Chapter 5
Output Feedback Robust Synchronization of Undirected Lur’e Networks

In this chapter we generalize the results on robust synchronization of undirected Lur’e networks by static, relative state information based protocols in Chapter 3 to the case that for each agent only relative measurement outputs are available. In this case dynamic protocols are used. We establish sufficient conditions under which a dynamic synchronization protocol exists for such networks. These conditions involve feasibility of a set of three LMI’s, reminiscent of the three LMI conditions in $H_\infty$ control by measurement output feedback. We show that, regardless of the size of the network, i.e. the number of agents, only these three LMI’s are involved. As a consequence, it turns out that our results are scalable. Our LMI’s only involve the linear parts of the dynamics of the individual agents. We assume that the interconnection topologies among these Lur’e agents are undirected and connected throughout this chapter. In the actual computation of the protocol matrices, the eigenvalues of the Laplacian matrix associated with the interconnection graph play a crucial role. In particular the matrices defining the protocol depend on the smallest nonzero eigenvalue and the largest eigenvalue of the Laplacian matrix. We validate our results by means of a numerical simulation example.

5.1 Introduction

In secure communication applications, output feedback based master-slave synchronization of two coupled Lur’e systems has been extensively studied. In [62], dynamic output feedback was used to recover a message signal in master-slave synchronization of Lur’e systems in the presence of measurement noise. Synchronization criteria for two static output coupled Lur’e systems with time delays were derived in [27]. In addition, in [15], it was assumed that the feedback nonlinearities are slope-restricted but also precisely known. The assumption that the feedback nonlinearities are known is often employed in observer-based output feedback stabilization of Lur’e systems. Thus the design of the above observer-based output feedback controllers does not deal with robustness, and hence addresses a different problem from the one addressed in this chapter. Our dynamical protocol is provided by a general linear dynamical system, which receives the weighted
relative measurement outputs and the weighted relative protocol states, and uses these to determine the diffusive coupling inputs to the agents. To the best of our knowledge, the present work constitutes the first result in which a treatment for output feedback based robust synchronization of Lur’e dynamical networks is given.

The remainder of this chapter is organized as follows. Section 5.2 formulates the output feedback robust synchronization problem we are interested in. Our main results are presented in Sections 5.3 and 5.4, respectively. Sufficient synchronization conditions are established and it is discussed how to compute a suitable dynamical protocol. A numerical simulation example is given in Section 5.5. Some concluding remarks together with suggestions for future work close this chapter.

5.2 Problem formulation

In this chapter, we consider an undirected multi-agent network of $N (\geq 2)$ identical Lur’e systems described by (see Fig. 5.1)

\[
\begin{align*}
\dot{x}_i &= A_p x_i + B_p u_i + E_p d_i \\
z_i &= C_p x_i \\
y_i &= M_p x_i \\
d_i &= -\phi(z_i, t)
\end{align*}
\]

where $x_i(t) \in \mathbb{R}^n$, $u_i(t) \in \mathbb{R}^m$, $z_i(t) \in \mathbb{R}^p$ and $y_i(t) \in \mathbb{R}^q$ are the state to be synchronized, the diffusive coupling input, the system output and the measurement output of agent $i$, respectively. The equation $d_i = -\phi(z_i, t)$ represents a time-varying, memoryless, nonlinear negative feedback loop. The function $\phi(\cdot, t)$ from $\mathbb{R}^p \times \mathbb{R}^+$ to $\mathbb{R}^p$ is unknown and can be any function from a set of functions to be specified later. $A_p$, $B_p$, $C_p$, $E_p$, and $M_p$ are known constant matrices of compatible dimensions. Without loss of generality, we assume that the dimensions $m$ and $q$ of the diffusive coupling inputs and the measurement outputs, respectively, are strictly
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less than the state space dimension $n$. In this case the rows of $B_p$ are linearly dependent and thus $B_p^\perp$ exists. Similarly, $(M_p^T)^\perp$ exists as well. Furthermore, $M_p$ is assumed to have full row rank. The interconnection topology among these agents is represented by a connected undirected graph $\mathcal{G}$ which is fixed.

In this chapter, the agents (5.1) in the network $\mathcal{G}$ are assumed to be interconnected by means of a distributed dynamic protocol of the form

$$
\begin{align*}
\dot{w}_i &= A_c w_i + B_c \sum_{j=1}^{N} a_{ij} (y_i - y_j) + D_c \sum_{j=1}^{N} a_{ij} (w_i - w_j), \quad i = 1, 2, \ldots, N, \\
u_i &= C_c w_i
\end{align*}
$$

(5.2)

where $w_i(t) \in \mathbb{R}^{n_c}$ is the protocol state for agent $i$, $A_c$, $B_c$, $C_c$ and $D_c$ are the parameter matrices of the protocol, and $\mathcal{A} = [a_{ij}]$ is the adjacency matrix of the graph $\mathcal{G}$. $n_c$, $A_c$, $B_c$, $C_c$ and $D_c$ need to be determined.

**Remark 5.1.** The dynamic protocol determines the information exchange among these agents, i.e. the copy of the protocol at agent $i$ receives the weighted relative measurement outputs and the weighted relative protocol states, uses these to determine the diffusive coupling input to agent $i$, and at the same time processes these quantities to determine the dynamics of its protocol state.

By interconnecting (5.1) and (5.2) we get the Lur'e dynamical network

$$
\begin{align*}
\begin{bmatrix}
\dot{x} \\
\dot{w}
\end{bmatrix}
&= 
\begin{bmatrix}
I_N \otimes A_p & I_N \otimes B_p C_c \\
\mathcal{L} \otimes B_c M_p & I_N \otimes A_c + \mathcal{L} \otimes D_c
\end{bmatrix}
\begin{bmatrix}
x \\
w
\end{bmatrix}
- 
\begin{bmatrix}
I_N \otimes E_p & 0
\end{bmatrix}
\Phi(z, t), \\
z = (I_N \otimes C_p) x
\end{align*}
$$

(5.3)

where $x = [x_1^T, x_2^T, \ldots, x_N^T]^T$, $w = [w_1^T, w_2^T, \ldots, w_N^T]^T$, $\Phi(z, t) = [\phi(z_1, t)^T, \phi(z_2, t)^T, \ldots, \phi(z_N, t)^T]^T$, $z = [z_1^T, z_2^T, \ldots, z_N^T]^T$, and $\mathcal{L}$ is the Laplacian matrix of the graph $\mathcal{G}$.

**Definition 5.1.** The network of agents (5.1) with the protocol (5.2) is robustly synchronized if $x_i(t) - x_j(t) \to 0$ and $w_i(t) - w_j(t) \to 0$ as $t \to \infty$, $\forall i, j = 1, 2, \ldots, N$, for all initial conditions and all unknown functions $\phi(\cdot, t)$ from a particular set of functions to be specified in the next sections.

### 5.3 Incremental passivity

In this section, we assume the set of unknown functions $\phi(\cdot, t)$ to consist of all functions that are incrementally passive. Recall from Chapter 3, the following
definition of incremental passivity for static systems of the form

\[ d = \phi(z, t) \]  

with input \( z(t) \in \mathbb{R}^p \) and output \( d(t) \in \mathbb{R}^p \).

