Chapter 2
Preliminaries

This chapter introduces the notation used in this thesis, some basic facts from algebraic graph theory, and some relevant important results in mathematical control theory.

2.1 Notation

Notation that is used frequently in this thesis is listed below. Since they are commonly used, we do not provide detailed definitions or explanations.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>( \mathbb{R} )</td>
<td>field of real numbers</td>
</tr>
<tr>
<td>( \mathbb{R}^n )</td>
<td>( n )-dimensional real Euclidean space</td>
</tr>
<tr>
<td>( \mathbb{R}^{m \times n} )</td>
<td>space of ( m \times n ) real matrices</td>
</tr>
<tr>
<td>( \mathbb{R}^+ )</td>
<td>field of nonnegative real numbers</td>
</tr>
<tr>
<td>( \mathbb{C} )</td>
<td>field of complex numbers</td>
</tr>
<tr>
<td>( \mathbb{C}^n )</td>
<td>( n )-dimensional complex Euclidean space</td>
</tr>
<tr>
<td>( \mathbb{C}^{m \times n} )</td>
<td>space of ( m \times n ) complex matrices</td>
</tr>
<tr>
<td>( \mathbf{0}<em>{n} ) (( \mathbf{1}</em>{n} ))</td>
<td>( n \times 1 ) column vector of all zeros (ones)</td>
</tr>
<tr>
<td>( |v|, |v|_\infty )</td>
<td>Euclidean norm, infinity norm of vector ( v )</td>
</tr>
<tr>
<td>( \mathbf{0}, \mathbf{I} )</td>
<td>zero matrix, identity matrix of compatible dimensions</td>
</tr>
<tr>
<td>( M^T )</td>
<td>transpose of real matrix ( M )</td>
</tr>
<tr>
<td>( M^* )</td>
<td>conjugate transpose of complex matrix ( M )</td>
</tr>
<tr>
<td>( \lambda_i(M) )</td>
<td>the ( i )th eigenvalue of real square matrix ( M )</td>
</tr>
<tr>
<td>( \lambda_{\min}(M) )</td>
<td>the smallest real part of the eigenvalues of square matrix ( M )</td>
</tr>
<tr>
<td>( \text{diag}{M_1, \cdots, M_m} )</td>
<td>diagonal matrix with diagonal entries ( M_1, \cdots, M_m )</td>
</tr>
<tr>
<td>( M &gt; (\geq) 0 )</td>
<td>positive (semi)definite matrix ( M )</td>
</tr>
<tr>
<td>( M &lt; (\leq) 0 )</td>
<td>negative (semi)definite matrix ( M )</td>
</tr>
<tr>
<td>( \otimes )</td>
<td>Kronecker product</td>
</tr>
<tr>
<td>( \mathcal{G} )</td>
<td>graph</td>
</tr>
<tr>
<td>( A )</td>
<td>adjacency matrix</td>
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<tr>
<td>( \mathcal{L} )</td>
<td>Laplacian matrix</td>
</tr>
<tr>
<td>( \min(\cdot) )</td>
<td>minimum</td>
</tr>
<tr>
<td>( \max(\cdot) )</td>
<td>maximum</td>
</tr>
<tr>
<td>( \sup(\cdot) )</td>
<td>supremum</td>
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</table>
Matrices, if not explicitly stated, are assumed to have compatible dimensions. The Kronecker product of an $n_1$ by $m_1$ matrix $M_1 = [m_{ij}]$ and an $n_2$ by $m_2$ matrix $M_2$ is the $n_1n_2$ by $m_1m_2$ matrix $M_1 \otimes M_2$ defined by

$$
M_1 \otimes M_2 = \begin{bmatrix}
m_{11}M_2 & m_{12}M_2 & \cdots & m_{1m_2}M_2 \\
m_{21}M_2 & m_{22}M_2 & \cdots & m_{2m_2}M_2 \\
\vdots & \vdots & \ddots & \vdots \\
m_{n_11}M_2 & m_{n_12}M_2 & \cdots & m_{n_1m_2}M_2
\end{bmatrix}.
$$

An important property of the Kronecker product is that

$$(M_1 \otimes M_2)(M_3 \otimes M_4) = (M_1M_3) \otimes (M_2M_4).$$

An $n$ by $n$ matrix $M = [m_{ij}]$ is called reducible if the indices $1, 2, \ldots, n$ can be divided into two disjoint nonempty sets $\{i_1, i_2, \ldots, i_u\}$ and $\{j_1, j_2, \ldots, j_v\}$ with $u + v = n$ such that $m_{i_\alpha j_\beta} = 0$ for all $\alpha = 1, 2, \ldots, u$ and $\beta = 1, 2, \ldots, v$. A square matrix is reducible if and only if it can be placed into a block upper-triangular form by simultaneous row/column permutations. A square matrix that is not reducible is said to be irreducible.

