3} \left(1 + \frac{a_K^2}{c_K g_K}\right), \\
\kappa^2 &= \frac{GM_K}{(g_K + c_K)^3 g_K^3} \left[4g_K^2 + 4g_K^2 c_K - (3g_K + c_K) R_g^2\right],
\end{align*}
\]

(4.16)

with

\[g_K = \left(a_K^2 + R_g^2\right)^{1/2}.\]

These frequencies are only a function of $L_z$ through $R_g$, because the perturbations are around circular orbits. Similar to spherical systems, $0.5 \leq \Omega_g/\kappa \leq 1$ (given by the limiting cases of a spherical harmonic oscillator and a Kepler potential in the symmetry plane). Notice that $\Omega_g$ does not describe fully the motion in $\phi$, and instead the actions and frequencies are (Dehnen 1999; Binney & Tremaine 2008)

\[
\begin{align*}
\Omega_R &= \kappa \\
\Omega_z &= \nu \\
\Omega_\phi &= \frac{\partial \kappa}{\partial J_\phi} J_R + \Omega_g \\
J_R &= \frac{E_R}{\kappa} \\
J_z &= \frac{E_z}{\nu} \\
J_\phi &= L_z.
\end{align*}
\]
where

\[ E_R = \frac{1}{2} P_R^2 + \frac{1}{2} \kappa^2 x^2 \]  
\[ E_z = \frac{1}{2} P_z^2 + \frac{1}{2} v^2 z^2. \]  

(4.18)

The difference is that \( \Omega_g \) is not the azimuthal frequency when the orbit is not exactly circular (i.e. \( J_R \neq 0 \)).

We computed the frequencies for one example case of a stream similar to the Carina-like progenitor of Buist & Helmi (2015) on a rather circular orbit evolved in a Kuzmin-Kutuzov potential with \( M_{\text{iso}} = 1.58 \times 10^{11} M_\odot \), \( r_{\text{iso}} = 22.9 \) and \( q_K = 1.39 \). In the left panel of Fig. 4.19 we show the stream in Cartesian coordinates, in the middle panel we show the stream in \( \Omega_R-\Omega_\phi \) space and in the right panel we show the stream in \( \Omega_\phi-\Omega_z \) space. The red line in all three panels indicates the (direction of the) progenitor orbit, and in the two right panels we indicate with a blue dashed line the frequencies assuming \( J_R = 0 \), i.e. when \( \Omega_\phi = \Omega_g \). This limiting case is strictly 1-dimensional, because then \( \Omega_\phi \), \( \kappa \) and \( v \) are solely functions of \( L_z \). This implies that the spreads \( \Delta \Omega_\phi \), \( \Delta \Omega_R \) and \( \Delta \Omega_z \) for a compact ensemble of particles will all be proportional to \( \Delta L_z \), and hence the particles will on a straight line in frequency space for \( \Delta L_z \ll L_z \).

The stream is closely aligned with the line corresponding to \( J_R = 0 \) but less so with the direction of the progenitor’s orbit. We see the curvature caused by the non-zero value of \( J_R \) most clearly at the endpoints of the stream in both projections in frequency space. The condition for such a stream to be on a straight line is therefore \( \Delta J_R \sim 0 \), but this is also necessary for the epicyclic approximation to work, and for that also \( \Delta J_z \sim 0 \). We conclude that similar to the spherical case, the frequencies are on a straight line when the spreads in the actions are small.

### 4.5.3 Expansion in Staeckel potentials

More generally an orbit in an axisymmetric Staeckel potential will be described by 3 frequencies, \( \Omega_\lambda \) and \( \Omega_\nu \) (which for sufficiently extended orbits, correspond approximately to the radial and vertical motion respectively) and \( \Omega_\phi \) (azimuthal motion). In Fig. 4.20 we show two streams integrated on the orbits from our test-particle simulations in the Kuzmin-Kutuzov potential and with a ‘Sculptor’-like progenitor but here we took \( q_K = 1.39 \). The top panels show that the two projections of the 3-D action distributions for each of the streams (yellow and blue) are very different in shape and orientation. These action distributions map onto the 3-D frequency space, whose projections are shown in the middle and bottom panels of this figure. It is evident from this figure that the distributions in frequency are still nearly 1-D, but begin to depict a small amount of curvature, which is attributed to the fact that the progenitors are larger in comparison to those shown for example in Fig. 4.18.