**Definition 5.2.** [42] The system (5.4) is called **incrementally passive** if the function \( \phi(\cdot, t) \) satisfies

\[ (z_1 - z_2)^T (\phi(z_1, t) - \phi(z_2, t)) \geq 0 \]

for all \( z_1, z_2 \in \mathbb{R}^p \) and \( t \in \mathbb{R}^+ \).

As noted before in Chapter 3, in general, incremental passivity is stronger than the property of passivity, which is defined by

\[ z^T \phi(z, t) \geq 0 \]

for all \( z \in \mathbb{R}^p \) and \( t \in \mathbb{R}^+ \). Passivity implies incremental passivity for linear systems, and also for monotonically increasing static nonlinearities [58].

We first establish sufficient conditions for the protocol (5.2) to robustly synchronize the network of agents (5.1). Subsequently we discuss how to compute a suitable dynamic protocol.

The following theorem gives conditions under which the distributed dynamic protocol (5.2) robustly synchronizes the network (5.1). Recall that \( \lambda_2, \cdots, \lambda_N \) are the nonzero Laplacian eigenvalues of the graph \( \mathcal{G} \).

**Theorem 5.2.** Let \( A_c \in \mathbb{R}^{n_c \times n_c}, B_c \in \mathbb{R}^{n_c \times q}, C_c \in \mathbb{R}^{m \times n_c}, D_c \in \mathbb{R}^{n_c \times n_c} \). If there exists a positive definite matrix \( P \in \mathbb{R}^{(n+n_c) \times (n+n_c)} \) such that

\[ P(A + BH_i M) + (A + BH_i M)^T P < 0 \]  

for all \( i = 2, \cdots, N \), and

\[ PE = C^T \]  

where \( A = \begin{bmatrix} A_p & 0 \\ 0 & 0_{p \times n_c} \end{bmatrix}, B = \begin{bmatrix} B_p & 0 \\ 0 & I_{n_c} \end{bmatrix}, H_i = \begin{bmatrix} 0 & \lambda_i C_c \\ B_c & A_c + \lambda_i D_c \end{bmatrix}, \)

\( M = \begin{bmatrix} M_p & 0 \\ 0 & I_{n_c} \end{bmatrix}, E = \begin{bmatrix} E_p \\ 0_{n_c \times p} \end{bmatrix}, C = \begin{bmatrix} C_p & 0_{p \times n_c} \end{bmatrix} \), then the network of agents (5.1) with the protocol (5.2) is robustly synchronized, i.e. the Lur'e network (5.3) is synchronized for all incrementally passive \( \phi(\cdot, t) \).

**Proof.** Let \( \mathcal{U} \) be an orthogonal matrix such that \( \mathcal{U}^T \mathcal{U} = \Lambda \), where \( \Lambda = \text{diag}(0, \lambda_2, \cdots, \lambda_N) \). All notation introduced in Chapter 2 will be used without redefinitions or statements throughout this chapter. Let

\[ \begin{bmatrix} \tilde{x} \\ \tilde{w} \end{bmatrix} = \begin{bmatrix} \mathcal{U}^T \otimes I_n \\ \mathcal{U}^T \otimes I_{n_c} \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \]
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and

\[
\begin{bmatrix}
\bar{x} \\
\hat{w}
\end{bmatrix} = \begin{bmatrix}
U_2^T \otimes I_n \\
U_2^T \otimes I_{n_c}
\end{bmatrix} \begin{bmatrix}
x \\
w
\end{bmatrix},
\]

where \( \bar{x} = [\bar{x}_1^T, \bar{x}_2^T, \ldots, \bar{x}_N^T]^T \), \( \hat{w} = [\hat{w}_1^T, \hat{w}_2^T, \ldots, \hat{w}_N^T]^T \), \( \bar{x} = [\bar{x}_2^T, \ldots, \bar{x}_N^T]^T \) and \( \hat{w} = [\hat{w}_2^T, \ldots, \hat{w}_N^T]^T \). Denote

\[
\bar{w} = (\bar{w}_2^T, \ldots, \bar{w}_N^T)^T = (\bar{\Lambda}^{-1} \otimes I_{n_c}) \hat{w}
\]

where \( \bar{\Lambda} = \text{diag}(\lambda_2, \ldots, \lambda_N) \). It follows from [65], Lemma 3.2 that \( x_i(t) - x_j(t) \to 0 \) and \( w_i(t) - w_j(t) \to 0 \) as \( t \to \infty \), \( \forall \ i, j = 1, 2, \ldots, N \), if and only if \( \bar{x}(t) \to 0 \) and \( \bar{w}(t) \to 0 \) as \( t \to \infty \). The dynamics of \( \bar{x} \) and \( \bar{w} \) is given by

\[
\begin{bmatrix}
\dot{\bar{x}} \\
\dot{\bar{w}}
\end{bmatrix} = \begin{bmatrix}
I_{N-1} \otimes A_p & \bar{\Lambda} \otimes B_pC_c \\
I_{N-1} \otimes B_cM_p & I_{N-1} \otimes A_c + \bar{\Lambda} \otimes D_c
\end{bmatrix} \begin{bmatrix}
\bar{x} \\
\bar{w}
\end{bmatrix} - \begin{bmatrix}
U_2^T \otimes E_p \\
0
\end{bmatrix} \Phi(z,t). \quad (5.7)
\]

Hence the robust synchronization of \( x \) and \( w \) is equivalent to the global asymptotic stability of \( \bar{x} \) and \( \bar{w} \), respectively.

By Lemma 2.1, we have

\[
\bar{x}^T (U_2^T \otimes C_p^T) \Phi(z,t) = \bar{z}^T (U_2U_2^T \otimes C_p^T) \Phi(z,t) = \bar{z}^T (I_N \otimes C_p^T) (U_2U_2^T \otimes I_p) \Phi(z,t) = \bar{z}^T (U_2U_2^T \otimes I_p) \Phi(z,t) = \frac{1}{N} \sum_{1 \leq i < j \leq N} (z_i - z_j)^T (\phi(z_i,t) - \phi(z_j,t)) \geq 0.
\]

Let \( P > 0 \) in (5.5) and (5.6) be appropriately partitioned as

\[
P = \begin{bmatrix}
P_1 & P_2 \\
P_2^T & P_3
\end{bmatrix}.
\]

Then (5.6) holds if and only if \( P_1E_p = C_p^T \) and \( P_2^T E_p = 0 \). Define a positive definite matrix \( P_4 \) by

\[
P_4 = \begin{bmatrix}
I_{N-1} \otimes P_1 & I_{N-1} \otimes P_2 \\
I_{N-1} \otimes P_2^T & I_{N-1} \otimes P_3
\end{bmatrix}.
\]

