Linear conic programming: genericity and stability
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Chapter 5

Copositive programming and the copositive cone*

Many combinatorial optimization problems such as computing the clique number, the stability number and the chromatic number of a graph can be formulated exactly as linear optimization problems over the copositive cone. For general surveys on copositive programming, we refer to [Düer10, Bur12, BSM03].

Definition 5.1. A clique in an undirected graph $\Gamma$ is a subset $S$ of the vertex set such that for every two vertices in $S$, there exists an edge connecting the two. The clique number is the largest possible size of a clique in $\Gamma$.

For example, in the following graph, the clique number is four.

Finding the clique number can be formulated as the following copositive program (see [MS65])

$$\omega(\Gamma) = \min\{\lambda \mid \lambda(E - A_\Gamma) - E \in \mathcal{OP}^k\},$$

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83
where $A_{\Gamma} \in S^k$ is the adjacency matrix of the graph $\Gamma$ with $k$ vertices and $E$ is the matrix of all ones. The corresponding dual problem is

$$\max \quad \langle E, Y \rangle$$
$$\text{s.t.} \quad \langle E - A_{\Gamma}, Y \rangle = 1$$
$$Y \in CP^k$$

As $(E - A_{\Gamma})$ is a nonnegative matrix whose diagonal components are all one, we have $(E - A_{\Gamma}) \in \text{int} \ COP^k$. Thus, the feasible set of the dual problem is a cross section of the completely positive cone. This implies that the dual Slater condition holds. Moreover, for a big enough $\lambda$, one can show that the primal Slater condition holds for the clique problem as well. By Corollary 2.14, the primal and the dual optimal solutions exist and the duality gap is zero for the clique problem. As $n = 1$ or equivalently $\dim \mathcal{L} = 1$, the primal optimal solution is of first order.

Note that since $\omega(\Gamma)$ is an integer, the clique number can be computed by checking copositivity of at most $k$ matrices. It has been shown in [MK87] that checking copositivity is a NP-hard problem. Even though checking copositivity in general is computationally hard problem, there are particular cases where one can identify or verify copositivity in polynomial time. For example, checking copositivity of a tridiagonal matrix can be done in linear time, see [Bom00]. Also it is known that the set of copositive matrices with entries from \{0, ±1\} can be identified easily (see [HP73, HH69]).

In this chapter, we are interested in some particular cases where copositivity can be checked efficiently. By no means is this chapter an exhaustive study; it is rather a small collection of easily solvable cases.

### 5.1 Copositive matrices with exactly one positive eigenvalue

One can check positive semidefiniteness of a matrix by checking its eigenvalues. On the other hand, one cannot decide whether a matrix is copositive or not based only on its eigenvalues. In this section, we study some
5.1. Copositive matrices with exactly one positive eigenvalue

subsets of the copositive cone which can be characterized easily, depend-
ning on the number of negative eigenvalues of the matrix.

Clearly, checking copositivity of a matrix \( M \in \mathcal{S}^k \) is equivalent to verifying if the quadratic form \( x^T M x \) is nonnegative over the standard simplex \( \{ x \in \mathbb{R}_+^k \mid \sum_{i=1}^k x_i = 1 \} \).

Note that the quadratic form given by \( M \) can be simplified by transforming it into diagonal form. To this end, let the eigenvalues of \( M \) be ordered as \( \lambda_1 \leq \cdots \leq \lambda_k \). Since \( M \in \mathcal{S}^k \), we can decompose \( M \) into \( M = Q \Lambda Q^T \), where \( \Lambda = \text{Diag}(\lambda_1, \ldots, \lambda_k) \) and \( Q \) is an orthogonal matrix whose columns are the corresponding eigenvectors. Denote the linearly independent columns of \( Q^T \) as \( q^1, \ldots, q^k \in \mathbb{R}^k \), and define \( Q := \text{conv}\{q^1, \ldots, q^k\} \) and \( y := Q^T x \). With these notations, we have

\[
M \in \mathcal{COP}^k \iff y^T \Lambda y \geq 0 \text{ for all } y \in Q.
\]  

Note that \( Q \) is a polytope. Actually, \( Q \) is the image of the standard simplex, which is a base of \( \mathbb{R}_+^k \), under the linear mapping \( x \mapsto Q^T x \), whence \( Q \) is also a simplex.

