Linear conic programming: genericity and stability

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Chapter 3

Genericity and Stability*

In this section, we study genericity and stability results for linear conic programs. Informally, a property is stable if the property remains satisfied under a small perturbation of the problem data, and one expects generic properties to be stable and to hold at a random instance of a problem.

Let us specify the conic problem parameter space first. Throughout the thesis, we assume that the cone $\mathcal{K}$ (and thus $\mathcal{K}^*$) and $n, k$ are arbitrarily fixed for results concerning genericity. Then the parameters of a pair of conic problems $(P)$ and $(D)$ are $A_1, \ldots, A_n \in S^k$, $B \in S^k$ and $c \in \mathbb{R}^n$. As before (see p.9), we identify $S^k \equiv \mathbb{R}^m$ with $m = \frac{1}{2}k(k + 1)$ and denote $A$ as the $m \times n$ matrix with columns $A_i$. Thus the set of problem instances of $(P)$ and $(D)$ is given by

$$\mathcal{P} := \{(A, B, c) \in \mathbb{R}^{m\times n} \times \mathbb{R}^m \times \mathbb{R}^n\} \equiv \mathbb{R}^{m\cdot n+m+n} \quad (3.1)$$

endowed with some norm.

**Definition 3.1.** We say that a property is generic, if it holds for a subset $\mathcal{P}_g$ of the set $\mathcal{P}$ of problem instances such that $\mathcal{P}_g$ is open in $\mathcal{P}$ and $\mathcal{P} \setminus \mathcal{P}_g$ has Lebesgue measure zero.

A property is said to be weakly generic if it holds for a subset $\mathcal{P}_g$ such that $\mathcal{P} \setminus \mathcal{P}_g$ has Lebesgue measure zero.

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*Submitted as

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Chapter 3. Genericity and Stability of properties of conic problems

Genericity implies both density and stability of “nice” problem instances. On the other hand, it is well known that density and openness do not imply genericity.

Throughout this section, we assume that \( n \leq m \) holds. For the case \( n > m \) the genericity results can be summarized by the following statement:

**Proposition 3.2.** Generically for the case \( n > m \), we have an unbounded primal \((P)\) and an infeasible dual \((D)\). In other words, generically strong duality holds with \( v_P = v_D = +\infty \).

**Proof.** To prove this, we use the well-known fact that (see e.g. [JJT00, Ex. 7.3.23])

\[
\text{a matrix } U \in \mathbb{R}^{t_1 \times t_2} \text{ with } t_1 \geq t_2 \text{ generically has full rank } t_2. \quad (3.2)
\]

We first show that wrt. \((c, A)\)

\[
\text{generically, the system } c = A^T Y \text{ has no solution } Y \in \mathbb{R}^m. \quad (3.3)
\]

Indeed, by (3.2) the matrix \( U := [A^T \ c] \in \mathbb{R}^{n \times (m+1)} \) generically has rank \( m + 1 \) whence \( Uz = 0 \) does not allow a nonzero solution. This means that generically the system in (3.3) is infeasible.

To show that \((P)\) is generically unbounded we consider the system \( Ax = B, \ c^T x = \tau \), any solution of which yields a primal feasible \( x \) with objective value \( \tau \). Again, by statement (3.2), generically, the matrix \( U := \begin{bmatrix} A \\ c^T \end{bmatrix} \in \mathbb{R}^{(m+1) \times n} \) has full rank \( m+1 \), so \( Ax = B, \ c^T x = \tau \) is solvable for any \( \tau \) (and \( B \)). \( \square \)

In Assumption 2.6, for conic programs, we take the matrices \( A_1, \ldots, A_n \) to be linearly independent, i.e., they span an \( n \)-dimensional linear space in \( \mathcal{S}_k \). Applying the standard result (3.2) for the matrix \( A \) \((m \geq n)\), we obtain that Assumption 2.6 is satisfied generically.

Suppose that a property is weakly generic. Any neighborhood of a problem parameter \((A, B, c) \in \mathcal{P}\) contains a parameter \((A', B', c') \in \mathcal{P}_g\), otherwise set \( \mathcal{P} \setminus \mathcal{P}_g \) does not have measure zero. Thus, weak genericity
implies density of the set $\mathcal{P}_g$. Clearly, the other direction is not true, i.e. density of a set $S$ does not imply that $S$ has full measure. For example, consider $S := \mathbb{Q} \subset \mathbb{R}$.