Consider a quadratic Lyapunov function candidate

\[
V(\bar{x}, \bar{w}) = \frac{1}{2} \begin{bmatrix}
\bar{x} \\
\bar{w}
\end{bmatrix}^T P_4 \begin{bmatrix}
\bar{x} \\
\bar{w}
\end{bmatrix}.
\]
Obviously, $V$ is positive definite and radially unbounded. The time derivative of $V$ along the trajectories of the system (5.7) is given by

$$
\dot{V}(\bar{x}, \bar{w}) = \begin{bmatrix} \bar{x} \\ \bar{w} \end{bmatrix}^T P_d \begin{bmatrix} \dot{\bar{x}} \\ \dot{\bar{w}} \end{bmatrix} = \begin{bmatrix} I_{N-1} & P_1 A_p \\ +I_{N-1} & P_2 B_c M_p \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{w} \end{bmatrix} + \begin{bmatrix} \bar{\Lambda} \otimes P_1 B_p C_c \\ +I_{N-1} \otimes P_2 A_c \\ +\bar{\Lambda} \otimes P_3 D_c \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{w} \end{bmatrix} - \begin{bmatrix} U_2^T \otimes P_2 E_p \end{bmatrix} \Phi(z, t)
$$

$$
= \sum_{i=2}^{N} \begin{bmatrix} \bar{x}_i \\ \bar{w}_i \end{bmatrix}^T \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} \begin{bmatrix} \bar{x}_i \\ \bar{w}_i \end{bmatrix} + \lambda_i P_1 B_p C_c + P_2 A_c + \lambda_i P_3 D_c \\
$$

$$
\leq \sum_{i=2}^{N} \begin{bmatrix} \bar{x}_i \\ \bar{w}_i \end{bmatrix}^T P(A + BH_i M) \begin{bmatrix} \bar{x}_i \\ \bar{w}_i \end{bmatrix}
$$

$$
= \frac{1}{2} \sum_{i=2}^{N} \begin{bmatrix} \bar{x}_i \\ \bar{w}_i \end{bmatrix}^T \begin{bmatrix} P(A + BH_i M) + (A + BH_i M)^T P \end{bmatrix} \begin{bmatrix} \bar{x}_i \\ \bar{w}_i \end{bmatrix}
$$

which is negative definite. Thus the system (5.7) is globally asymptotically stable, i.e. the Lur’e network (5.3) is robustly synchronized. This completes the proof. □

Below we will discuss conditions for the existence of protocol matrices $A_c$, $B_c$, $C_c$, $D_c$ and a common solution $P > 0$ of (5.5) and (5.6) in Theorem 5.2. Our first theorem gives necessary conditions.

**Theorem 5.3.** Assume that there exists a positive integer $n_c$, $A_c \in \mathbb{R}^{n_c \times n_c}$, $B_c \in \mathbb{R}^{n_c \times q}$, $C_c \in \mathbb{R}^{m \times n_c}$, $D_c \in \mathbb{R}^{n_c \times n_c}$ and a positive definite matrix $P \in \mathbb{R}^{(n+n_c) \times (n+n_c)}$ such that (5.5) and (5.6) hold for all $i = 2, \ldots, N$. Then there exist positive definite matrices $X_p$ and $Y_p$ of size $n \times n$ such that

$$
B_p^\perp (A_p X_p + X_p A_p^T) B_p^\perp < 0,
$$

(5.8)
\[ E_p = X_p C_p^T, \]  
(5.9)

\[ M_p^{T\perp} (Y_p A_p + A_p^T Y_p) (M_p^{T\perp})^T < 0, \]  
(5.10)

\[ Y_p E_p = C_p^T, \]  
(5.11)

\[ Y_p - X_p^{-1} \geq 0. \]  
(5.12)

**Proof.** Define \( X := P^{-1} \). We get

\[(A + BH_i M)X + X(A + BH_i M)^T < 0\]

for all \( i = 2, \ldots, N \) and thus

\[B^\perp (AX + XA^T) B^\perp < 0.\]

Similarly, we have

\[M^{T\perp} (YA + A^T Y) (M^{T\perp})^T < 0,\]

where \( Y := P \). Partition

\[X = \begin{bmatrix} X_p & X_{pc} \\ X_p^T & X_c \end{bmatrix}, \quad Y = \begin{bmatrix} Y_p & Y_{pc} \\ Y_p^T & Y_c \end{bmatrix}.\]

Note that \( B^\perp = \begin{bmatrix} B_p^\perp & 0 \\ \ast & \ast \end{bmatrix}, \quad M^{T\perp} = \begin{bmatrix} M_p^{T\perp} & 0 \\ \ast & \ast \end{bmatrix},\)

\[AX + XA^T = \begin{bmatrix} A_p X_p + X_p A_p^T & \ast \\ \ast & \ast \end{bmatrix}, \]

\[YA + A^T Y = \begin{bmatrix} Y_p A_p + A_p Y_p^T & \ast \\ \ast & \ast \end{bmatrix}.\]

Then we obtain (5.8) and (5.10). We also have \( E = XC^T \) and \( YE = C^T \), which imply (5.9) and (5.11), respectively. Furthermore, \( XY = I \) implies that \( X_p Y_p + X_{pc} Y_{pc} = I \) and \( X_p Y_{pc} + X_{pc} Y_c = 0 \). Thus

\[Y_p - X_p^{-1} = Y_{pc} Y_{pc}^{-1} Y_{pc}^T \geq 0,\]

i.e. (5.12) holds. \( \square \)

We will now show that the necessary conditions obtained in Theorem 5.3 above
are almost sufficient. In fact, if we replace the inequality (5.12) by its strict version
\[ Y_p - X_p^{-1} > 0, \]  
(5.13)
we obtain sufficient conditions for the existence of \( A_c, B_c, C_c, D_c \) and \( P > 0 \) such that (5.5) and (5.6) hold for all \( i = 2, \cdots, N \).

Our following protocol design is inspired by the measurement output feedback \( H_\infty \)-optimization controller construction for general linear systems in [52].

**Theorem 5.4.** There exists a positive integer \( n_c \) and matrices \( A_c \in \mathbb{R}^{n_c \times n_c}, B_c \in \mathbb{R}^{n_c \times q}, C_c \in \mathbb{R}^{m \times n_c}, D_c \in \mathbb{R}^{n_c \times n_c}, P > 0 \in \mathbb{R}^{(n+n_c) \times (n+n_c)} \) such that (5.5) and (5.6) hold for all \( i = 2, \cdots, N \) if there exist positive definite matrices \( X_p \) and \( Y_p \) of size \( n \times n \) such that (5.8), (5.9), (5.10), (5.11) and (5.13) hold, respectively. Suitable \( A_c, B_c, C_c, D_c \) and \( P \) are computed as follows.

- Choose \( r_2 > 0 \) such that
  \[ A_p X_p + X_p A_p^T - 2r_2 \lambda_2 B_p B_p^T < 0; \]  
(5.14)
- Choose \( r_1 > 0 \) such that
  \[ Y_p A_p + A_p^T Y_p - 2r_1 M_p^T M_p < 0; \]  
(5.15)
- Define \( F := -r_2 B_p X_p^{-1} \) and \( G := -r_1 Y_p^{-1} M_p^T; \)
- Define
  \[ R'_F := (A_p + \lambda_i B_p F)^T X_p^{-1} + X_p^{-1} (A_p + \lambda_i B_p F), \ i = 2, \cdots, N, \]
and
  \[ R_G := (A_p + G M_p) Y_p^{-1} + Y_p^{-1} (A_p + G M_p)^T; \]
- Choose a real number \( \tau \in (0, 1) \) such that
  \[ Y_p R_G Y_p < (1 - \tau) R'_F; \]
- Define
  \[ Z_p := Y_p - X_p^{-1}, \]
  \[ \tilde{G} := Z_p^{-1} Y_p G, \]
  \[ \Delta_1 := \tau Z_p^{-1} (A_p^T X_p^{-1} + X_p^{-1} A_p), \]
  \[ \Delta_2 := -Z_p^{-1} ((1 - \tau) F^T B_p^T X_p^{-1} - \tau X_p^{-1} B_p F); \]
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Choose
\[ P := \begin{bmatrix} Y_p & -Z_p \\ -Z_p & Z_p \end{bmatrix}, \]
\[ A_c := A_p + \tilde{G}M_p + \Delta_1, \]
\[ B_c := -\tilde{G}, \]
\[ C_c := F, \]
\[ D_c := B_p F + \Delta_2. \]

Obviously, the protocol has the same state dimension as the agents, i.e. \( n_c = n \).