**Definition 5.2.** We say that the Perron-Frobenius property holds for a matrix \( M \in \mathcal{S}^k \) if \( M \) has a positive dominant eigenvalue with a corresponding nonnegative eigenvector.

It is shown in [HH69] that a copositive matrix has a dominant positive eigenvalue. The Perron-Frobenius property of copositive matrices was studied in [JR05], [HH69] and [Bom08]. In particular, it is known (e.g. [JR05, Theorem 11]) that if an indefinite matrix with exactly one positive eigenvalue is copositive, then it has the Perron-Frobenius property. In this section, we show that the converse is true for matrices with nonnegative diagonal elements.

As any positive semidefinite matrix is copositive, we may and do assume in the following that \( M \) has a negative eigenvalue. Moreover, this section deals with an indefinite matrix with exactly one positive eigenvalue, i.e.

\[
\lambda_1 \leq \cdots \leq \lambda_{k-1} \leq 0 < \lambda_k \text{ and } \lambda_1 < 0.
\]  

85
As we mentioned earlier, an indefinite matrix $M \in \mathcal{COP}^k$ satisfying (5.2) has the Perron-Frobenius property. However, in the case $\lambda_1 = 0$, the Perron-Frobenius property might not be fulfilled. For instance, consider ([JR05, page 280])

$$M = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$ 

In this case, $\lambda_1 = 0$ and $\lambda_2 = 2$ and $M$ has a nonnegative eigenvector corresponding to $\lambda_1$, but not to $\lambda_2$.

We study further copositive matrices satisfying (5.2).

**Proposition 5.3.** Let $M$ be an indefinite matrix with exactly one positive eigenvalue. Then $M$ is copositive if and only if the following two conditions both hold:

(a) $M_{ii} \geq 0$ for all $i = 1, \ldots, k$ and

(b) the Perron-Frobenius property holds for $M$.

**Proof.** Let $M \in \mathcal{COP}^k$ be indefinite and satisfy (5.2). Then it is well known that $M_{ii} \geq 0$ for all $i$ and the Perron-Frobenius property holds for $M$. Thus, one implication is clear.

Now let us consider the other direction. Suppose properties (a) and (b) hold. Define $\text{Pos}(M) := \{ z \in \mathbb{R}^k \mid z^T \Lambda z \geq 0 \}$. Obviously, $\text{Pos}(M)$ is a cone. Let

$$\text{Pos}^+(M) := \{ z \in \mathbb{R}^k \mid z_k \geq 0, z^T \Lambda z \geq 0 \}$$

and

$$\text{Pos}^-(M) := \{ z \in \mathbb{R}^k \mid z_k \leq 0, z^T \Lambda z \geq 0 \}.$$

It is clear that

$$\text{Pos}(M) = \text{Pos}^+(M) \cup \text{Pos}^-(M).$$

We claim that both $\text{Pos}^+(M)$ and $\text{Pos}^-(M)$ are convex cones and prove this for $\text{Pos}^+(M)$. Obviously, $\text{Pos}^+(M)$ is a cone. Define a function of $(k-1)$ variables as

$$f(z_1, \ldots, z_{k-1}) := |\lambda_k|^{-\frac{1}{2}} \left( \sum_{i=1}^{k-1} |\lambda_i| z_i^2 \right)^{\frac{1}{2}}.$$
5.1. Copositive matrices with exactly one positive eigenvalue

It is easy to see that $f$ is convex. With this, we have

\[
\text{Pos}^+(M) = \{ z \in \mathbb{R}^k \mid z_k \geq 0, \lambda_k z_k^2 \geq |\lambda_1| z_1^2 + \cdots + |\lambda_{k-1}| z_{k-1}^2 \} \\
= \{ z \in \mathbb{R}^k \mid z_k \geq f(z_1, \ldots, z_{k-1}) \} = \text{epi}(f).
\]

So $\text{Pos}^+(M)$ is the epigraph of a convex function, and hence convex. An analogous argument shows that $\text{Pos}^-(M)$ is a convex cone.