As in the case of a spherical potential, we may understand the nearly 1-D distribution and its orientation in frequency space using Eqs. (4.12) and (4.13). For the orbits explored in Fig. 4.20 the largest eigenvalue of the Hessian matrix is approximately 10 times bigger than the other two eigenvalues. On the other hand, for the streams explored in the spherical potential this ratio is generally significantly larger. This difference explains why the streams are not so perfectly aligned with \( \hat{e}_1 \) as in the spherical case, and to-
4.6 Action-angle behaviour of streams in approximate potentials of varying shape

Thus far we have discussed mostly from a theoretical point of view what to expect in angle and frequency space in the true potential, and why streams are on straight lines. This section uses the test-particle simulations of Sec. 4.4 to explore in a constrained way

Together with the anisotropic distribution of the actions present in the progenitor initially, may explain the small amount of curvature we observe.

Nonetheless we may conclude that the additional freedom for having 3 independent actions and, hence, frequencies (as is the case for most orbits in non-spherical potentials), does not change the straight line appearance of streams.
what happens when the wrong potential is used to compute the angles and frequencies. To this end we will look into the test-particle simulations that were run in the axisymmetric Staeckel potential but we will now assume a spherical potential in Sec. 4.6.1, a Staeckel potential but with a different flattening parameter ($q'_K = 1/q_K$) in Sec. 4.6.2, and a spherical potential with a more dissimilar radial mass distribution in Sec. 4.6.3.

### 4.6.1 Spherical approximation

We now compute the angles and frequencies for the test-particle simulations using the isochrone potential. This potential is the spherical limit of the Kuzmin-Kutuzov potential, and its characteristic parameters are defined in Sec. 4.4.1. The results are shown in Fig. 4.21, where we do not include the $\Omega_r-\Omega_\phi$ space as it is redundant in the spherical case. Instead, the behaviour in the angles $\theta_\phi$ and $\theta_\psi$ is relevant since these additionally depend on the physical location of the particles (which encodes information about the potential in which they were evolved as discussed earlier in Sec. 4.5).

In the top panel of Fig. 4.21 we show the resulting frequency distributions. We find that quite a few streams have become thicker or longer in the spherical approximation to the potential in comparison to their behaviour in the true axisymmetric potential. Some of the streams seem quite irregular such as S140, a characteristic that is also seen in some of the Aquarius streams, e.g. stream S107 from halo Aq-A and stream S125 from halo Aq-D (see Fig. 4.9). One noteworthy stream is S103, which looks very regular and is thinner than before, but that has become much longer instead. As a consequence of the distortions, it is generally more difficult to fit straight lines to the frequency distribution, and therefore also the determination of their slopes is harder (see e.g. S122 and S138).

For comparison we show in Fig. 4.22 how two of the progenitor orbits as a function of time map onto frequency space if the frequencies are computed at each time step in the approximate spherical potential. These distributions oscillate in both directions, while if the potential had been truly spherical they would collapse into a single point. The size of these oscillations is determined by the behaviour of the energy of the isochrone potential $E_{\text{iso}}$ and the (instantaneous value of the) total angular momentum $L$, or in other words by how well a spherically symmetric potential represents the true mass distribution. This comparison helps to understand why some streams are very thick in frequency space, as this is effectively a reflection of the behaviour of the trajectory.

In the central panel of Fig. 4.21 we show the $\theta_r-\theta_\phi$ space. Streams such as S103 and S121 appear quite similar to their counterparts in Fig. 4.15 plotted in their natural (true) angle space. Notice that when the straight line fits look reasonable, the slopes in this spherical angles space and in the corresponding Staeckel angles are similar, differing by $\sim 0.05$. Like for the Aquarius streams, the slopes in $\theta_r-\theta_\phi$ span a range between $\sim 0.4-0.7$. In all cases, the streams have become thicker. The energy gradient along the streams seems especially discontinuous at some locations, and this is mostly present in the $\theta_r$ direction. A closer inspection shows that for some streams such as S140 and S121 the energy gradient is somewhat better retained.