For weak genericity results, one can arbitrarily fix $A \in \mathbb{R}^{m \times n}$ and prove that the property holds for all $(B, c)$ from a weakly generic set $S(A) \subset \mathbb{R}^{m+n}$. We emphasize that this implies that the property holds for almost all problem instances in the whole space $\mathcal{P} = \{(A, B, c)\}$. Indeed, under this assumption for any fixed $A \in \mathbb{R}^{m \times n}$ the property holds on the whole $\mathbb{R}^{m+n}$ except for the set $S(A)^C := \mathbb{R}^{m+n} \setminus S(A)$ of Lebesgue measure $\mu(S(A)^C) = 0$ in $\mathbb{R}^{m+n}$. But then by Fubini’s theorem the property holds for $(A, B, c) \in \mathbb{R}^{m \times n} \times \mathbb{R}^m \times \mathbb{R}^n$ except for a set of measure $\int_{\mathbb{R}^{m \times n}} \mu(S(A)^C) \, dA = 0$.

Concerning openness, however, we have to be careful: If for any fixed $A$, a property holds for any $(B, c)$ from an open set $S(A) \subset \mathbb{R}^{m+n}$, then this property need not hold for an open set in $\mathcal{P}$. There are also properties which are not weakly generic but stable. From a numerical viewpoint, stability is crucial.

**Definition 3.3.** We say that a property is stable at an instance $\overline{Q} := (\overline{A}, \overline{B}, \overline{c}) \in \mathcal{P}$ if the property is satisfied for all $Q := (A, B, c) \in \mathcal{P}$ in an open neighborhood of $\overline{Q}$.

Genericity of properties like strong duality, nondegeneracy, strict complementarity and uniqueness of solutions of linear conic programs have been discussed before. For the SDP case, it is indicated in [Sha97, p. 310] that the Slater condition (Mangasarian-Fromovitz condition) is generic. Alizadeh, Haeberly, and Overton [AHO97] as well as Shapiro [Sha97] specifically discuss generic properties of semidefinite programs (SDP). Pataki and Tunçel [PT01] derive weak genericity results on strict complementarity, uniqueness, and nondegeneracy for general linear conic programs. Recently in [BDL11] Bolte, Daniilidis and Lewis gave special full genericity results with respect to the parameter $c$ under the extra assumption that the cone $\mathcal{K}$ is a semialgebraic set.

In the thesis, we attempt to prove our genericity results with techniques which are as basic as possible. Genericity of Slater’s condition
will be proven by topological arguments. Weak genericity of uniqueness is shown based on the classical result that Lipschitz-functions (convex functions) are differentiable almost everywhere. For weak genericity of nondegeneracy and strict complementarity more sophisticated techniques from geometric measure theory are still needed (see [PT01]).

3.1 Genericity of Slater’s condition and strong duality

In most applications, one is concerned with the cases where optimal solutions of \((P)\) and \((D)\) are attained with zero duality gap. In this section, we illustrate that Slater’s condition and strong duality are generic properties.

Genericity of the primal or dual Slater condition can be derived in many different ways. For example, genericity of nondegeneracy (or uniqueness of an optimal solution) implies genericity of Slater’s condition by Proposition 2.21. However, we give an independent easy proof of the fact that in conic programming the Slater condition holds generically. To do so we only make use of the result that the boundary of a convex set has measure zero.

Lemma 3.4. Let \(\mathcal{T}\) be a full-dimensional closed convex set in \(\mathbb{R}^s\). Then the boundary of \(\mathcal{T}\) has \(s\)-dimensional Lebesgue measure zero.

Proof. We repeat here the elegant proof of [Lan86]. Consider an open ball \(B_\varepsilon(p)\) with center \(p \in \text{bd } \mathcal{T}\) and radius \(\varepsilon > 0\). Since there exists a hyperplane supporting the convex set \(\mathcal{T}\) at \(p\), at least half of the ball does not contain points of \(\mathcal{T}\). Therefore,

\[
\limsup_{\varepsilon \to 0} \frac{\mu(\mathcal{T} \cap B_\varepsilon(p))}{\mu(B_\varepsilon(p))} \leq \frac{1}{2}.
\]

On the other hand, Lebesgue’s density theorem (see e.g., [Fau02]), says that for almost all points \(p\) of the Lebesgue measurable set \(\mathcal{T}\) we have
that
\[
\lim_{\varepsilon \to 0} \frac{\mu(\mathcal{T} \cap B_\varepsilon(p))}{\mu(B_\varepsilon(p))} = 1.
\]
This immediately implies that \( \text{bd } \mathcal{T} \) has measure zero.

Roughly speaking, Slater’s condition says that the feasible set of the problem is not entirely contained in the boundary of the convex cone.

**Theorem 3.5.** Let matrices \( A_i \) \((i = 1, \ldots, n)\) be given arbitrarily and denote \( A \) as the matrix with columns \( A_i \). Then there exists a generic subset \( S_1 \subset \mathbb{R}^n \) (open with complement of measure zero), such that for any \( c \in S_1 \) precisely one of the following alternatives holds for the corresponding problem instance of \((D)\):

1. either the feasible set of \((D)\) is empty, i.e., \( \{Y \in \mathcal{K}^* \mid A^T Y = c\} = \emptyset \),
   or
2. Slater’s condition holds for \((D)\), i.e., \( \{Y \in \text{int } \mathcal{K}^* \mid A^T Y = c\} \neq \emptyset \).