### 2.2 Algebraic graph theory

This section provides some requisite definitions and results from algebraic graph theory. For more details we refer to the standard reference [21] on algebraic graph theory. Since we use a graph to describe the interconnections among the agents in a network, the Laplacian matrix associated with the graph and in particular its Laplacian eigenvalues will be given. In addition, concepts of (general) algebraic connectivity of graphs that are frequently involved in synchronization criteria for multi-agent networks will be defined.

#### 2.2.1 Graphs

Graphs are used naturally to describe interconnections among agents in networks. A node and an edge in the graph represent an agent and the interconnection between two adjacent agents in the network, respectively. A (directed) graph $G = (V, E)$ consists of a finite, nonempty node set $V = \{1, 2, \ldots, N\}$ and an edge set $E \subset V \times V$, see Fig. 2.1. In this thesis, we will only consider graphs that are simple, i.e. graphs that do not contain self-loops $(i, i)$, $i = 1, 2, \ldots, N$, and that have at most one edge between any two different nodes. The graph $G$ is undirected if the property $(i, j) \in E \iff (j, i) \in E$ holds for all $i, j = 1, 2, \ldots, N$ and $i \neq j$, see Fig. 2.1b. Obviously, undirected graphs can be viewed as special directed graphs.
A directed path from node $i_0$ to node $i_l$ is a sequence of directed edges of the form $(i_{p-1}, i_p), p = 1, \cdots, l$. An undirected path can be defined likewise. A path with no repeated nodes is called a simple path. All paths in this thesis are assumed to be simple. A directed graph is strongly connected if there is a directed path from node $i$ to node $j$ for any $i \neq j$ in this graph. An undirected graph is connected if there is an undirected path connecting nodes $i$ and $j$ for any $i \neq j$. A directed tree on $N$ nodes is a directed graph with $N - 1$ edges, which has a node, called root, with the property that there is a directed path from it to each of the other nodes in the graph. A directed spanning tree in graph $G$ is a subgraph that is a directed tree and that has the same node set as the graph $G$ and part of the edges of $G$. A (strongly) connected subgraph of a (directed) graph is called a (strongly) connected component in this graph.

2.2.2 Adjacency and Laplacian matrices

The adjacency matrix $A$ associated with a graph $G$ is defined as $[A]_{ij} = a_{ij}$ if $(j, i) \in \mathcal{E}$ and $[A]_{ij} = 0$ otherwise, where $a_{ij} > 0$ is the edge weight of $(j, i)$. In this thesis, the existence of edge $(i, j)$ means that node/agent $j$ can receive the relative information with respect to node/agent $i$, i.e. node $i$ is a neighbor of node $j$. If all positive $a_{ij}$’s are ones, then it is an unweighted graph; otherwise, it is called weighted. Since the graph $G$ is simple, $a_{ii} = 0$ for all $i = 1, 2, \cdots, N$. The in-degree of node $i$ is defined by $d_i = \sum_{j=1}^{N} a_{ij}$. $\mathcal{D} := \text{diag}(d_1, d_2, \cdots, d_N)$ is named the in-degree matrix of the graph $G$. The Laplacian matrix $\mathcal{L}$ of the graph $G$ is defined by $\mathcal{L} := \mathcal{D} - A$.

Examples: The graph shown in Fig. 2.1a is directed and contains a directed spanning tree; the graph in Fig. 2.1b is undirected and connected. Assume that both of them are unweighted. The adjacency matrices associated with the two
graphs are, respectively,

\[ A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}. \]

The Laplacian matrices of the two graphs are, respectively,

\[ L_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \quad \text{and} \quad L_2 = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 3 & -1 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & -1 & -1 & 3 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{bmatrix}. \]

### 2.2.3 Properties

After modeling the network by a graph, its Laplacian matrix will play an important role in subsequent analysis and synthesis. In this subsection we will review the most important properties of the Laplacian matrix of a graph.