In the bottom panels of Fig. 4.21 we show the angles $\theta_r-\theta_\psi$. The comparison of $\theta_x-\theta_y$ to $\theta_r-\theta_\psi$ is less straightforward because $\Omega_\psi$ and $\Omega_y$ differ by a factor 2 in the spherical limit. Therefore we prefer to plot instead $2\Delta\theta_\psi$. The streams in this space look very similar to those in the true $\theta_x-\theta_y$ space, but unlike what we found for the streams in the Aquarius haloes the fitted slopes are not close to the value 0.5 (or 1.0 if scaled).
Figure 4.21: Test-particle streams from Fig. 4.14–4.16 with the approximate frequencies \( \Delta \Omega_r - \Delta \Omega_\phi \) (top panels) and angles \( \Delta \theta_r - \Delta \theta_\phi \) (middle panels) and \( \Delta \theta_r - 2 \Delta \theta_\phi \) (bottom panels), computed in the limiting spherical isochrone potential. The colours represent the energy gradient in the isochrone potential. The panels have been centred on the progenitor position. The insets indicate the fitted slopes for the approximate potential (black), and the slopes for the true potential when using the corresponding angles (green): the two will only be equal in the spherical limit. The errors were found by bootstrapping the fit 20 times.
We conclude that using a spherical approximation to the potential leads to an increase of the spread in frequency space, by making the streams longer, wider and/or distorted, but the energy gradient remains quite intact in this space. The streams in angle space often look much more well-behaved, although they do become thicker. This good-behaviour probably reflects that the spherical potential has the right average enclosed mass. We find that the fitted slopes in angle and frequency space may look reasonable, but they deviate significantly from each other indicating that the true potential in which the streams have evolved has not been used for the angle and frequency computations.

### 4.6.2 Axisymmetric Staeckel potential with different flattening

In this section we compute the angles and frequencies for the Kuzmin-Kutuzov potential, but now assuming a flattening $q'_K = 1/q_K$, which reverses the axis lengths $a_K$ and $c_K$ compared to the true potential. For the simulations of Sec. 4.4 we used a prolate potential, and inverting $q_K$ results in an oblate shape. The results are shown in Figs. 4.23 and 4.24, and qualitatively are very similar to what we saw in the previous section.

In this Staeckel potential, all our orbits have three independent frequencies, i.e. two independent frequency planes, and the overall behaviour of the streams is very similar in these planes, apart from slope differences, as can be seen by comparing Figs. 4.23 and 4.24. The frequency distributions are even more broadened than when assuming a spherical potential, and for some of the streams such as S56 also more extended (i.e. beyond the boundaries of the box, which we retained for easier comparison, and which is centred on the progenitor’s centre of mass). This broadening appears to be asymmetric with respect to the progenitor. As in the previous section, the energy gradient in frequency space remains almost intact because the spherically averaged enclosed mass is the same as in the true potential. On the other hand, the energy gradient in angle space almost cannot be discerned.

In both angle spaces we see that some of the streams depict small scale wiggles, such as streams S86, S118 and S140, and these are more apparent in $\theta_\lambda-\theta_\nu$ space. This is probably because the flattening of the potential is much more different from the true value (and than the spherical shape assumed in the previous section, although close
4.6: Behaviour of Streams in Approximate Potentials of Varying Shape

Figure 4.23: Test-particle streams from Fig. 4.14-4.16 with the frequencies and angles in the $\lambda$-\$\phi$ projections computed for a Staeckel potential with flattening $q_K' = 1/q_K$. The colours represent the energy gradient with the most bound particles in yellow and the least bound particles in blue.

Inspection of Fig. 4.21 shows that some subtle wiggles are also apparent for the same streams in this case). Wiggles such as these are therefore an indication of a significant departure from the true 3-dimensional shape of the potential in which the streams have evolved.

4.6.3 Spherical approximation with incorrect radial form

We now focus on the impact of computing the actions and angles in a spherical potential whose radial dependence is quite different from the Kuzmin-Kutuzov potential in the spherical limit. We explore an NFW potential that has the same enclosed mass and mass gradient at $r_{\text{fix}} = 50$ kpc (see also Fig. 4.10).

In Fig. 4.25 we show the resulting distribution of these angles and frequencies for the streams evolved in the axisymmetric Kuzmin-Kutuzov potential of Sec. 4.4. At first sight, many of the streams look very similar to those in Fig. 4.21, which corresponds to the spherical isochrone mass distribution computations.
In frequency space we see that some of the streams are longer and thinner (see e.g. S140 and S118). We can also compare the slopes derived from fitting straight lines to these distributions to those computed in the isochrone potential, such as for S86 and S103, and we find that they are not the same. In angle space the differences are more subtle, and the typical difference between the fitted slopes between the isochrone and NFW angles is $\sim 0.05$, which is comparable to the estimated error in the fitting. The energy gradient for the isochrone and NFW angles is not always the same, with the isochrone potential showing more fluctuations in the $\theta_r$ direction. Overall we find that the frequencies are the most sensitive to the potential, but the differences are small because the slope and the enclosed mass at $r_{\text{fix}} = 50$ kpc are equal for the NFW and isochrone profiles.