An analogous result holds for the primal program \((P)\), i.e., there is a generic subset \( \tilde{S}_1 \) of \( \mathbb{R}^m \) such that for any \( B \in \tilde{S}_1 \) either the corresponding program \((P)\) is infeasible or \((P)\) satisfies the Slater condition.

**Proof.** For the case of program \((D)\), note that the set
\[
S := \{c = A^T Y \mid Y \in \mathcal{K}^*\} \subset \mathbb{R}^n
\]
is a convex set with \( \text{dim } S =: l \leq n \). We define \( S_1 := \text{int } S \cup (\mathbb{R}^n \setminus \text{cl } S) \). As a union of two open sets, \( S_1 \) is clearly open. Note that for \( c \in (\mathbb{R}^n \setminus \text{cl } S) \) the alternative (1) holds, i.e., the feasible set is empty. If \( l < n \) (i.e., \( A \) does not have full rank \( n \)), then the statement is true. So we can assume \( \text{dim } S = n \), and since by Lemma 3.4 the set \( \text{bd } S = \mathbb{R}^n \setminus S_1 \) has measure zero, it is sufficient to show that for \( c \in \text{int } S \) the Slater condition holds (alternative (2)).

So let \( c \in \text{int } S \) be given. By assumption there exists some \( Y_0 \in \mathcal{K}^* \) for which \( A^T Y_0 = c \) holds. Consider the affine space \( Y_0 + \ker A^T \). If \( Y_0 + \ker A^T \cap \text{int } \mathcal{K}^* \neq \emptyset \), then Slater’s condition holds and we are done.
So assume by contradiction that $Y_0 + \ker A^T \cap \text{int} \mathcal{K}^* = \emptyset$. This implies in particular that $Y_0 \in \text{bd} \mathcal{K}^*$, and since $\text{int} \mathcal{K}^* \neq \emptyset$, there exists a separating hyperplane with normal vector $N$ such that

$$
\langle N, Y \rangle \geq \langle N, Y_0 \rangle \quad \text{for all } Y \in \mathcal{K}^* \quad \text{and} \quad N \perp \ker A^T. \quad (3.4)
$$

(Here we have used that the affine space $Y_0 + \ker A^T$ “meets $\mathcal{K}^*$ tangentially” at $Y_0$.)

Since $c \in \text{int} S$, there exists an open neighborhood $\emptyset \neq U_\varepsilon(c) \subset \text{int} S$ of $c$ and by continuity of the mapping $A^T Y$ there exists an open neighborhood $\emptyset \neq U_\delta(Y_0)$ of $Y_0$ such that $A^T U_\delta(Y_0) \subset U_\varepsilon(c)$. The separating hyperplane divides $U_\delta(Y_0)$ into two parts. Take a point $Y_1 \in U_\delta(Y_0)$ such that $\langle N, Y_1 \rangle < \langle N, Y_0 \rangle$. By construction, $c_1 := A^T Y_1 \in U_\varepsilon(c) \subset \text{int} S$. So there must exist a pre-image $\tilde{Y}_1 \in \mathcal{K}^*$ with $A^T \tilde{Y}_1 = c_1$, i.e., $\tilde{Y}_1 = Y_1 + \tilde{Y}_0$ with $\tilde{Y}_0 \in \ker A^T$. Altogether using $\langle N, \tilde{Y}_0 \rangle = 0$ and (3.4), we attain the contradiction

$$
\langle N, Y_0 \rangle \leq \langle N, \tilde{Y}_1 \rangle = \langle N, Y_1 + \tilde{Y}_0 \rangle = \langle N, Y_1 \rangle < \langle N, Y_0 \rangle.
$$

This concludes the proof for problem $(D)$.

For the primal case we proceed as follows. We note that $\mathbb{R}^m$ allows an orthogonal decomposition

$$
\mathbb{R}^m = \text{im} A \oplus \ker A^T, \quad B = B_1 \oplus B_2 \quad \text{for} \quad B \in \mathbb{R}^m
$$

where $B_2$ is the projection $\text{proj}_{\ker A^T} B$ of $B \in \mathbb{R}^m$ onto the linear space $\ker A^T$. Let $Q \in \mathbb{R}^{m \times m}$ be the matrix representation of this projection, i.e., $B_2 = \text{proj}_{\ker A^T} B = QB$. We now consider the convex cone $R := Q \mathcal{K}$. As before we have

$$
QB \in \ker A^T \setminus \text{cl } R \quad \Rightarrow \quad \{ B - Ax \mid x \in \mathbb{R}^n \} \cap \mathcal{K} = \emptyset
$$

and we can show (with $\text{int } R$ relative to $\ker A^T$)

$$
QB \in \text{int } R \quad \Rightarrow \quad \{ B - Ax \mid x \in \mathbb{R}^n \} \cap \text{int } \mathcal{K} \neq \emptyset.
$$

Here again $\text{bd } R$ has measure zero and thus $R_1 := \text{int } R \cup (\ker A^T \setminus \text{cl } R)$ is relatively open in $\ker A^T$ with $\ker A^T \setminus R_1$ of measure zero in $\ker A^T$. 38
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Consequently, the set $\tilde{S}_1 := \text{im } A \oplus R_1$ is open in $\mathbb{R}^m$ with $\mathbb{R}^m \setminus \tilde{S}_1$ of measure zero in $\mathbb{R}^m$. By construction, for $B \in \tilde{S}_1$, precisely one of the two alternatives holds.

