This appendix shows that if $F$ is the analysis operator for a frame corresponding to the unit sphere $S_V$ in the Hilbert space $V$, $F^+u$ with $u = Fa \lor Fb \lor \cdots$ equals a weighted average of $a$, $b$, etc., with the weights corresponding to the angle over which each vector is maximal. Although extremely natural, we will see that this statement is much less trivial than it might initially appear.

Geometric algebra and its associated calculus [68, 70] are used throughout this appendix. Although a comprehensive discussion of this topic is way beyond the scope of this appendix, for understanding the argument below it is important to know that one can have so-called “$n$-blades” of varying degree $n$, with a 0-blade being a scalar (associated with a point), a 1-blade being a vector associated with a line, a 2-blade being associated with a hyperplane and a 3-blade being associated with a volume, and so on (until we reach the $d$-blade, with $d$ the dimensionality of the space). It is possible to compute the norm of blades, as well as invert them (with respect to the so-called geometric product, which is associative but not commutative).

A particularly important blade is the pseudoscalar $I$, representing a unit (hyper)volume (so a 3-blade in 3D space). If we have a vector $v \in V$, then $v I$ is associated with the hyperplane through the origin perpendicular to $v$. It is also important to know that $I^{-1}$ equals $I$ or $-I$, depending on the dimensionality of the space.

Our first order of business is to find $F^+$. Analogous to the argument in Section 3.1.2, we can see that the frame is tight. To find the associated frame coefficient, we first note that $F^+F = \int_{S_V} s \circ 2 \parallel ds\parallel$ can be considered a degree-2 identity tensor multiplied by the frame coefficient, and that $I_2 \cdot I_2 = d$. We can now see that

$$\frac{1}{d} I_2 \cdot (F^+F) = \frac{1}{d} I_2 \cdot \int_{S_V} s \circ 2 \parallel ds\parallel = \frac{1}{d} \int_{S_V} I_2 \cdot s \circ 2 \parallel ds\parallel = \frac{A}{d}.$$  

Here $V$ is a $d$-dimensional Hilbert space, and $S_V$ is the unit sphere in $V$; $\parallel ds\parallel$ represents the “magnitude” of the infinitesimal surface element of the unit sphere at $s$, such that $\int_{S_V} \parallel ds\parallel$ equals the surface area $A = \frac{d \pi^{d/2}}{\Gamma(d/2+1)}$ of the unit sphere. Note that $\frac{A}{d} = \frac{\pi^{d/2}}{\Gamma(d/2+1)} = O$, the volume of the unit ball. If we write $\bar{a} = F^+u$, we can now see that

$$\bar{a} = \frac{1}{O} F^+u = \frac{1}{O} \int_{S_V} u s \parallel ds\parallel.$$
Figure B.1: **LEFT:** Four vectors \((a_1, a_2, \ldots)\) are shown, along with the circles resulting from plotting their projections on all directions and the arcs (on the outer ring) over which different points are considered maximal. The supremum \((u)\) of the four lifted vectors is represented by the dashed curve. **RIGHT:** The partition of the unit sphere and ball for the same \(u\). Note that for the purposes of integration the boundaries are oriented, so \(L_{31}\) contains the same points as \(L_{13}\), but is assigned opposite orientation.

Since \(s\) is normal to the surface of the unit sphere at \(s\), we have \(s \parallel ds\| = ds \Gamma^{-1}\) \cite[eq. 6.149]{68}, and thus:

\[
\bar{a} = \frac{1}{O} \int_{S_V} u_s ds \Gamma^{-1}.
\]

Now, suppose that there is a partition of the unit sphere \(S_V\) (where the regions overlap at (and only at) their boundaries), such that each (closed) region \(S^{(i)} (i \in I)\) corresponds to a vector \(a_i\). A region is said to correspond to a vector \(a_i\) if for every \(s \in S^{(i)}\), we have that \(u_s = s \cdot a_i\). This allows us to write the integral above as a sum of integrals over the \(S^{(i)}\):

\[
\bar{a} = \frac{1}{O} \sum_{i \in I} \int_{S^{(i)}} s \cdot a_i ds \Gamma^{-1}.
\]

Each of these integrals integrates over a patch of the unit \(d\)-sphere. If we denote by \(B^{(i)}\) the set \(\{\lambda s \mid \lambda \in [0, 1] \text{ and } s \in S_d^{(i)}\}\) (see Fig. B.1b), then the patch \(S^{(i)}\) is part of the boundary of \(B^{(i)}\). Also, the \(B^{(i)}\) form a partition of the unit ball \(B_V\). Recognizing that \(a_i = \nabla_s (s \cdot a_i)\), and employing the fundamental theorem of calculus \cite[eq. 6.142]{68}, we can
thus rewrite the above in terms of integrals over the $B^{(i)}$ and the parts of their boundary that are not covered by $S^{(i)}$:

$$\bar{a} = \frac{1}{O} \sum_{i \in I} \left[ \int_{B^{(i)}} a_i \, dv - \int_{\partial B^{(i)} \setminus S^{(i)}} v \cdot a_i \, dv \right] I^{-1}$$

$$= \frac{1}{O} \sum_{i \in I} \left[ \int_{B^{(i)}} a_i \|dv\| I - \int_{\partial B^{(i)} \setminus S^{(i)}} v \cdot a_i \, dv \right] I^{-1}$$

$$= \frac{1}{O} \sum_{i \in I} |B^{(i)}| a_i - \frac{1}{O} \sum_{i \in I} \int_{\partial B^{(i)} \setminus S^{(i)}} v \cdot a_i \, dv \, I^{-1}. \quad (B.1)$$