**Proof.** Obviously, by (5.9) and (5.11), the proposed \( P = \begin{bmatrix} Y_p & -Z_p \\ -Z_p & Z_p \end{bmatrix} \) satisfies (5.6). Next we will show that (5.5) also holds for all \( i = 2, \cdots, N \).

By Finsler’s lemma \([24]\), (5.8) and (5.10) imply that there exist \( r_2 > 0 \) and \( r_1 > 0 \) such that (5.14) and (5.15) hold, respectively. Thus we have
\[ R^i_F \leq (A_p + \lambda_2 B_p F)^T X_p^{-1} + X_p^{-1} (A_p + \lambda_2 B_p F) < 0 \]
for all \( i = 2, \cdots, N \), and \( R_G < 0 \). Since we choose \( \tau \in (0, 1) \) such that \( Y_p R_G Y_p < (1 - \tau) R^N_F \), and \( R^N_F \leq R^{N-1}_F \leq \cdots \leq R^2_F \), we get \( Y_p R_G Y_p < (1 - \tau) R^i_F \) for all \( i = 2, \cdots, N \). Note that such \( \tau \) always exists and the largest Laplacian eigenvalue \( \lambda_N \) is involved.

Denote \( A_i := A + BH_i M \). Then (5.5) holds if and only if
\[ \tilde{P} \tilde{A}_i + \tilde{A}_i^T \tilde{P} < 0, \quad i = 2, \cdots, N, \tag{5.16} \]
where
\[ \tilde{P} = S^T P S = \begin{bmatrix} X_p^{-1} & 0 \\ 0 & Z_p \end{bmatrix}, \]
\[ \tilde{A}_i = S^{-1} A_i S \]
\[ = \begin{bmatrix} A_p + \lambda_i B_p C_c & -\lambda_i B_p C_c \\ A_p - B_c M_p + \lambda_i B_p C_c - A_c - \lambda_i D_c & -\lambda_i B_p C_c + A_c + \lambda_i D_c \end{bmatrix}, \]
and \( S = S^{-1} = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \). By straightforward computation, the \((1, 1)\) block of the left hand of (5.16) turns out to be \( R^i_F \). The \((2, 1)\) block can be computed to be equal to
\[ Z_p (A_p - B_c M_p + \lambda_i B_p C_c - A_c - \lambda_i D_c) - \lambda_i C_c^T B_p^T X_p^{-1} \]
\[ = Z_p \left\{ A_p + Z_p^{-1} Y_p G M_p + \lambda_i B_p F - \right\}. \]
\[ [A_p + Z_p^{-1}Y_pGM_p + \tau Z_p^{-1} (A_p^T X_p^{-1} + X_p^{-1} A_p)] \\
- \lambda_i [B_p F - Z_p^{-1} ((1 - \tau)F^T B_p^T X_p^{-1} - \tau X_p^{-1} B_p F)] \]
\[- \lambda_i F^T B_p^T X_p^{-1} \]
\[ = - \tau R_F^i. \]

The (2, 2) block can be computed to be equal to \( Y_p R_G Y_p - R_F^i + 2\tau R_F^i \), see (5.17).

\[ Z_p(-\lambda_iB_pC_c + A_c + \lambda_iD_c) + (-\lambda_iB_pC_c + A_c + \lambda_iD_c)^T Z_p \]
\[ = Z_p [-\lambda_iB_p F + A_p + Z_p^{-1}Y_pGM_p + \Delta_1 + \lambda_i(B_p F + \Delta_2)] \]
\[ + [h(-\lambda_iB_p F + A_p + Z_p^{-1}Y_pGM_p + \Delta_1 + \lambda_i(B_p F + \Delta_2)))]^T Z_p \]
\[ = Z_p A_p + A_p^T Z_p + Y_pGM_p + M_p^T G^T Y_p + Z_p(\Delta_1 + \lambda_i\Delta_2) + (\Delta_1 + \lambda_i\Delta_2)^T Z_p \]
\[ = Z_p A_p + A_p^T Z_p + Y_pGM_p + M_p^T G^T Y_p \]
\[ + Z_p [\tau Z_p^{-1} (A_p^T X_p^{-1} + X_p^{-1} A_p) - \lambda_i Z_p^{-1} ((1 - \tau)F^T B_p^T X_p^{-1} - \tau X_p^{-1} B_p F)] \]
\[ + [\tau Z_p^{-1} (A_p^T X_p^{-1} + X_p^{-1} A_p) - \lambda_i Z_p^{-1} ((1 - \tau)F^T B_p^T X_p^{-1} - \tau X_p^{-1} B_p F)]^T Z_p \]
\[ = Z_p A_p + A_p^T Z_p + Y_pGM_p + M_p^T G^T Y_p + 2\tau (A_p^T X_p^{-1} + X_p^{-1} A_p) \]
\[ - \lambda_i(1 - \tau)F^T B_p^T X_p^{-1} + \lambda_i\tau X_p^{-1} B_p F - \lambda_i(1 - \tau)X_p^{-1} B_p F + \lambda_i F^T B_p^T X_p^{-1} \]
\[ = (Y_p - X_p^{-1}) A_p + A_p^T (Y_p - X_p^{-1}) + Y_pGM_p + M_p^T G^T Y_p \]
\[ + 2\tau R_F^i - \lambda_i F^T B_p^T X_p^{-1} - \lambda_i X_p^{-1} B_p F \]
\[ = Y_p R_G Y_p - R_F^i + 2\tau R_F^i \]
(5.17)

Thus the left hand of (5.16) equals
\[ \begin{bmatrix}
R_F^i \\
-\tau R_F^i \\
-\tau R_F \\
Y_p R_G Y_p - R_F^i + 2\tau R_F^i
\end{bmatrix} \]
for all \( i = 2, \ldots, N \). The latter equals
\[ \begin{bmatrix}
(1 - \tau)R_F^i \\
0 \\
0 \\
Y_p R_G Y_p - (1 - \tau)R_F^i
\end{bmatrix} + \tau \begin{bmatrix}
1 & -1 \\
-1 & 1
\end{bmatrix} \otimes R_F^i \]
for all \( i = 2, \ldots, N \). Obviously, the first term above is negative definite and the second one is negative semi-definite since \( \begin{bmatrix}
1 & -1 \\
-1 & 1
\end{bmatrix} \succeq 0 \) and \( R_F^i < 0 \). Therefore, (5.16) and also (5.5) hold for all \( i = 2, \ldots, N \). This completes the proof. \( \square \)

Remark 5.5. Note that there is a gap between the necessary conditions obtained in Theorem 5.3 and the sufficient conditions obtained in Theorem 5.4, i.e. we need the strict inequality (5.13) to hold instead of the non-strict one. The conditions in Theorem 5.3 are quite close to necessary and sufficient conditions for Theorem 5.4.
However, at this moment it is unclear how to close this gap. This is an interesting problem for future research.