From property (a), we have $0 \leq M_{ii} = (q^i)^T \Lambda q^i$, so $q^i \in \text{Pos}(M)$ for all $i$. Note that the last components of the $q^i$’s make up the eigenvector corresponding to the largest eigenvalue $\lambda_k$. Using property (b), we have that the $q^i_k$ have the same sign for all $i$. This implies that either $q^i \in \text{Pos}^+(M)$ for all $i$ or $q^i \in \text{Pos}^-(M)$ for all $i$. If $q^i \in \text{Pos}^+(M)$ for all $i$, then using convexity of $\text{Pos}^+(M)$ we have that $Q \subseteq \text{Pos}^+(M)$ and hence $M \in \text{COP}^k$. If $q^i \in \text{Pos}^-(M)$ for all $i$, then using similar arguments we have that $Q \subseteq \text{Pos}^-\!\!(M)$ which implies $M \in \text{COP}^k$. \qed

It is well known that by the Perron-Frobenius theorem, nonnegative matrices have the Perron-Frobenius property. In our case, the copositive matrices satisfying (5.2) turn out to be nonnegative.

**Proposition 5.4.** Let $M$ be an indefinite matrix with exactly one positive eigenvalue. Then

\[
M \in \text{COP}^k \iff M \in \mathcal{N}^k.
\]

**Proof.** Clearly, $\mathcal{N}^k \subset \text{COP}^k$, so one implication is trivial. To prove the converse, let $M \in \text{COP}^k$ and pick arbitrary indices $i, j$. We need to show that $0 \leq M_{ij} = (q^i)^T \Lambda q^j$. Since by Proposition 5.3 the Perron-Frobenius property holds for $M = Q \Lambda Q^T$, the last column of $Q$ is nonnegative, so $q^i_k \geq 0$ and $q^j_k \geq 0$.

From Proposition 5.3(a), we have $0 \leq (q^l)^T \Lambda q^l = \sum_{v=1}^k \lambda_v (q^l_v)^2$ for $l = i, j$. Using $\lambda_1 \leq \cdots \leq \lambda_{k-1} \leq 0 < \lambda_k$, this implies

\[
(\lambda_k)^{\frac{1}{2}} q^l_k \geq \left( \sum_{v=1}^{k-1} |\lambda_v| (q^l_v)^2 \right)^{\frac{1}{2}} \quad \text{for} \quad l = i, j. \tag{5.3}
\]
By multiplying these two inequalities and using Cauchy-Schwarz, we get

\[
\lambda_k q_k^i q_k^j \geq \left( \sum_{v=1}^{k-1} |\lambda_v| (q_v^i)^2 \right)^{\frac{1}{2}} \left( \sum_{v=1}^{k-1} |\lambda_v| (q_v^j)^2 \right)^{\frac{1}{2}}
\geq |\lambda_1| q_1^i q_1^j + \cdots + |\lambda_{k-1}| q_{k-1}^i q_{k-1}^j.
\]

This is equivalent to

\[
0 \leq \sum_{v=1}^{k} \lambda_v q_v^i q_v^j = (q^i)^T \Lambda q^j = M_{ij}
\]

which shows that \( M \in \mathcal{NN}^k \).

By combining Propositions 5.3 and 5.4, we obtain the following theorem:

**Theorem 5.5.** Let \( M \) be an indefinite symmetric matrix with exactly one positive eigenvalue. Then the following are equivalent:

(a) The Perron-Frobenius property holds for \( M \) and \( M_{ii} \geq 0 \) for all \( i \),

(b) \( M \) is nonnegative,

(c) \( M \) is copositive.

Furthermore, we identify whether or not the copositive/nonnegative matrices satisfying (5.2) are on the boundary of the copositive cone.

**Corollary 5.6.** Let \( M \) be an indefinite matrix with exactly one positive eigenvalue. Then \( M \in \text{bd} \mathcal{COP}^k \) if and only if \( M \in \mathcal{NN}^k \) and there exists an index \( i \) such that \( M_{ii} = 0 \).

**Proof.** If \( M_{ii} = 0 \), then \((e_i)^T M e_i = 0\) where \( e_i \) is the \( i \)-th unit vector, so \( M \in \text{bd} \mathcal{COP}^k \). To show the converse, let \( M \in \text{bd} \mathcal{COP}^k \), i.e., there exists \( x \geq 0 \), \( x \neq 0 \) such that \( x^T M x = 0 \). By Theorem 5.5, we have \( M \in \mathcal{NN}^k \). Then we obtain

\[
0 = x^T M x \geq \sum_{i=1}^{k} M_{ii} x_i^2.
\]

As \( x \neq 0 \), there exists an index \( i \) such that \( M_{ii} = 0 \). \( \square \)
5.2. Copositive matrices with exactly one negative eigenvalue

Consider a quadratic programming problem

$$\min x^T M x + b^T x \quad \text{s. t.} \quad Ax \leq 0 \quad (QP)$$

where $b \in \mathbb{R}^k$, $M \in \mathcal{S}^k$ and $A$ is an $n \times k$ matrix. The problem is known to be NP-hard even in the case $M$ has only one negative eigenvalue, see [PV91].