Here $\partial B^{(i)}$ denotes the boundary of $B^{(i)}$ and $|B^{(i)}|$ denotes the volume of $B^{(i)}$, $|S^{(i)}|$ will be used to denote the area of $S^{(i)}$. From the construction of the $B^{(i)}$ it follows that $|B^{(i)}|/O = |S^{(i)}|/A$ (recall that $O$ is the volume of the unit sphere, while $A$ is its surface area), thus

$$\bar{a} = \frac{1}{A} \sum_{i \in I} |S^{(i)}| a_i - \frac{1}{O} \sum_{i \in I} \int_{\partial B^{(i)} \setminus S^{(i)}} v \cdot a_i \, dv \, I^{-1}. \quad (B.1)$$

This shows that $\bar{a}$ is a convex (linear) combination of the $a_i$, minus another term. It can be shown that in general this other term does not vanish.

To eliminate the unwanted term from Eq. (B.1) we take a closer look at the boundaries of the $B^{(i)}$. As mentioned before, the $B^{(i)}$ form a partition of the unit $d$-ball. Also, we can see that $S^{(i)}$ is the only part of the boundary of $B^{(i)}$ that lies on the unit sphere. This leads us to conclude that the sets $\partial B^{(i)} \setminus S^{(i)}$ consist entirely of boundaries between adjacent $B^{(i)}$ (see Fig. B.1b). So, if we denote the boundary between $B^{(i)}$ and $B^{(j)}$ by $L_{ij}$, and ignore the factor $-(1/O)$, we get:

$$\sum_{i \in I} \int_{\partial B^{(i)} \setminus S^{(i)}} v \cdot a_i \, dv \, I^{-1} = \sum_{i \in I} \sum_{j \neq i} \int_{L_{ij}} v \cdot a_i \, dv \, I^{-1}.$$

These two nested sums essentially cover all pairs $(i,j)$ such that $k \neq l$. As $L_{ij}$ and $L_{ij}$ cover the same points with opposite orientation, we can combine the term for $(i,j)$ and $(j,i)$:

$$\sum_{i \in I} \sum_{j \neq i} \int_{L_{ij}} v \cdot a_i \, dv \, I^{-1}$$

$$= \sum_{(i,j)|i<j} \int_{L_{ij}} v \cdot a_i \, dv \, I^{-1} + \int_{L_{ij}} v \cdot a_j \, dv \, I^{-1}$$

$$= \sum_{(i,j)|i<j} \int_{L_{ij}} v \cdot a_i \, dv \, I^{-1} - \int_{L_{ij}} v \cdot a_j \, dv \, I^{-1}$$

$$= \sum_{(i,j)|i<j} \int_{L_{ij}} v \cdot (a_i - a_j) \, dv \, I^{-1}. \quad (B.2)$$

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Figure B.2: Left: The same three points and associated partition as before are shown, together with the resulting point $\bar{a}$. By construction $L_{13}$ is perpendicular to $a_3 - a_1$ and $\bar{a}$ can be computed using Eq. (B.3). Right: The same three points are shown, but now with a partition for which Eq. (B.2) is non-zero. The point $b$ is the convex linear combination that would result from Eq. (B.3), while $\bar{a}$ is computed using Eq. (B.1). The red arrow shows the difference between the two points. For comparison, the grey point shows the position of $\bar{a}$ in the left situation.

Clearly, the above is zero if $v \cdot (a_i - a_j) = 0$ or, equivalently, $v \cdot a_i = v \cdot a_j$, for all $i, j \in I$ and $v \in L_{ij}$. For this to be true all $v \in L_{ij}$ must lie on a hyperplane through the origin perpendicular to $a_i - a_j$ (such a hyperplane is also referred to as the hyperplane dual to $a_i - a_j$), see Fig. B.2. Due to the construction of the $B^{(i)}$ this is equivalent to demanding that the boundaries between the $S^{(i)}$ lie on such hyperplanes. Equivalently, $u$ must be continuous in the sense that $v \cdot a_i$ equals $v \cdot a_j$ at the boundary between $S^{(i)}$ and $S^{(j)}$.

To recap, suppose we have a $u$ such that the unit sphere can be partitioned into regions $S^{(i)}$ with associated $a_i$, such that $u_s = s \cdot a_i$ for all $i \in I$ and $s \in S^{(i)}$. The above shows that, if for all adjacent $S^{(i)}$ and $S^{(j)}$, the boundary between them lies on the hyperplane dual to $a_i - a_j$ (in other words: if $u$ is continuous), the vector $\bar{a} = F^+ u$ can be computed as the convex linear combination

$$\bar{a} = (1/A) \sum_{i \in I} |S^{(i)}| a_i. \hspace{1cm} (B.3)$$

This can be seen to apply to joins and meets of lifted vectors, and in this case can be linked to the convex hull of those vectors (see Fig. B.2 and Section 5.3).