5.4 Incremental sector boundedness

In this section we assume $\phi(\cdot, t)$ to be any one from a given set of incrementally sector bounded functions. Recall the following definition from Chapter 3:

Definition 5.3. Let $S_1, S_2 \in \mathbb{R}^{p \times p}$ be real symmetric matrices with $0 \leq S_1 < S_2$. $\phi(\cdot, t)$ is called incrementally sector bounded within sector $[S_1, S_2]$ if it satisfies

$$[d_1 - d_2 - S_1(z_1 - z_2)]^T [d_1 - d_2 - S_2(z_1 - z_2)] \leq 0$$

for all $z_1, z_2 \in \mathbb{R}^p$ and $t \in \mathbb{R}^+$. Where $d_1 = \phi(z_1, t)$, $d_2 = \phi(z_2, t)$.

Remark 5.6. The condition of incremental sector boundedness is related to the ordinary sector boundedness: $(\phi(z, t) - S_1 z)^T(\phi(z, t) - S_1 z) \leq 0$, $\forall z \in \mathbb{R}^p$ and $t \in \mathbb{R}^+$. Obviously, incremental sector boundedness implies sector boundedness and thus is a stronger condition. The incremental version seems to be a natural assumption here since in the synchronization context one deals with 'stabilizing' the relative states. It is easily seen that in the SISO case incremental sector boundedness is equivalent to a slope-restrictedness condition, see [73]. If a SISO nonlinear function has finite slopes, as for example saturation and deadzone nonlinearities, we can verify whether the function satisfies an incremental sector boundedness condition using its slopes easily. In general, we have to check this condition for given nonlinearities case-by-case.

We first establish sufficient conditions for the protocol (5.2) to synchronize the network of agents (5.1). Subsequently we discuss how to compute a suitable dynamic protocol.

The following theorem gives sufficient conditions on a given protocol (5.2) to synchronize the network of agents (5.1) against all incrementally sector bounded nonlinearities within given sectors. Recall that $\lambda_2, \cdots, \lambda_N$ are the nonzero Laplacian eigenvalues of the graph $G$.

Theorem 5.7. Let $n_c$ be a positive integer, $A_c \in \mathbb{R}^{n_c \times n_c}$, $B_c \in \mathbb{R}^{n_c \times q}$, $C_c \in \mathbb{R}^{m \times n_c}$ and $D_c \in \mathbb{R}^{n_c \times n_c}$. If there exists a positive definite matrix $P \in \mathbb{R}^{(n+n_c) \times (n+n_c)}$ and a positive real number $\tau$ such that

$$\begin{bmatrix}
    P(A + BH_i M) + (A + BH_i M)^T P \\
    -\tau C^T (S_1 S_2 + S_2 S_1) C \\
    -E^T P + \tau (S_1 + S_2) C \\
    -PE + \tau C^T (S_1 + S_2) \\
    -2\tau I_p
\end{bmatrix} < 0$$

(5.19)
for all \( i = 2, \ldots, N \), where \( A = \begin{bmatrix} A_p & 0 \\ 0 & 0_{n_c \times n_c} \end{bmatrix}, B = \begin{bmatrix} B_p & 0 \\ 0 & I_{n_c} \end{bmatrix} \),

\[
H_i = \begin{bmatrix} 0 & \lambda_i C_c \\ B_c & A_c + \lambda_i D_c \end{bmatrix},
\]

\[
M = \begin{bmatrix} M_p & 0 \\ 0 & I_{n_c} \end{bmatrix},
\]

\[
C = \begin{bmatrix} C_p & 0 \end{bmatrix},
\]

\[
E = \begin{bmatrix} E_p \\ 0_{n_c \times p} \end{bmatrix},
\]

then the network of agents (5.1) with the protocol (5.2) is synchronized, i.e. the Lur'e network (5.3) is synchronized for all incrementally sector bounded \( \phi(\cdot, t) \) within sector \( [S_1, S_2] \).

**Proof.** Similar to Theorem 5.2, partition \( P > 0 \) in (5.19) appropriately as

\[
P = \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix}.
\]

Define a positive definite matrix \( P_4 \) by

\[
P_4 = \begin{bmatrix} I_{N-1} \otimes P_1 & I_{N-1} \otimes P_2 \\ I_{N-1} \otimes P_2^T & I_{N-1} \otimes P_3 \end{bmatrix}.
\]

In order to study stability of (5.7), we consider the same quadratic Lyapunov function candidate

\[
V(\bar{x}, \bar{w}) = \frac{1}{2} \begin{bmatrix} \bar{x} \\ \bar{w} \end{bmatrix}^T P_4 \begin{bmatrix} \bar{x} \\ \bar{w} \end{bmatrix}.
\]

Obviously, \( V \) is positive definite and radially unbounded. The time derivative of \( V \) along the trajectories of (5.7) is given by

\[
\dot{V}(\bar{x}, \bar{w}) = \begin{bmatrix} \bar{x} \\ \bar{w} \end{bmatrix}^T P_4 \begin{bmatrix} \dot{\bar{x}} \\ \dot{\bar{w}} \end{bmatrix} =
\]

\[
\begin{bmatrix} I_{N-1} \otimes P_1 A_p + I_{N-1} \otimes P_2 B_c M_p & \bar{\Lambda} \otimes P_1 B_p C_c + I_{N-1} \otimes P_2 A_c + I_{N-1} \otimes P_2 D_c \\ I_{N-1} \otimes P_2^T A_p + I_{N-1} \otimes P_3 B_c M_p & \Lambda \otimes P_2^T B_p C_c + I_{N-1} \otimes P_3 A_c + \bar{\Lambda} \otimes P_3 D_c \end{bmatrix}
\]

\[
\begin{bmatrix} \bar{x} \\ \bar{w} \end{bmatrix} - \begin{bmatrix} \bar{U}_2^T \otimes I_p E_p \\ \bar{U}_2^T \otimes P_2^T E_p \end{bmatrix} \Phi
\]

\[
\begin{bmatrix} I_{N-1} \otimes P_1 E_p \\ I_{N-1} \otimes P_2^T E_p \end{bmatrix} (\bar{U}_2^T \otimes I_p) \Phi =
\]
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By Lemma 2.1, we have

\[
[(U_2^T \otimes I_p) \Phi - (U_2^T \otimes I_p) (I_N \otimes S_1)z]^T
\]

\[
[(U_2^T \otimes I_p) \Phi - (U_2^T \otimes I_p) (I_N \otimes S_2)z]
\]

\[
=[(I_N \otimes S_1)z]^T (U_2U_2^T \otimes I_p) [\Phi - (I_N \otimes S_2)z]
\]

\[
=\frac{1}{N} \sum_{1 \leq i < j \leq N} [\phi(z_i, t) - \phi(z_j, t) - S_1(z_i - z_j)]^T
\]