Remark 3.6. The Slater conditions for ($P$) and ($P_0$) are clearly equivalent. Also the genericity result for ($D$) in Theorem 3.5 wrt. parameter $c$ can be translated to the following corresponding result for ($D_0$): Let $\mathcal{L}$ be given. Then there exists a generic subset $Q_1 \subset \mathbb{R}^m$ such that for any $C \in Q_1$ precisely one of the following alternatives holds for the corresponding problem instance of ($D_0$):

(1′) either the feasible set of ($D_0$) is empty, or

(2′) Slater’s condition holds for ($D_0$), i.e.,

$$\{Y \mid Y \in (\mathcal{L}^\perp + C') \cap \text{int } \mathcal{K}^* \} \neq \emptyset.$$ 

To see this, similar to the second part of the proof of Theorem 3.5, consider the orthogonal decomposition

$$\mathbb{R}^m = \mathcal{L}^\perp \oplus \mathcal{L}, \quad C = C_1 \oplus C_2 \quad \text{for} \quad C \in \mathbb{R}^m.$$ 

Let $P \in \mathbb{R}^{m \times m}$ be the matrix representation of the projection $\text{proj}_\mathcal{L}$ onto $\mathcal{L}$, and let $C_2 = PC = \text{proj}_\mathcal{L} C$. Then as in the the proof of Theorem 3.5 above we consider the convex cone $S := PK^* \subset \mathcal{L}$ and the set (relative to $\mathcal{L}$)

$$S_1 = \text{int } S \cup (\mathcal{L} \setminus \text{cl } S),$$

which is relatively open with $\mathcal{L} \setminus S_1$ of measure zero. Note that for $PC \in \text{int } S$ the alternative (2′) holds and for $PC \in \mathcal{L} \setminus \text{cl } S$ the condition (1′) is true. So the set $Q_1 = \mathcal{L}^\perp \oplus S_1$ is the required generic set in $\mathbb{R}^m$.

It is well-known that strong duality always holds in linear programming (unless both programs are infeasible) but strong duality need not hold in general conic programming as illustrated in Example 2.11. However, as we shall see, strong duality is a generic property.

From Proposition 2.13, we have that for given $A, B$ the primal Slater condition implies that 2nd moment cone $\mathcal{N}_2$ is closed, and so is the 1st
moment cone $\mathcal{N}_1$. Under these assumptions, we derive that $C_1 = C_2$ and by applying Table 2.3, the duality gap is zero for all $c$, i.e. strong duality holds for all $c$. Also, it is well-known that the primal Slater condition implies that the dual optimal solution of a feasible dual problem is attained, see e.g. [ADS13, Lemma 3.1]. So the genericity of Slater’s condition in Theorem 3.5 leads to the following genericity result of strong duality (similar to [STO07]):

**Corollary 3.7.** Let $A \in \mathbb{R}^{m \times n}$ be given arbitrarily. Then with the generic subset $\tilde{S}_1 \subset \mathbb{R}^m$ from Theorem 3.5 the following holds for $B \in \tilde{S}_1$:

- either the feasible set of $(P)$ is empty,
- or $(P)$ is strictly feasible and for any $c \in \mathbb{R}^n$ we have $v_P = v_D$, meaning that if $(D)$ is infeasible, then $v_P = v_D = +\infty$, and if $(D)$ is feasible, then $v_P = v_D$ is finite and the minimum value of $(D)$ is attained.

An analogous result holds for the dual program $(D)$ wrt. $c \in S_1 \subset \mathbb{R}^n$ (with $S_1$ from Theorem 3.5).

The Slater condition is trivially stable. Indeed, assume that $X = B - \sum_{i=1}^n x_i A_i$ is a Slater point for $(P)$ with respect to the data $(A, B, c)$. Then also the point $X := B - \sum_{i=1}^n \bar{x}_i A_i$ is a Slater point after any sufficiently small perturbation $(A, B, c)$ of $(\bar{A}, \bar{B}, \bar{c})$.

Note that in Corollary 3.7, a generic set was constructed for problem instances either with infeasible $(P)$ or with strong duality. It follows from Corollary 3.7 that strong duality is stable at $(\bar{A}, \bar{B}, \bar{c})$ under Slater’s condition. In general, the property “strong duality” can be unstable as feasibility (of its dual problem) might be not a stable property.