### 4.6.4 The effects of self-gravity

In reality the progenitors of streams will initially be bound by their self-gravity. This implies that particles that become unbound because the internal gravitational pull is
Figure 4.25: Test-particle streams from Fig. 4.14-4.16 with the frequencies $\Delta \Omega_r - \Delta \Omega_\phi$ (top panels) and angles $\Delta \theta_r - \Delta \theta_\phi$ (middle panels) and $\Delta \theta_r - 2 \Delta \theta_\phi$ (bottom panels), computed with the best fitting NFW potential at $r_{\text{fix}} = 50$ kpc to the limiting spherical isochrone potential (see Fig. 4.10). The colours represent the energy gradient in the NFW potential. The panels have been centred on the progenitor position. The insets indicate the fitted slopes for the approximate potential (black), and the slopes of the true angles (green), which are only equal in the spherical case. The errors were found by bootstrapping the fit 20 times.
not large enough, will typically be released at specific points along the orbit (close to pericentre), rather than continuously as modelled thus far. This results in the leading and trailing arms being offset from each other in configuration space (Johnston 1998). Another feature is a gap in energy-angular momentum space (Gibbons et al. 2014). The process of disruption also causes particles to define a bow-tie structure in action (and energy-angular momentum) space. Since one of our goals is to understand the properties of streams in a cosmological context as provided by the Aquarius N-body simulations, we attempt here to establish what the effect of self-gravity is on the distribution of particles in frequency and angle spaces.

To this end, we simulated the evolution of a $3.7 \times 10^8 \, M_\odot$ progenitor for 10 Gyr on an orbit with $r_{apo} = 23.6$ kpc and $r_{peri} = 10.2$ kpc in a spherical NFW potential with $M_s = 1.5 \times 10^{11} \, M_\odot$ and $r_s = 12$ kpc using an N-body code that uses a quadrupole expansion to model the internal gravitational potential of the system (Helmi & White 2001). Fig. 4.26 shows snapshots for $t = 0$ and $t = 3.5$ Gyr. The left panels plot the projection of the stream on the progenitor’s orbital plane, where the red particles correspond to those that are (still) bound. The stream has already spread out significantly by $t = 3.5$ Gyr because the progenitor is large and the orbit is rather confined to the inner regions of the host, which results in fast evolution. By this time, the progenitor has almost completely dissolved, as only a few particles are marked in red.

Figure 4.26: Projections of configuration, frequency, angle and action space at $t = 0$ (top panels) and $t = 3.5$ Gyr (bottom panels) for an N-body stream evolved in a spherical NFW potential. The first column shows the projection of the stream on the orbital plane of the progenitor. In the left and central-left panels particles still bound to the progenitor are plotted in red. In the third column we have overlaid the distribution of frequencies at $t = 10$ Gyr in grey. In the panels on the right we also show the final distribution of actions at $t = 10$ Gyr in grey. The actions, angles and frequencies have been centred on the coordinates of the centre of mass of the progenitor and the colours represent the energy gradient.

In the second panel we show the angle distribution initially and at 3.5 Gyr, with the particles bound to the progenitor marked in red. By 3.5 Gyr many wraps fill up the angle space, to which we fitted parallel straight lines and whose slope is shown in the inset.
The frequency distribution at both times is shown in the third column panels, which is overlaid onto the distribution at \( t = 10 \) Gyr in grey. We learn from this that at \( t = 3.5 \) Gyr, the distribution has reached its final (bow-tie like) shape, and the evolution of the particles is simply \( \theta \sim \Omega t \) by this point in time.

The expected gap near the progenitor in action space forms slowly and is only subtle at the final time (see Gibbons et al. 2014, for a more detailed discussion of the process). Furthermore, we find no significant offset in configuration space between the leading and trailing arms in our simulation, nor any epicyclic oscillations in the stream. This is most likely related to the fast disruption of the progenitor as a consequence of its low density contrast with respect to the host. This is quite different to what is seen for N-body simulations of globular clusters whose disruption process is slower because these are strongly bound gravitational systems (Küpper et al. 2010, 2012).