\[
[\phi(z_i, t) - \phi(z_j, t) - S_2(z_i - z_j)] \leq 0.
\]
On the other hand,

\[
\frac{1}{2} \begin{bmatrix} \bar{x} \\ \left( U_2^T \otimes I_p \right) \Phi \end{bmatrix}^T \begin{bmatrix} I_{N-1} \otimes C_p^T (S_1 S_2 + S_2 S_1) C_p & -I_{N-1} \otimes C_p^T (S_1 + S_2) \\ -I_{N-1} \otimes (S_1 + S_2) C_p & 2I_{(N-1)p} \end{bmatrix} \begin{bmatrix} \bar{x} \\ \left( U_2^T \otimes I_p \right) \Phi \end{bmatrix} = \left( \left( U_2^T \otimes I_p \right) \Phi - (I_{N-1} \otimes S_1) \bar{x} \right)^T \left( \left( U_2^T \otimes I_p \right) \Phi - (I_{N-1} \otimes S_2) \bar{x} \right)
\]

\[
\leq 0,
\]
equivalently,

\[
\frac{1}{2} \sum_{i=1}^N \begin{bmatrix} \bar{x}_i \\ \bar{\phi}_i \end{bmatrix}^T \begin{bmatrix} C_p^T (S_1 S_2 + S_2 S_1) C_p & -C_p^T (S_1 + S_2) \\ -C_p (S_1 + S_2) C_p & 2I_p \end{bmatrix} \begin{bmatrix} \bar{x}_i \\ \bar{\phi}_i \end{bmatrix} \leq 0,
\]

and also

\[
\frac{1}{2} \sum_{i=2}^N \begin{bmatrix} \bar{\phi}_i \\ \bar{\phi}_i \end{bmatrix}^T \begin{bmatrix} C_p^T (S_1 S_2 + S_2 S_1) C_p & -C_p^T (S_1 + S_2) \\ -C_p (S_1 + S_2) C_p & 2I_p \end{bmatrix} \begin{bmatrix} \bar{\phi}_i \\ \bar{\phi}_i \end{bmatrix} \leq 0.
\]

Now using (5.19), it can be seen that \( \dot{V} (\bar{x}, \bar{w}) \) is negative definite for all \( \bar{x} \) and \( \bar{w} \) satisfying (5.7). Thus the system (5.7) is globally asymptotically stable for all incrementally sector bounded \( \phi(\cdot, t) \) within sector \([S_1, S_2]\), i.e. the Lur'e network (5.3) is robustly synchronized. This completes the proof. \( \square \)

Theorem 5.7 can be given an interpretation in terms of dissipativity synthesis as follows. By taking the Schur complement, (5.19) is equivalent to

\[
\begin{bmatrix}
P \left( A - \frac{1}{2} E (S_1 + S_2) C + BH_i M \right) + \left( A - \frac{1}{2} E (S_1 + S_2) C + BH_i M \right)^T P - PE + \frac{1}{2} \tau C^T (S_2 - S_1)^2 C & -2\tau I_p \\
-E^T P & -2\tau I_p
\end{bmatrix} < 0.
\]  

(5.20)
Thus, by applying the bounded real lemma for given $A_c, B_c, C_c, D_c$, the existence of $P > 0$ and $\tau > 0$ such that (5.19) holds for all $i = 2, \cdots, N$ is equivalent to the condition that the $N - 1$ dynamic feedback controllers

$$\begin{align*}
\dot{w}_i &= A_c w_i + \lambda_i B_c y_i + \lambda_i D_c w_i \\
u_i &= C_c w_i
\end{align*}$$

render the single system

$$\begin{align*}
\dot{x} &= \left(A_p - \frac{1}{2} E_p (S_1 + S_2) C_p \right) x + B_p u + E_p d \\
z &= \sqrt{\frac{\tau}{2}} (S_2 - S_1) C_p x \\
y &= M_p x
\end{align*}$$

dissipative with respect to the supply rate $s(d, z) = \tau d^T d - z^T z$ with the common storage function

$$\begin{bmatrix} w_i \\ x \end{bmatrix}^T P \begin{bmatrix} w_i \\ x \end{bmatrix}.$$ 

Note that Theorem 5.7 states that the protocol (5.2) synchronizes the network of agents (5.1) if the $N - 1$ matrix inequalities (5.19) have a common solution $(P, \tau)$ with $P > 0$ and $\tau > 0$. In the following theorem we establish necessary and sufficient conditions for the existence of a protocol (5.2) such that the corresponding matrix inequalities (5.19) (or (5.20)) indeed have such common solution $(P, \tau)$.

Before doing this, we first note that the unknown $\tau$ can be removed: By defining $\bar{P} := \frac{1}{2\tau} P$ and dividing (5.20) by $2\tau$, we see that (5.20) has a solution $(P, \tau)$ if and only if the inequalities

$$\begin{bmatrix}
\bar{P} \left( A - \frac{1}{2} E(S_1 + S_2) C + B H_i M \right) + \\
\left( A - \frac{1}{2} E(S_1 + S_2) C + B H_i M \right)^T \bar{P} - \bar{P} E \\
+ \frac{1}{4} C^T(S_2 - S_1)^2 C \\
-E^T \bar{P} \\
-I_p
\end{bmatrix} < 0, \ i = 2, \cdots, N \quad (5.21)
$$

have a common solution $\bar{P}$. Below we will focus on the condition (5.21) instead of (5.19) or (5.20). Our following protocol design is inspired by the measurement output feedback $H_\infty$ controller construction for general linear systems in [52]. It is important to note that, whereas (5.21) involves a common solution to $N - 1$ inequalities, the conditions obtained below only involve three inequalities, regardless of the number of agents.

**Theorem 5.8.** Let $n_c$ be a positive integer. There exist $A_c \in \mathbb{R}^{n_c \times n_c}$, $B_c \in \mathbb{R}^{n_c \times q}$,
$C_c \in \mathbb{R}^{m \times n_c}, D_c \in \mathbb{R}^{n_c \times n_e}$ and $\bar{P} > 0$ such that (5.21) holds for all $i = 2, \cdots, N$ if and only if there exist matrices $X_p > 0$, $Y_p > 0$ of size $n \times n$ such that

$$[B_p] \begin{bmatrix} A_p X_p + \frac{1}{4} E_p E_p^T & X_p \bar{C}_p^T \end{bmatrix} \begin{bmatrix} C_p X_p \\ -I \end{bmatrix} [B_p]^{\top} < 0, \quad (5.22)$$

$$[M_p^{\top}] \begin{bmatrix} Y_p A_p \bar{A}^T Y_p + \frac{1}{4} \bar{C}_p^T \bar{C}_p \end{bmatrix} \begin{bmatrix} E_p^T Y_p \\ -I \end{bmatrix} [M_p^{\top}]^{\top} < 0, \quad (5.23)$$

$$Y_p - \frac{1}{4} X_p^{-1} > 0, \quad (5.24)$$

where $\bar{A}_p := A_p - \frac{1}{2} E_p (S_1 + S_2) C_p$, $\bar{C}_p := (S_2 - S_1) C_p$.

In this case, suitable $A_c$, $B_c$, $C_c$, $D_c$ and $\bar{P}$ are obtained as follows.