By combining the results in Corollary 3.7 for the primal and dual programs we obtain:

**Corollary 3.8.** Let $A \in \mathbb{R}^{m \times n}$ be given arbitrarily. Then with the generic subsets $S_1 \subset \mathbb{R}^n$, $\tilde{S}_1 \subset \mathbb{R}^m$ from Theorem 3.5, for any $(B, c) \in \tilde{S}_1 \times S_1$ precisely one of the following alternatives holds:
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(1) Both feasible sets of \((P)\) and \((D)\) are empty.

(2) Precisely one of the feasible sets of \((P)\) or \((D)\) is empty and 
\[ v_P = v_D = \pm \infty. \]

(3) Both \((P)\) and \((D)\) are feasible and for both problems the optimal value is attained with \(v_P = v_D\).

A corresponding result for self-dual problems \((P_0), (D_0)\) holds wrt. to a generic set \(\mathbb{S}_1 \times Q_1 \subset \mathbb{R}^m \times \mathbb{R}^m\) of parameters \((B, C)\) (cf. Remark 3.6).

In Corollary 3.8, matrix \(A\) is arbitrarily fixed and the parameters are \((B, c)\). In [STO07], the authors consider \(A\) as a parameter, and they define that universal duality is said to hold with respect to \(A\), if for any \((B, c)\) the equality \(v_P = v_D\) holds for \((P)\) and \((D)\). Moreover, the following is shown in [STO07, Theorem 4.5, Theorem 4.7].

**Theorem 3.9.** There is a generic subset \(S \subset \mathbb{R}^{m \times n}\) such that for any \(A \in S\) universal duality holds.

The main difference between the statements in Corollary 3.8 and Theorem 3.9 is that by taking \(A\) as a parameter in the generic set \(S\) of Theorem 3.9, the case that both the primal and the dual problem are infeasible is excluded. In Corollary 3.8, for fixed \(A\) one cannot exclude generically in \((B, c)\) the infeasibility of both programs \((P)\) and \((D)\).

Let us illustrate this difference by an example.

**Example 3.10.** Consider the LP:

\[
(P) \quad \max \ c^T x \quad \text{s.t.} \quad B - Ax \geq 0
\]

with
\[
c = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

The corresponding dual problem is

\[
(D) \quad \min \ B^T Y \quad \text{s.t.} \quad A^T Y = c, \quad Y = (y_1, y_2, y_3) \geq 0.
\]

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The primal and dual feasibility conditions are:

\[ x_2 \leq 0, \quad x_2 \geq 1, \quad x_1 \leq 0 \]

and

\[ y_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + y_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix} + y_3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad y_i \geq 0. \]

Both programs are infeasible, and for fixed \( A \) this property is stable with respect to small perturbations of \( c, B \). So in Corollary 3.8, the alternative (1) cannot be excluded generically. However, according to the genericity concept in Theorem 3.9 (where \( A \) is the parameter) a generic perturbation of the matrix \( A \) above makes either \( (P) \) or \( (D) \) feasible.

The notion of universal duality goes back to Duffin [DJK83]. Under the assumption that \( (P) \) is feasible, in [DJK83] the parameters \( (A, B) \in \mathbb{R}^{m \times n} \times \mathbb{R}^m \) are said to yield primal uniform LP duality for \( (P) \) and \( (D) \), if for any \( c \in \mathbb{R}^n \) either \( \mathcal{F}_D = \emptyset \) and \( v_P = v_D = \infty \); or \( v_P = v_D \) is finite and a solution of \( (D) \) exists. As we mentioned in Table 2.3, if the primal problem is feasible and 1st moment cone \( \mathcal{N}_1 \) is closed, then strong duality holds for all \( c \in \mathbb{R}^n \), i.e. the primal uniform LP duality holds.

By construction, moment cones are linear images of the cone \( \mathcal{K}^* \). In [BM10], the following genericity result on closedness of the linear image of a cone is shown.

**Theorem 3.11.** Let \( k \in \mathbb{N} \) and let \( \mathcal{K} \subset \mathbb{R}^m \) be a closed convex cone. Then the set

\[ S_1 := \mathbb{R}^{k \times m} \setminus \text{int}\{T \in \mathbb{R}^{k \times m} \mid TK \text{ is closed}\} \]

has Lebesgue measure zero.

Utilizing Theorem 3.11, this offers another approach to show (weak) genericity of primal uniform LP duality similar to Theorem 3.9.
3.2 Weak genericity of uniqueness of optimal solutions

We now study the genericity of uniqueness of optimal solutions of conic programs. We derive this result by using the fact that convex functions are differentiable almost everywhere\(^1\).