By the Gershgorin circle theorem \([22]\), it is easy to check that all Laplacian eigenvalues have nonnegative real parts. It is well known that \(L \mathbf{1}_N = 0_N\), i.e. \(\mathbf{1}_N\) is a right eigenvector associated with the Laplacian eigenvalue 0. The algebraic multiplicity of 0 as an eigenvalue of the Laplacian matrix is the number of (strongly) connected components in the corresponding graph.

#### Undirected Graphs:

Let \(G\) be an undirected graph with \(N\) nodes, where \(N \geq 2\). The undirected graph \(G\) is connected if and only if its Laplacian eigenvalue 0 has geometric multiplicity one, see e.g. \([33]\). In this case, the eigenvalues of the Laplacian matrix \(L\) associated with \(G\) can be ordered as

\[ \lambda_1 = 0 < \lambda_2 \leq \cdots \leq \lambda_N. \]

Furthermore, there exists an orthogonal matrix \(U = \begin{bmatrix} \sqrt{N} \mathbf{1}_N & U_2 \end{bmatrix}\), where \(U_2 \in \mathbb{R}^{N \times (N-1)}\), such that

\[ U^T L U = \text{diag}(0, \lambda_2, \cdots, \lambda_N). \]

It is obvious that \(U_2^T U_2 = I_{N-1}\) and \(U_2^T U_2^T = I_N - \frac{1}{N} \mathbf{1}_N \mathbf{1}_N^T\). We denote \(\Lambda := \text{diag}(0, \lambda_2, \cdots, \lambda_N)\), which can be partitioned as \(\Lambda = \begin{bmatrix} 0 & o_{(N-1) \times 1} \\ o_{1 \times (N-1)} & \bar{\Lambda} \end{bmatrix}\), where \(\bar{\Lambda} := \text{diag}(\lambda_2, \cdots, \lambda_N)\).

The following lemma will play a crucial role in Chapters 3 and 5.
Lemma 2.1. For any two vectors \( a = [a_1^T, a_2^T, \ldots, a_N^T]^T \) and \( b = [b_1^T, b_2^T, \ldots, b_N^T]^T \), where \( a_i, b_i \in \mathbb{R}^n \), \( i = 1, 2, \ldots, N \), we have

\[
a^T \left( U_2 U_2^T \otimes I_n \right) b = \frac{1}{N} \sum_{1 \leq i < j \leq N} (a_i - a_j)^T (b_i - b_j).
\]

Proof.

\[
a^T \left( U_2 U_2^T \otimes I_n \right) b = a^T \left( I_N - \frac{1}{N} 1_N 1_N^T \right) b
\]

\[
= \sum_{i=1}^{N} a_i^T b_i - \frac{1}{N} \sum_{i=1}^{N} \left( \sum_{j=1}^{N} a_j^T \right) b_i
\]

\[
= \sum_{i=1}^{N} \left( a_i^T - \frac{1}{N} \sum_{j=1}^{N} a_j^T \right) b_i
\]

\[
= \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} (a_i - a_j)^T b_i
\]

\[
= \frac{1}{2N} \sum_{i=1}^{N} \sum_{j=1}^{N} (a_i - a_j)^T (b_i - b_j)
\]

\[
= \frac{1}{N} \sum_{1 \leq i < j \leq N} (a_i - a_j)^T (b_i - b_j).
\]

Definition 2.1. [19] For an undirected graph with Laplacian matrix \( L \), its algebraic connectivity is defined as its second smallest Laplacian eigenvalue. This real number is equal to

\[
\lambda_2(L) = \min_{x^T 1_N = 0, x \neq 0_N} \frac{x^T L x}{x^T x}.
\]

The magnitude of this value reflects how well connected the overall graph is, and has been widely used in analysing the synchronizability and robustness of networks.