- Choose $r_2 > 0$ such that
  $$[\bar{A}_p X_p + X_p \bar{A}^T + \frac{1}{4} E_p E_p^T - 2 r_2 \lambda_2 B_p B_p^T \bar{C}_p^T \bar{C}_p X_p \bar{C}_p^T \bar{C}_p^T < 0; \quad (5.25)$$

- Choose $r_1 > 0$ such that
  $$[Y_p \bar{A}_p + \bar{A}^T Y_p + \frac{1}{4} \bar{C}_p^T \bar{C}_p - 2 r_1 M_p^{\top} M_p \bar{Y}_p \bar{Y}_p - \frac{1}{4} Y_p - \frac{1}{4} X_p^{-1} \bar{C}_p^T \bar{C}_p Y_p - \frac{1}{4} E_p E_p^T < 0; \quad (5.26)$$

- Define $F := - r_2 B_p^T X_p^{-1}$, $G := - r_1 Y_p^{-1} M_p^{\top}$;

- Define
  $$R_F^i := (\bar{A}_p + \lambda_i B_p F)^T X_p^{-1} + X_p^{-1} (\bar{A}_p + \lambda_i B_p F) + \frac{1}{4} X_p^{-1} E_p E_p^T X_p^{-1} + \bar{C}_p^T \bar{C}_p, \quad i = 2, \cdots, N,$$

  and
  $$R_G := (\bar{A}_p + G M_p) Y_p^{-1} + Y_p^{-1} (\bar{A}_p + G M_p)^T + \frac{1}{4} Y_p^{-1} \bar{C}_p^T \bar{C}_p Y_p^{-1} + E_p E_p^T;$$

- Choose a real number $k \in (0, 1)$ such that
  $$Y_p R_G Y_p < \frac{1}{4} (1 - k) R_F^N;$$
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- Define $Z_p := Y_p - \frac{1}{4} X_p^{-1}$, $\bar{G} := Z_p^{-1} Y_p G$, 
  
  $\Delta_1 := \frac{1}{4} k Z_p^{-1} \left( \bar{A}_p^T X_p^{-1} + X_p^{-1} \bar{A}_p + \frac{1}{4} X_p^{-1} E_p E_p^T X_p^{-1} + \bar{C}_p^T \bar{C}_p \right)$, 

  $\Delta_2 := -\frac{1}{4} Z_p^{-1} \left( (1 - k) F^T B_p^T X_p^{-1} - k X_p^{-1} B_p F \right)$;

- Choose $A_c := \bar{A}_p + \frac{1}{4} E_p E_p^T X_p^{-1} + \bar{G} M_p + \Delta_1$, $B_c := -\bar{G}$, $C_c := F$, 
  $D_c := B_p F + \Delta_2$, 
  
  $\bar{P} := \begin{bmatrix} Y_p & -Z_p \\ -Z_p & Z_p \end{bmatrix}$.

Obviously, this protocol has the same state dimension as the agents, i.e. $n_c = n$.

**Proof.** (only if) Define $X := \frac{1}{4} \bar{P}^{-1}$. We get

$$
\begin{bmatrix}
(\bar{A} + BH_i M) X + X (\bar{A} + BH_i M)^T + \frac{1}{4} EE^T \bar{C} \bar{C}^T \\
CX
\end{bmatrix} < 0, \quad i = 2, \ldots, N,
$$

where $\bar{A} := A - \frac{1}{2} E(S_1 + S_2) C$, $\bar{C} := (S_2 - S_1) C$, and thus

$$
\begin{bmatrix}
B^\perp & \bar{A} X + X \bar{A}^T + \frac{1}{4} EE^T \bar{C} \bar{C}^T \\
0 & CX
\end{bmatrix} < 0.
$$

Similarly, we have

$$
\begin{bmatrix}
M^T & Y A + A^T Y + \frac{1}{4} C^T C \bar{Y} \bar{E} \\
0 & E^T \bar{Y}
\end{bmatrix} < 0,
$$

where $Y = \bar{P}$. Partition

$$
X = \begin{bmatrix} X_p & X_{pc} \\ X_{pc} & X_c \end{bmatrix}, \quad Y = \begin{bmatrix} Y_p & Y_{pc} \\ Y_{pc} & Y_c \end{bmatrix},
$$

appropriately. Note that $[B] \perp = [B^\perp \begin{bmatrix} 0 & I \\ 0 & 1 \end{bmatrix}]$, 

$B^\perp = [B_p^\perp \begin{bmatrix} 0 & I \\ 0 & 1 \end{bmatrix}]$, 

Thus we obtain (5.22) and (5.23). Furthermore, $XY = \frac{1}{4} I$ implies that $X_p Y_p + X_{pc} Y_{pc} = \frac{1}{4} I$ and $X_p Y_{pc} + X_{pc} Y_c = 0$. Thus $Y_p - \frac{1}{4} X_p^{-1} = Y_{pc} Y_{pc}^{-1} Y_{pc}^T \geq 0$. Since (5.22) and (5.23) are strict, we can always perturb $X_p$ and $Y_p$ slightly so that (5.22) and (5.23) still hold, but also the strict inequality (5.24) holds.
(if) By Finsler’s lemma [24], (5.22) and (5.23) imply that there exist \( r_2 > 0 \) and \( r_1 > 0 \) such that (5.25) and (5.26) hold, respectively. By taking the Schur complement, (5.25) is equivalent to

\[
\bar{A}_p X_p + X_p \bar{A}^T_p + \frac{1}{4} E_p E_p^T - 2r_2 \lambda_2 B_p B_p^T + X_p \bar{C}^T_p \bar{C}_p X_p < 0.
\]

Thus we have

\[
R^i_F \leq \left( \bar{A}_p + \lambda_2 B_p F \right)^T X_p^{-1} + X_p^{-1} \left( \bar{A}_p + \lambda_2 B_p F \right) + \frac{1}{4} X_p^{-1} E_p E_p^T X_p^{-1} + \bar{C}^T_p \bar{C}_p < 0
\]

for all \( i = 2, \cdots, N \), and similarly, \( R_G < 0 \). Since we choose \( k \in (0, 1) \) such that \( Y_p R_G Y_p < (1 - k) R^N_F \), together with \( R^N_F \leq R^{N-1}_F \leq \cdots \leq R^2_F \), we get \( Y_p R_G Y_p < (1 - k) R^i_F, \forall \ i = 2, \cdots, N \). Note that such \( k \) always exists.

Denote \( A_i := \bar{A} + BH_i M, i = 2, \cdots, N \). Then (5.21) holds if and only if

\[
\bar{P} \bar{A}_i + \bar{A}_i^T \bar{P} + \frac{1}{4} C^T \bar{C} + \bar{P} G \bar{P}^T \bar{P} < 0, \ i = 2, \cdots, N,
\]

(5.27)

where \( \bar{P} = S^T \bar{P} S = \begin{bmatrix} \frac{1}{4} X^{-1}_p & 0 \\ 0 & Z_p \end{bmatrix} \), \( \bar{A}_i = S^{-1} A_i S \), \( \bar{E} = S^{-1} E \), \( \bar{C} = \bar{C} S \) and

\[
S = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix}.
\]

We first compute

\[
\bar{A}_i = \begin{bmatrix} \bar{A}_p + \lambda_i B_p C_c & -\lambda_i B_p C_c \\ \bar{A}_p - B_c M_p + \lambda_i B_p C_c - A_c - \lambda_i D_c & -\lambda_i B_p C_c + A_c + \lambda_i D_c \end{bmatrix},
\]

\[
\bar{C} = \begin{bmatrix} \bar{C}_p & 0 \\ 0 & \bar{C}_p \end{bmatrix}, \bar{E} = \begin{bmatrix} E_p \\ E_p \end{bmatrix}.
\]