In this section, let us fix \((A, B)\) and consider our primal problem \((SIP_P) = P(c)\) in SIP form with \(c\) as a parameter:

\[
P(c) \quad \max c^T x \quad \text{s.t.} \quad (B - Ax)^T Y \geq 0 \quad \text{for all} \; Y \in Z := \mathcal{K}^*.
\]

Note that in this case the feasible set \(\mathcal{F}_P\) is fixed and the problem depends on parameter \(c\) only. Let us denote the optimal value function as \(v_P(c)\) and the set of primal optimal solutions as

\[
\mathcal{F}_P^*(c) := \{x \in \mathcal{F}_P \mid c^T x = v_P(c)\}.
\]

Let us denote the subdifferential of the function \(v_P(c)\) as \(\partial v_P(c)\). For the proof of the following results, see [GL88, page 262] for (1), and [GL88, Theorem 2.1] for (2).

**Theorem 3.12.** Let \(B\) and \(A\) be such that \(\mathcal{F}_P \neq \emptyset\). Then the following holds:

1. \(v_P(c)\) is a proper closed convex function of \(c\) on its effective domain \(\mathcal{D}_P\).
2. \(\partial v_P(c) = \mathcal{F}_P^*(c)\).

By using Theorem 3.12 and Rademacher’s theorem for convex functions we are ready to prove the weak genericity of uniqueness in conic programming and obtain at the same time an alternative proof for the genericity of the Slater condition.

**Proposition 3.13.** Let \(A\) and \(B\) be such that \(\mathcal{F}_P \neq \emptyset\). Then for almost all \(c \in \mathbb{R}^n\) the following alternative holds:

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\(^1\)The author would like to thank A. Shapiro for pointing out the approach.
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- either the dual problem \((D)\) is infeasible and \(v_P(c) = +\infty\), or
- the Slater condition holds for \((D)\) and the solution of \((P)\) is unique.

A corresponding dual result holds with respect to \(B\) (for fixed \(A, c\)).

**Proof.** Let \(A, B\) be such that \(\mathcal{F}_P \neq \emptyset\). Let \(\mathcal{D}_P\), with boundary \(\text{bd} \mathcal{D}_P\), be the (convex) effective domain of the convex function \(v_P(c)\) from Theorem 3.12. We distinguish the following three cases for \(c \in \mathbb{R}^n\):

(i) \(c \in \text{bd} \mathcal{D}_P\),
(ii) \(c \notin \text{cl} \mathcal{D}_P\),
(iii) \(c \in \text{int} \mathcal{D}_P\).

By Lemma 3.4, case (i) occurs on a set of measure zero in \(\mathbb{R}^n\). In case (ii), in view of the relation

\[
\text{The dual problem is feasible } \Rightarrow \quad c \in \mathcal{D}_P
\]

we get that the dual problem is infeasible. In this case, unboundedness of \(P(c)\) follows from Table 2.3 and the first alternative holds.

In case (iii), we use the fact that the convex function \(v_P(c)\) defined on the open set \(\text{int} \mathcal{D}_P\) is differentiable for almost all \(c \in \text{int} \mathcal{D}_P\) (see e.g. [Roc70, Theorem 25.5]). This implies that for these values of \(c\) the subgradient \(\partial v_P(c) = \mathcal{F}_P^*(c) = \{\nabla v_P(c)\}\) is a singleton by Theorem 3.12 (2). Moreover, in this case, combining (2.5) and (2.4), the Slater condition holds for the dual problem.

The proof of the dual statement is similar.

A uniqueness result similar to the statement in Proposition 3.13 can also be found in [BDL11], even for more general convex programs. Also, another different proof is given in [PT01, Theorem 3] which is based on deeper results from geometric measure theory.

By combining the statements of Proposition 3.13 for the primal and dual we obtain:

**Corollary 3.14.** Let \(A \in \mathbb{R}^{m \times n}\) be given arbitrarily. Then for almost all \((c, B) \in \mathbb{R}^n \times \mathbb{R}^m\) the following holds: If both \((P)\) and \((D)\) are feasible, then both satisfy the Slater condition and both have unique optimal solutions \(\overline{X}\) and \(\overline{Y}\).
3.3 Nondegeneracy and strict complementarity

We now discuss weak genericity of nondegeneracy and strict complementarity of optimal solutions along the lines given in [PT01]. In this section, we therefore consider problems in self-dual form \((P_0)\) and \((D_0)\) as in [PT01]. Let \(\overline{X}\) and \(\overline{Y}\) denote primal and dual optimal solutions.