At 3.5 Gyr, the distribution of angles and frequencies follow each other very closely, as quantified by the slope of the fitted straight lines, and this is in agreement with the findings of Sanders & Binney (2014). The difference between test-particle and N-body simulations is especially seen in the bow-tie shape in action space and frequency space, but it is almost absent in angle space. The dynamics of streams is otherwise mostly the same. Furthermore, the slopes of the straight lines along which the stream is distributed in angle and in frequency space are the same, in agreement with the results by Sanders & Binney (2013b,a). Only for very massive progenitors (\( \sim 10^{10} \) M\(_\odot\)) this picture may change because of interactions between the stream and the progenitor (Choi et al. 2009), or because the overall potential of the halo changes while the progenitor system is in the process of disruption (Vera-Ciro & Helmi 2013; Gómez et al. 2015).

### 4.7 Discussion and conclusions

We have studied the behaviour of streams in fully cosmological N-body simulations of the formation of stellar haloes from the Aquarius project (Springel et al. 2008). These stellar haloes were produced by tagging dark matter particles in these otherwise dark-matter only simulations according to the GALFORM semi-analytic galaxy formation model (Cooper et al. 2010; Lowing et al. 2015). From these haloes we selected the stream-like objects and used the ‘tagged’ particles to build up a catalogue of stellar streams. Our interest was to understand how these streams, evolved in a fully cosmological time-dependent framework behave, particularly when studied in action-angle space. Since the Aquarius haloes’ potential is not analytic, we explored as a first approximation their behaviour when a spherical NFW potential was assumed. For the best fitting NFW mass distribution we computed the streams’ angles and frequencies.

We have found that many of the streams in the Aquarius haloes show several wraps in angle space that appear to be on relatively straight lines, as reported in other works for streams evolved in static or evolving smooth potentials (Sanders & Binney 2013b; Buist & Helmi 2015). However in many cases these lines are not parallel. We also found patchy features and wiggly behaviour in angle space. In frequency space, often the structures are very broad but relatively linear and depict some amount of irregularity. The width of these streams and features are typically larger than what we have seen before in simple simulations of streams evolved in a smooth spherical potential.

To understand the nature of the various features, we proceeded to explore how vari-
ous deviations from spherical symmetry could be affecting the behaviour of streams. We have been able to demonstrate that, independently of the form of the host potential, if the angles and frequencies are computed self-consistently then the streams are expected to be along straight lines in frequency and angle space. This is because the Hessian of the Hamiltonian generally has one eigenvalue that is much greater than the others and this dictates largely the direction in which the streams will expand (Tremaine 1999). The exact direction depends also on the action distribution. These results are valid provided the progenitor of the stream is relatively compact in phase-space, and we have demonstrated this explicitly for streams evolved in spherical potentials, for an almost circular stream in an axisymmetric potential using the epicyclic approximation, and for streams in an axisymmetric Staeckel potential.

We next focused on why streams evolved in a particular potential but whose angles and frequencies have been computed in a different approximate potential are still on straight lines, as we have found for the Aquarius simulations. To this end we ran a set of simulations with test-particles in an axisymmetric prolate Kuzmin-Kutuzov potential, which is of Staeckel form. We computed the angles and frequencies in this potential, and found again the characteristic linear appearance of streams in these spaces. Next we assumed different forms of the potential and computed the angles and frequencies for those cases as well. We found that even if this procedure is not self-consistent, streams are still distributed along relatively straight lines. However, we found in frequency space that they were typically thicker and somewhat distorted, and in angle space that they began to depict a wiggly behaviour. For example we found that using a potential with the wrong flattening (spherical or oblate, instead of prolate) has a strong effect on the size of streams in frequency space. Differences in the angles and frequencies distributions for spherical potentials of different radial form remained subtle provided the enclosed mass was approximately correct within in the radial extent probed by the streams orbits.

In all cases, the energy gradient along the stream seems almost intact in frequency space (similar to what was seen for the Aquarius streams) but clearly distorted or broken in angle space. The straight lines that we fit to the angle distributions differ in slope, even when the potential is assumed to be spherical, contrary to expectations. This is the clearest indication that the shape assumed for the angle computation is incorrect and does not correspond to that of the potential in which the streams were evolved. Finally, we also investigated what happens to the actions and angles in a simulation with self-gravity. The largest difference is that during the disruption process of the progenitor, the action distribution of the particles that eventually form the stream is altered, but otherwise the dynamics of streams are the same as in test-particle simulations.