By straightforward computation, the (1, 1) block of the left hand side of (5.27) turns out to be \( \frac{1}{4} R^i_F \). The (2, 1) and (2, 2) blocks can be computed to be equal to \( -\frac{1}{4} k R^i_F \) and \( Y_p R_G Y_p - \frac{1}{4} R^i_F + \frac{1}{2} k R^i_F \), respectively. The detailed computation can be found in the Appendix. Thus the left hand side of (5.27) equals

\[
\begin{bmatrix} \frac{1}{4} R^i_F & -\frac{1}{4} k R^i_F \\ -\frac{1}{4} k R^i_F & Y_p R_G Y_p - \frac{1}{4} R^i_F + \frac{1}{2} k R^i_F \end{bmatrix}
\]

for all \( i = 2, \cdots, N \). The latter equals

\[
\begin{bmatrix} \frac{1}{4} (1 - k) R^i_F & 0 \\ 0 & Y_p R_G Y_p - \frac{1}{4} (1 - k) R^i_F \end{bmatrix} + \frac{1}{4} k \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \otimes R^i_F
\]

for all \( i = 2, \cdots, N \). Obviously, the first term above is negative define and the
second term is negative semi-define since $k > 0$, $\left[ \begin{smallmatrix} 1 & -1 \\ 1 & 1 \end{smallmatrix} \right] \succeq 0$ and $R_i^T < 0$. Therefore, (5.27) and also (5.21) hold for all $i = 2, \cdots, N$. This completes the proof. \hfill \square

Remark 5.9. Theorem 5.8 indeed provides the steps to design a protocol (5.2) that robustly synchronizes the network. The state space dimension of the protocol is equal to that of the agents, and the computational steps in the theorem yield the parameter matrices $A_c, B_c, C_c$ and $D_c$ after solving the three matrix inequalities (5.22), (5.23) and (5.24). As a side result, this theorem also gives an explicit expression for a common solution $\bar{P} > 0$ to the LMI's (5.21).

5.5 Simulation examples

In this section we will give a numerical example to illustrate the results obtained in this chapter. For the same reason explained in the previous chapters, the incremental passivity case is omitted here. As in [74], a Chua’s circuit model is taken as the individual agent dynamics:

$\begin{align*}
\dot{x} &= A_p x + B_p u + E_p z \\
z &= C_p x \\
y &= M_p x \\
d &= -\phi(z)
\end{align*}$

where $x = [x_1, x_2, x_3]^T$, $A_p = \begin{bmatrix} -3.2 & 10 & 0 \\ 1 & -1 & 1 \\ 0 & -14.87 & 0 \end{bmatrix}$, $M_p = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $B_p = [1, 1, 0]^T$, $C_p = [1, 0, 0]$, $E_p = [-2.95, 0, 0]^T$, and $\phi(z) = |x_1 + 1| - |x_1 - 1|$. Clearly, $\phi(z)$ is incrementally sector bounded within sector $[0, 2]$. We will consider a connected undirected network of 50 such agents. The interconnection topology is randomly generated, and its second smallest and largest Laplacian eigenvalues are computed to be 0.1784 and 7.3237, respectively. All the computation below can be done easily using the Matlab LMI Control Toolbox.

First, by (5.22), we compute a suitable $X_p$ as

$X_p = \begin{bmatrix} 0.9529 & 0.6375 & 0.1156 \\ 0.6375 & 0.7789 & 0.0440 \\ 0.1156 & 0.0440 & 1.9730 \end{bmatrix}$. 

Then (5.24) becomes linear and together with (5.23), a suitable $Y_p$ is computed to be

$$
Y_p = \begin{bmatrix}
3.3329 & -0.2383 & -0.0022 \\
-0.2383 & 3.3960 & -0.7572 \\
-0.0022 & -0.7572 & 3.1047
\end{bmatrix}.
$$

Following the rest of the steps in Theorem 5.8, the protocol parameter matrices are computed to be:

$$
A_c = \begin{bmatrix}
-233.4831 & 31.4925 & -0.3681 \\
26.3295 & -269.3691 & 0.4859 \\
8.0665 & -84.8507 & -0.1220
\end{bmatrix},
$$

$$
B_c = \begin{bmatrix}
239.1310 & -23.4507 \\
-23.4507 & 264.3036 \\
-7.7791 & 68.4690
\end{bmatrix},
$$

$$
C_c = \begin{bmatrix}
-229.7929 & -499.9370 & 24.6057
\end{bmatrix},
$$

$$
D_c = \begin{bmatrix}
-236.7923 & -515.1647 & 25.3552 \\
-249.9538 & -543.7989 & 26.7645 \\
-4.2914 & -9.3364 & 0.4595
\end{bmatrix}.
$$

Let $x_i = [x_{i1}, x_{i2}, x_{i3}]^T$ and $w_i = [w_{i1}, w_{i2}, w_{i3}]^T$ be the agent state and the protocol state, $i = 1, 2, \cdots, 50$. Denote $X_j := [x_{1j}, x_{2j}, \cdots, x_{50j}]^T$ and $W_j := [W_{1j}, W_{2j}, \cdots, W_{50j}]^T$ for $j = 1, 2, 3$. Choose the initial states as $x_i(0) = 0.1[i, i, i]^T$, $w_i(0) = 0_3$, $i = 1, 2, \cdots, 50$. The first components of the trajectories of the network (5.3), i.e. $X_1$, are plotted in Fig. 5.2, where four plots are given using different time scales to visualize the synchronization process more vividly. Clearly, the network reaches synchronization and the trajectories are bounded. The first components of the trajectories of the protocol dynamics, i.e. $W_1$, are plotted in Fig. 5.3. As in [65], it can be shown that the dynamic part of the protocol in fact acts as an observer for the weighted sum of the relative states based on the weighted sum of the relative outputs $\sum_{j=1}^{N} a_{ij}(y_i - y_j)$. Thus, for each $i$, the protocol state $w_i$ is an estimate of the weighted sum of the relative states $\sum_{j=1}^{N} a_{ij}(x_i - x_j)$ and the protocol state converges to zero.

### 5.6 Conclusions

In this chapter we have generalized the results in Chapter 3 on robust synchronization of homogeneous Lur'e networks by static, relative state based protocols to the
case that only relative measurement outputs are available. In this case dynamic protocols are used. We have given sufficient conditions under which a dynamic protocol exists that synchronizes the network. These conditions involve feasibility of a set of three LMI’s, reminiscent of the three LMI conditions in $H_\infty$ control by
measurement feedback. It is important to note that only three LMI’s are involved, regardless of the size of the network, i.e. the number of agents. In this sense, our results are scalable. The LMI’s only involve the linear part of the dynamics of the individual agents. Only in the computation of the protocol matrices the eigenvalues of the Laplacian matrix associated with the interconnection graph play a role. In particular, the matrices defining the protocol highly depend on the smallest nonzero eigenvalue and the largest eigenvalue of the Laplacian matrix. We have validated our results by means of a numerical simulation example involving synchronization of a grid of Chua’s circuits.