Directed Graphs:

Let \( G \) be a directed graph with \( N \) nodes, where \( N \geq 2 \). The graph \( G \) is strongly connected if and only if its Laplacian matrix \( L \) is irreducible; the graph \( G \) contains a directed spanning tree if and only if zero is a simple Laplacian eigenvalue. Note that except zero Laplacian eigenvalue(s), the other Laplacian eigenvalues associated with a directed graph might be in general strictly complex. Thus the value in
Definition 2.1 is not suitable to measure how well connected a directed graph is.
The general algebraic connectivity for directed graphs is defined below.

**Definition 2.2.** For a strongly connected directed graph with Laplacian matrix $L$, its general algebraic connectivity is defined to be the real number

$$c = \min_{x^T \xi = 0, x \neq 0} \frac{x^T (\Xi L + L^T \Xi) x}{2x^T \Xi x},$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_N)^T$, $\Xi = \text{diag}(\xi_1, \xi_2, \ldots, \xi_N)$, $\xi^T L = 0_N^T$ with $\xi_i > 0$, $\forall i = 1, 2, \ldots, N$, and $\sum_{i=1}^N \xi_i = 1$.

We have $c > 0$, see Corollary 2 in [71]. Besides, the condition $\sum_{i=1}^N \xi_i = 1$ on the left eigenvector $\xi$ associated with the unique zero Laplacian eigenvalue makes this left eigenvector unique as well. Thus the value in the above definition is fixed for a given strongly connected graph.

Assume that a graph $G$ contains a directed spanning tree and has $p(\geq 2)$ strongly connected subgraphs. Then the Laplacian matrix $L$ of the graph $G$ can be written in its Frobenius normal form (see e.g. [6]):

$$L = \begin{bmatrix}
L_{11} & 0 & \cdots & 0 \\
L_{21} & L_{22} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
L_{p1} & L_{p2} & \cdots & L_{pp} \\
\end{bmatrix}, \quad (2.1)$$

where $L_{qq} \in \mathbb{R}^{n_q \times n_q}$ are irreducible, $q = 1, 2, \cdots, p$, with $n_1 + n_2 + \cdots + n_p = N$. Let $L_{qq} = L_q + D_q$, where $L_q$ is the Laplacian matrix associated with the $q$th strongly connected subgraph in the graph $G$. Obviously, $L_{11}$ is associated with the strongly connected subgraph which is composed of all the roots in this graph and $D_1 = 0$. It is easy to see that the diagonal matrices $D_q \geq 0$ and $D_q \neq 0$, $q = 2, \cdots, p$. Intuitively, there is information flow from the first subgraph to the others. In addition, $p = 1$ if and only if $G$ is a strongly connected graph.

Note that $L$ and $L$ represent the same graph $G$. To obtain $L$, we can relabel the nodes in the graph. Of course, this does not change its topological structure and thus our following statements based on $L$ hold for $L$ as well.

Below we give the definition of general algebraic connectivity for the strongly connected subgraphs.

**Definition 2.3.** [71] For a graph containing a directed spanning tree with its Laplacian matrix in the form of (2.1), the general algebraic connectivity of the $q$th
strongly connected subgraph, \( q = 2, \ldots, p \), is defined to be the real number

\[
c_q = \min_{x \neq 0_{n_q}} \frac{x^T (\Xi_q L_{qq} + L_{qq}^T \Xi_q) x}{2x^T \Xi_q x}
= \lambda_{\min} \left( \frac{1}{2} \sqrt{\Xi_q^{-1}} (\Xi_q L_{qq} + L_{qq}^T \Xi_q) \sqrt{\Xi_q^{-1}} \right),
\]

where \( \xi_q = (\xi_{q1}, \ldots, \xi_{qn_q})^T, \Xi_q = \text{diag}(\xi_{q1}, \ldots, \xi_{qn_q}), \xi_q^T L_q = 0^T_{n_q} \) with \( \xi_{qr} > 0 \), \( \forall \ r = 1, \ldots, n_q \), and \( \sum_{r=1}^{n_q} \xi_{qr} = 1, \sqrt{\Xi_q} = \text{diag} \left( \sqrt{\xi_{q1}}, \ldots, \sqrt{\xi_{qn_q}} \right) \).

Denote the general algebraic connectivity of the first strongly connected subgraph by \( c_1 \), see Definition 2.2. We have \( c_q > 0, \forall \ q = 1, 2, \ldots, p \), see Lemma 14 in [71]. So, the general algebraic connectivity for a graph containing a directed spanning tree can be defined as follows.