In [PT01], the authors fix the parameter \(\mathcal{L}\) and consider the set of problem instances where both primal and dual problems are feasible with zero duality gap

\[
\mathcal{D}(\mathcal{L}) := \{(C, B) \mid \text{the corresponding } \mathcal{F}_{P_0} \neq \emptyset, \mathcal{F}_{D_0} \neq \emptyset \text{ and } v_{P_0} = v_{D_0}\}.
\]

Furthermore, in this set of parameters, the set of problem instances which have a pair of strictly complementary optimal solutions is denoted as

\[
\mathcal{D}(\mathcal{L}) := \{(C, B) \in \mathcal{D}(\mathcal{L}) \mid \exists \text{ strictly complementary } \overline{X} \in \mathcal{F}_{P_0}, \overline{Y} \in \mathcal{F}_{D_0}\}.
\]

Using a deep result from geometric measure theory [PT01, Theorem 3], the authors obtain weak genericity of strict complementarity.

**Theorem 3.15.** [PT01, Proposition 2] For fixed \(\mathcal{L}\), the set \(\mathcal{D}(\mathcal{L}) \setminus \mathcal{D}(\mathcal{L})\) has \(\text{dim}(\overline{\mathcal{D}(\mathcal{L})})\)-dimensional Hausdorff measure zero.

Before going further, let us shortly introduce the Hausdorff measure. For any subset \(S\) of \(\mathbb{R}^l\), let us define the diameter of \(S\)

\[
diam(S) = \sup\{|x - y| \mid x, y \in S\}.
\]

For \(S \subseteq \mathbb{R}^l\) and for some \(\delta > 0\), a \(\sigma\)–cover of the set \(S\) is a family of countably many sets \(S_j\) with \(diam(S_j) \leq \delta\) such that \(S \subseteq \bigcup_{j=1}^{\infty} S_j\). For a nonnegative \(t \in \mathbb{R}\), the \(t\)-dimensional Hausdorff measure of \(S\) is defined as

\[
H^t(S) := v(t) \lim_{\delta \to 0} \inf \left\{ \sum_{j=1}^{\infty} (\text{diam}(S_j)/2)^t \mid \{S_j\} \text{ is a } \delta\text{-cover of } S \right\},
\]
where \( v(t) := \frac{\pi^{t/2}}{\Gamma(t/2+1)} \). For an integer \( t \), the term \( v(t) \) is the volume of the Euclidean unit ball in \( \mathbb{R}^t \). It is well-known that the \( n \)-dimensional Hausdorff measure and the Lebesgue measure in \( \mathbb{R}^n \) coincide (see, e.g., [Mor95, Corollary 2.8]). One of the advantages of the Hausdorff measure is the possibility of measuring lower dimensional subsets in \( \mathbb{R}^l \). For example, the one-dimensional Hausdorff measure of a simple curve in \( \mathbb{R}^l \) is equal to the length of the curve, for a proof and further details see [Fed69, Mor95].

Now let us turn back to genericity of strict complementarity. We have shown in Corollary 3.8 that the gap-free set \( \mathcal{D}(\mathcal{L}) \subset \mathbb{R}^m \times \mathbb{R}^m \) is not of measure zero and has non-empty interior in the space \( \mathbb{R}^m \times \mathbb{R}^m \). Thus, Theorem 3.15 can be stated as follows

**Theorem 3.16.** For fixed \( \mathcal{L} \), the set \( \overline{\mathcal{D}}(\mathcal{L}) \setminus \mathcal{D}(\mathcal{L}) \) has Lebesgue measure zero.

From Lemma 2.25, under strict complementarity, dual uniqueness of an optimal solution implies primal nondegeneracy. Using similar arguments as [PT01, p.456], we combine Corollary 3.8, Theorem 3.16, Corollary 3.14 and Lemma 2.25 in order to prove that the primal nondegeneracy condition is weakly generical at solutions of \( (P_0) \).

Note that Lemma 2.25(c) does not hold for \( X \) and \( Y \) interchanged unless \( \mathcal{K} \) is facially exposed. However, if we define strict complementarity as in (2.18), then Lemma 2.25(b) holds for \( X \) and \( Y \) interchanged. Analogous to (2.17) following [PT01], one can show that (2.18) is a generic property. Thus, using the same arguments, weakly generically at solutions of \( (D_0) \) the nondegeneracy condition holds. Combining everything, we have:

**Corollary 3.17.** Let \( \mathcal{L} \) be given arbitrarily. Then for almost all \( (C, B) \in \mathbb{R}^{2m} \) the following is true: If the corresponding programs \( (P_0), (D_0) \) are both feasible, then there exist unique optimal solutions \( \overline{X} \) of \( (P_0) \) and \( \overline{Y} \) of \( (D_0) \). These solutions are nondegenerate and satisfy the strict complementarity condition.
3.4. Genericity results in the SDP case

Remark 3.18. With the same projection trick as in Remark 3.6, the genericity result of Corollary 3.17 for \((P_0), (D_0)\) can directly be translated to the following statement for the programs in the form \((P), (D)\):

Let \(A \in \mathbb{R}^{m \times n}\) be arbitrary. Then for almost all \((c, B) \in \mathbb{R}^n \times \mathbb{R}^m\) we have that if \((P)\) and \((D)\) are both feasible, then there exist unique optimal solutions \(\overline{X}\) of \((P)\) and \(\overline{Y}\) of \((D)\). Moreover, \(\overline{X}\) and \(\overline{Y}\) are both nondegenerate and satisfy the strict complementarity condition.