Taking $b = 0$ and $A = -I$, we have that the optimal value of (QP) is nonnegative if and only if matrix $M$ is copositive. In the particular case when $M$ has exactly one negative eigenvalue, it is shown that checking copositivity is solvable in polynomial time, see [Din96, Theorem 4.3.1] \(^1\). Moreover, an algorithm to test copositivity of a matrix with exactly one negative eigenvalue is proposed in [Din96, Algorithm 4.2.1]. In this algorithm, the author considers the quadratic form on a hyperplane such that the corresponding so-called projected Hessian on the hyperplane has exactly one negative eigenvalue less than $M$. More specifically, when $M$ has exactly one negative eigenvalue, a vector $h \in \mathbb{R}^k$ can be found such that $x^T M x$ is convex on hyperplane $h^T x = 1$. Then the author shows that minimizing the quadratic form $x^T M x$ over the set

$$\{ x \in \mathcal{F}_{QP} \mid x_1 \geq 0 \text{ and } h^T x = 1 \}$$

is a convex quadratic programming problem. If the optimal solution of this subproblem is nonnegative or the problem is infeasible, then it is proven that $M$ is copositive on the set $\{ x \in \mathcal{F}_{QP} \mid x_1 \geq 0 \}$. The remaining case $\{ x \in \mathcal{F}_{QP} \mid x_1 \leq 0 \text{ and } h^T x = 1 \}$ is analysed similarly. Then the results are combined to identify whether matrix $M$ is copositive or not.

The observations from Section 5.1 can be applied to matrices with exactly one negative eigenvalue as well. The approach in [Din96] and the one based on the method from Section 5.1 are not the same.

\(^1\)The author would like to thank I. Bomze for pointing out this thesis.
Chapter 5. Copositive programming and the copositive cone

So based on observations from Section 5.1, let us next investigate matrices with exactly one negative eigenvalue. Throughout this section, $M$ is an indefinite matrix with exactly one negative eigenvalue, i.e.

$$\lambda_1 < 0 \leq \lambda_2 \leq \cdots \leq \lambda_k \text{ and } 0 < \lambda_k.$$  \hspace{1cm} (5.4)

Similar to Section 5.1, we study copositivity of a matrix $M$ which satisfies (5.4) by transforming its quadratic form into diagonal form. Consider the cone $\text{Neg}(M) := \{ z \in \mathbb{R}^k \mid z^T \Lambda z \leq 0 \}$. Clearly, we have $\text{Neg}(M) = \text{Pos}(-M)$. So similarly as in the proof of Proposition 5.3, we can decompose $\text{Neg}(M)$ into two full-dimensional convex cones, i.e. $\text{Neg}(M) = \text{Neg}^+(M) \cup \text{Neg}^-(M)$ with

$$\text{Neg}^+(M) := \{ z \in \mathbb{R}^k \mid z_1 \geq 0, z^T \Lambda z \leq 0 \}$$

and

$$\text{Neg}^-(M) := \{ z \in \mathbb{R}^k \mid z_1 \leq 0, z^T \Lambda z \leq 0 \}.$$

From (5.1), it is clear that $M \in \mathcal{COP}^k \iff \mathcal{Q} \cap \text{int} \text{Neg}(M) = \emptyset$. Using $\text{Neg}^+(M) \cap \text{Neg}^-(M) = \{ z \in \mathbb{R}^k \mid z_i = 0 \text{ if } \lambda_i \neq 0 \}$ and $\lambda_1 < 0 < \lambda_k$, we have $\text{int} \text{Neg}^+(M) \cap \text{int} \text{Neg}^-(M) = \emptyset$. Therefore, we derive

$$M \in \mathcal{COP}^k \iff \mathcal{Q} \cap \text{int} \text{Neg}^+(M) = \emptyset \text{ and } \mathcal{Q} \cap \text{int} \text{Neg}^-(M) = \emptyset.$$  

Note that the sets $\text{Neg}^+(M)$ and $\text{Neg}^-(M)$ are second order cones and $\mathcal{Q}$ is a polytope, as displayed in Figure 5.1.