**Definition 2.4.** For a graph containing a directed spanning tree with its Laplacian matrix in the form of (2.1), its general algebraic connectivity is defined to be the real number

\[
c = \min_{1 \leq q \leq p} c_q,
\]

where \( c_q \)'s are given in Definitions 2.2 and 2.3.

It is easy to check that \( c \) in both Definitions 2.2 and 2.4 will reduce to \( \lambda_2(L) \) in Definition 2.1 if the graph is undirected. For this reason, \( c \) is named the general algebraic connectivity.

The role of general algebraic connectivities will be demonstrated in Chapter 4, where robust synchronization of directed Lur'e networks is studied.

## 2.3 Mathematical control theory

In this section, several important concepts and theorems from mathematical control theory will be reviewed. The Lyapunov stability theory is quite well known and will be omitted here.

### 2.3.1 Schur complement lemma

First the Schur complement lemma is given.

**Lemma 2.2.** [22] Let \( M \) be a symmetric matrix partitioned into blocks:

\[
M = \begin{bmatrix} M_1 & M_2 \\ M_2^T & M_3 \end{bmatrix},
\]
Assume that $M_3$ is positive (negative) definite. Then the following properties are equivalent:

- $M$ is positive (negative) definite;
- The Schur complement of $M_3$ in $M$, defined as the matrix $M_1 - M_2 M_3^{-1} M_2^T$, is positive (negative) definite.

We have a similar argument to $M_1$ and its Schur complement.

### 2.3.2 S-procedure

We often encounter the constraint that some quadratic function (or quadratic form) be negative whenever some other quadratic functions (or quadratic forms) are all negative, for example, in our considered incremental sector boundedness case in this thesis. In some cases, such constraint can be expressed as an LMI in the data defining the quadratic functions or forms; in other cases, we can form an LMI that is a conservative but often useful approximation of the constraint. The S-procedure can be used to deal with this kind of constraint problem. Below we only introduce the S-procedure involving quadratic forms and strict inequalities. For the one involving nonstrict inequalities, we refer to Subsection 2.6.3 in [5].

Let $M_1$ and $M_2$ be real symmetric matrices. Then $x^T M_1 x < 0$ for all $x \neq 0$ such that $x^T M_2 x \leq 0$ if there exists $\tau \geq 0$ such that $M_1 - \tau M_2 < 0$. Here is only one constraint. The general multiple constraint case can be found in [5].

The Schur complement lemma and the S-procedure are two critical tools in the theory of linear matrix inequalities, which will be widely used throughout this thesis.

### 2.3.3 Minimal left annihilator

Below we give the definition of minimal left annihilator of a given matrix.

**Definition 2.5.** [56] For a given matrix $B \in \mathbb{C}^{n \times m}$ with rank $r < n$, we denote by $B^\perp \in \mathbb{C}^{(n-r) \times n}$ any matrix of full row rank such that $B^\perp B = 0$. Any such matrix $B^\perp$ is called a minimal left annihilator of $B$.

Note that, for a given $B$, a minimal left annihilator $B^\perp$ exists if and only if $B$ has linearly dependent rows. The set of all such matrices is given by $B^\perp = TU_2^*$, where $T$ is any nonsingular matrix and $U_2$ is obtained from the singular value decomposition

$$B = [U_1 \quad U_2] \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^* \\ V_2^* \end{bmatrix}.$$ 

Thus, for a given $B$, $B^\perp$ is not unique. Throughout this paper, $B^\perp$ will denote any choice from this set of matrices.
2.3.4 Finsler’s lemma

A useful result in the application of linear matrix inequalities in robust control is the following lemma, known as Finsler’s lemma:

**Lemma 2.3.** [56] Let \( x \in \mathbb{R}^n \), \( M_1 \in \mathbb{R}^{n \times n} \) be symmetric, and \( M_2 \in \mathbb{R}^{m \times n} \) such that \( \text{rank}(M_2) < m \). Then the following statements are equivalent:

- \( x^T M_1 x < 0 \) for all \( x \neq 0 \) such that \( M_2 x = 0 \);
- \( M_2^+ M_1 (M_2^+)^T < 0 \);
- \( \exists \mu \in \mathbb{R} \) such that \( M_1 - \mu M_2 M_2^T < 0 \);
- \( \exists M_3 \in \mathbb{R}^{n \times m} \) such that \( M_1 + M_3 M_2 + M_2^T M_3^T < 0 \).