In conclusion, we have been able to reproduce and understand most of the features seen in the approximate angles and frequencies for the Aquarius streams, with the exception of the ‘noisy’ and ‘patchy’ appearance of the streams in angle and configuration space. We believe these can be attributed to interactions of a stream with dark matter substructures, which are known to give rise to disturbed morphologies (Bonaca et al. 2014). Such interactions may also introduce non-adiabatic time-dependent effects on streams that lead to the formation of gaps (Yoon et al. 2011; Carlberg 2013; Ngan & Carlberg 2014).

Finally, since the angle-frequency misalignments found for the Aquarius streams can mostly be attributed to using the wrong potential, this implies that they cannot be used determine the mass growth history of the Aquarius dark matter halos, as we had pro-
posed in Buist & Helmi (2015). This may be resolved with approximate schemes to com-pute the actions in a triaxial potential (e.g. Sanders & Binney 2014; Bovy 2014) and by using the distortions of sufficiently thin streams in angle and frequency space to further constrain the true potential. Once this is under control, measuring the angle-frequency misalignment to determine the evolution of the potential may be feasible.

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4.A Fitting algorithm

When the individual wraps of a stream are sufficiently distinct in angle space, we can fit straight lines to these and compare them with those fitted in frequency space. This comparison can be used to determine the potential and eventually its time evolution. In the Aquarius simulations the angle distributions are derived assuming a spherical potential. These distributions are rather noisy, making this process non-trivial and complex. Therefore, we first group particles into possible wraps, and then use a standard parallel line fitting algorithm on these groups. We also remove the progenitor before starting this procedure because we expect it not to follow the behaviour of the stream.

In Fig. 4.27 we show two examples: halo Aq-A’s S164 and the test-particle simulations’ S140 as they are being fitted. In case of the Aquarius haloes, we first use the binding energies of the particles in the stream to find the most bound particle and then centre angle and frequency space on this particle. If there is no bound structure, the centre is put at the location of the highest density in angle space. We generously removed the progenitor and particles in its surroundings by computing the total binding energy with a much higher mass per particle (\( \sim 20 \)). These bound particles are marked in red in the top-left panel of Fig. 4.27.

In the next step we fit a straight line to the remaining particles in each independent projection of frequency space using a simple least squares method. We show this line for the \( r-\phi \) projection of angle space in the left panels of Fig. 4.27. We then bin the data of this angle space projection in \( N \) bins horizontally and \( N/S(\Delta \Omega_\phi) \) vertically, where \( N = 40 \) for the test-particle streams and \( N = 30 \) for the Aquarius streams, unless there are fewer than 100 particles. We then ‘clean’ this image by emptying the bins that have fewer counts than the median in the image.

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We group the particles by connecting pixels that fall within the pattern of Fig. 4.28, which is elongated in the same direction as the stream, which is on an angle of 45 degrees if it follows the frequency distribution. This procedure results in the groups shown in the third column of Fig. 4.27. To take into account that the leading and trailing arms of the
Figure 4.27: Examples of the straight lines fitting routine for halo Aq-A’s S164 (top panels) and the test particle simulations’ S140. In the left panel we show the stream in angle space with possible bound particles in red and the straight line obtained by fitting the frequency distribution. The second panel shows the streams in bins, where the axis have been scaled such that the streams are oriented at 45 deg. The third panel shows the groupings found by the pattern filling algorithm. The last panel shows the the resulting straight lines determined using parallel fitting.

Figure 4.28: Pattern used to link pixels in the angle-space image of the stream.
stream can be different, we split the image in half at $\Delta \theta_r = 0$, unless the resulting groups have less than 100 particles. It is clear from the top row that sometimes structures are grouped together that should not be connected, but it is difficult to completely prevent this from happening without removing too many particles from the streams.

We then use a least-squares fit for parallel lines with different offsets. The result is optimised by running the full fitting procedure twice, because we can then use the slope fitted to the angles found in the first iteration when binning angle space. To estimate the errors in the fitting we bootstrap the data 20 times, which was found to be enough to get a reasonable estimate of the errors, although this does not fully reflect the error when the wraps in angle space overlap or if they are not on parallel lines.