![Figure 5.1](image)
5.2. Copositive matrices with exactly one negative eigenvalue

Hence, we have transformed the problem of checking copositivity into two convex feasibility problems. By choosing a suitable objective function, we can rephrase these as the following two convex optimization problems:

\[
\inf \quad ||x - z||^2 \\
\text{s.t.} \quad x \in \mathcal{Q} \\
\quad \quad \quad z \in \text{int Neg}^\pm(M)
\]

where Neg^\pm(M) stands for either Neg^+(M) or Neg^-(M). If for at least one of the sets Neg^+(M) and Neg^-(M) the optimal value of problem (5.5) is zero and the optimal solution is attained, then \( \mathcal{Q} \cap \text{int Neg}(M) \neq \emptyset \). In other words, \( M \) is not copositive. Otherwise, \( \mathcal{Q} \cap \text{int Neg}(M) = \emptyset \) holds and thus \( M \) is copositive.

Observe that the feasible set in (5.5) is not closed, which is disadvantageous for practical implementations. Considering the closure of the feasible set, it is easy to see that

\[
M \in \text{int COP}_k \iff \mathcal{Q} \cap \text{Neg}^+(M) = \emptyset \quad \text{and} \quad \mathcal{Q} \cap \text{Neg}^-(M) = \emptyset.
\]

Thus, to check if \( M \in \text{int COP} \), we have to solve the following two convex problems

\[
\min \quad ||x - z||^2 \\
\text{s.t.} \quad x \in \mathcal{Q} \\
\quad \quad \quad z \in \text{Neg}^+(M)
\]

where again Neg^\pm(M) stands for either Neg^+(M) or Neg^-(M). If the optimal value of (5.6) is strictly positive for both cases, then \( M \in \text{int } \mathcal{C} \), otherwise \( M \notin \text{int } \mathcal{C} \).

**Example 5.7.** Consider the following matrix from [BE13, Example 2.12] which has eigenvalues as studied in this section.

\[
M = \begin{bmatrix}
1 & 1.63 & 1 & -0.77 & -0.67 \\
1.63 & 1 & 0 & 0.32 & -0.82 \\
1 & 0 & 1 & -0.26 & -0.67 \\
-0.77 & 0.32 & -0.26 & 1 & 0.77 \\
-0.67 & -0.82 & -0.67 & 0.77 & 1
\end{bmatrix}
\]
We solved problems (5.6) for $M$ using SeDuMi [Stu99]. The optimal value is zero and $z = x = (0.008, 0.315, 0.258, 0.009, 0.41)^T$. Note that the obtained $x$ is a negative certificate which means $x^T M x = -0.0052 < 0$, so $M \notin \mathcal{COP}$.

In [Väl89], it was shown for $M \in S^k$ that if the order of the maximal positive definite principal submatrix is $(k - 1)$, then copositivity of the matrix can be verified by a convex quadratic program. The following example shows that an indefinite matrix with exactly one negative eigenvalue does not necessarily contain a maximal positive definite principal submatrix of order $(k - 1)$, so the two approaches are complementary.

**Example 5.8.** Consider the following matrix from [Väl89, Example 4.1]

$$
M = \begin{bmatrix}
1 & -1 & 1 & 2 & -3 \\
-1 & 2 & -3 & -3 & 4 \\
1 & -3 & 5 & 6 & -4 \\
2 & -3 & 6 & 5 & -8 \\
-3 & 4 & -4 & -8 & 16
\end{bmatrix}
$$

$M$ is indefinite and has exactly one negative eigenvalue, but the maximal positive semidefinite submatrix of $M$ is the leading $3 \times 3$ principal submatrix. So, in this example, the formulation of [Väl89] is not a convex quadratic problem.

We numerically tested copositivity of $M$ by solving (5.6). There is no negative certificate and the optimal values are very close to zero, so within a given accuracy $M$ is not strictly copositive. This can also be checked directly since $x^T M x = 0$ for $x = (1, 2, 1, 0, 0)$. In fact, it can be shown that $M$ is copositive by the methods from [DDGH13].