Note that in the above \( M_3 = \frac{1}{2} \mu M_2^T \) is one feasible solution. Finsler’s lemma will be applied to several problems in this thesis. By means of this lemma, suitable synchronizing protocol design can be done.

2.3.5 Input-to-state stability

Consider the system

\[
\dot{x} = f(x, u),
\]

where the state \( x(t) \in \mathbb{R}^n \), the input \( u(t) \in \mathbb{R}^m \) is essentially bounded, and the map \( f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n \) is assumed to be locally Lipschitz with \( f(0, 0) = 0 \).

Before moving on, we introduce the notations of class \( K \), class \( K_\infty \), class \( L \) and class \( KL \) functions.

**Definition 2.6.** A continuous function \( \alpha : [0, a) \rightarrow [0, \infty) \) is of class \( K \) if it is strictly increasing and \( \alpha(0) = 0 \). It belongs to class \( K_\infty \) if \( a = \infty \) and \( \alpha(r) \rightarrow \infty \) as \( r \rightarrow \infty \).

**Definition 2.7.** A continuous function \( \beta : [0, \infty) \rightarrow [0, \infty) \) is of class \( L \) if it is monotonically decreasing and \( \lim_{t \to \infty} \beta(t) = 0 \).

**Definition 2.8.** A continuous function \( \gamma(\cdot, \cdot) \) that is of class \( K \) with respect to its first argument and is of class \( L \) with respect to its second argument belongs to class \( KL \).

These notions are very important in Lyapunov stability analysis and in particular in input-to-state stability.

**Definition 2.9.** The system (2.2) is said to be locally input-to-state stable (ISS) if there exist a \( KL \) function \( \gamma \), a class \( K \) function \( \alpha \) and constants \( k_1 > 0, k_2 > 0 \) such that

\[
\|x(t)\| \leq \gamma(\|x_0\|, t) + \alpha(\sup_{t \geq 0} \|w_T(t)\|), \quad \forall t \geq 0, \ 0 \leq T \leq t
\]  

(2.3)
for all $x_0 \in D$ satisfying $\|x_0\| < k_1$, and all $u \in D_u$ satisfying $\sup_{t \geq 0} \|u_T(t)\| < k_2$. It is said to be globally ISS if (2.3) holds for all $x_0, u$, any initial state and any bounded input.

Assume that the system (2.2) is ISS. Obviously, $\dot{x} = f(x, 0)$ has an asymptotically stable equilibrium point at the origin.

**Theorem 2.4.** A continuous function $V : D \mapsto \mathbb{R}$ is an ISS Lyapunov function on $D$ for the system (2.2) if and only if there exist class $K$ functions $\alpha_1, \alpha_2, \alpha_3$ and $\sigma$ such that the following two conditions are satisfied

$$\alpha_1(\|x\|) \leq V(x(t)) \leq \alpha_2(\|x\|) \quad \forall \ x \in D, \ t \geq 0, \quad (2.4)$$

$$\frac{\partial V(x)}{\partial x} f(x, u) \leq -\alpha_3(\|x\|) + \sigma(\|x\|) \quad \forall \ x \in D, \ u \in D_u. \quad (2.5)$$

$V$ is a global ISS Lyapunov function if $D = \mathbb{R}^n$, $D_u = \mathbb{R}^m$ and $\alpha_1, \alpha_2, \alpha_3, \sigma \in K_\infty$.

Consider two cascade-connected systems shown in Fig. 2.2, where $\Sigma_1$ and $\Sigma_2$ are given respectively by

$$\Sigma_1 : \quad \dot{x} = f(x, u), \quad (2.6)$$

$$\Sigma_2 : \quad \dot{y} = g(y, x). \quad (2.7)$$

**Theorem 2.5.** Consider the cascade interconnection of the systems $\Sigma_1$ and $\Sigma_2$. If both the systems are input-to-state stable, then the composite system $\Sigma$

$$\Sigma : \quad u \rightarrow \begin{bmatrix} x \\ y \end{bmatrix}$$

is also input-to-state stable.

For details on input-to-state stability, we refer to [57].