Note that to assure uniqueness of the solution of \((P)\) in terms of the variable \(x \in \mathbb{R}^n\) we have to assume that \(A\) has full rank \(n\). Recall, however, that for \(m \geq n\) a matrix \(A \in \mathbb{R}^{m \times n}\) generically has full rank \(n\), see (3.2).

3.4 Genericity results in the SDP case

In this section, we illustrate an approach from [AHO97] which is different from the previous ones to show genericity results for semidefinite programming. Using techniques from differential topology or differential geometry, one can show directly that nondegeneracy and strict complementarity are weakly generic properties for SDP, see [AHO97]. In Proposition 2.21, we have shown that nondegeneracy implies the Slater condition. Thus, if nondegeneracy holds for almost all feasible problem instances, then so does Slater’s condition. This was also established in [Sha97, page 310]. Under strict complementarity, primal and dual optimal solutions are nondegenerate if and only if the primal and dual optimal solutions are unique, see Lemma 2.25. Thus, weak genericity of uniqueness directly follows from weak genericity of nondegeneracy and strict complementarity.

In the following, we investigate the boundary structure of the positive semidefinite cone and illustrate how a transversality theorem can be applied to show such genericity results, see [AHO97].
Consider the set

\[ D_r := \{ X \in S^k \mid \text{rank } X = r \} \]

and

\[ D_r^+ = S_+^k \cap D_r = \{ X \in S^k \mid X \in S_+^k \text{ and rank } X = r \}. \]

It is a standard result in differential topology that the sets \( D_r \) and \( D_r^+ \) are submanifolds in \( S^k \), see e.g. [GP74] and [HS95, Prop 2.1]. Let us sketch the proofs.

The set \( T_r := \{ X \in S^k \mid \text{rank } X \geq r \} \) is an open set in \( S^k \) and thus it is a smooth submanifold of \( S^k \). For \( X \in T_r \), we can rearrange its columns and rows such that \( X \) has the following form

\[ X = \begin{pmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{pmatrix} \]

where \( X_1 \) is an \( r \times r \) nonsingular matrix. Multiplying \( X \) by a nonsingular matrix, we obtain

\[ \begin{pmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{pmatrix} \begin{pmatrix} I & -X_1^{-1}X_2 \\ 0 & I \end{pmatrix} = \begin{pmatrix} X_1 & 0 \\ X_2^T & X_3 - X_2^T X_1^{-1} X_2 \end{pmatrix}. \]

Thus, \( X \) has rank \( r \) if and only if \( X_3 - X_2^T X_1^{-1} X_2 = 0 \). Consider a smooth map \( f : T_r \mapsto S^r \) given by

\[ f(X) = X_3 - X_2^T X_1^{-1} X_2. \]

As the partial derivative \( \nabla_{X_3} f(X) \) of \( f \) wrt. \( X_3 \) is equal to the identity, the function \( f \) is a submersion, i.e., a smooth map whose differential map \( df(X) \) is surjective for any \( X \in T_r \). Applying the Preimage theorem, we derive that \( f^{-1}(0) = D_r \) is a submanifold of \( T_r \).

The boundary of manifold \( D_r \) consists of lower rank matrices

\[ \text{rbd } D_r = D_0 \cup \ldots \cup D_{r-1}. \]

From continuity of eigenvalue functions, we can easily check that the boundary of \( S_+^k \) and the interior of \( S_+^k \) are

\[ \text{bd } S_+^k = D_0^+ \cup \cdots \cup D_{k-1}^+ \quad \text{and} \quad \text{int } S_+^k = D_k^+. \]
3.4. Genericity results in the SDP case

Using eigenvalue decomposition, one can directly check that

\[ D_r^+ = \{ ZZ^T \in S^k \mid Z \in \mathbb{R}^{k \times r} \text{ and rank } Z = r \}. \]

Thus, \( D_r^+ \) is a smooth embedded submanifold of \( S^k \) for all \( 0 \leq r \leq k \). Moreover, it is known that \( D_r^+ \) has one connected component, see \([HS95, \text{Prop } 2.1]\).

**Definition 3.19.** Let \( f : U \mapsto W \) be a smooth map between smooth manifolds \( U \) and \( W \) and let \( V \subset W \) be a submanifold. We say that \( f \) is transversal to \( V \) at a point \( x \in U \) if either \( f(x) \notin V \) or

\[ df(x)(\mathcal{T}_x U) + \mathcal{T}_{f(x)} V = \mathcal{T}_{f(x)} W. \]  

(3.5)

Here \( \mathcal{T}_x U \) denotes the tangent space to \( U \) at \( x \) and \( df(x) \) denotes the differential of \( f \) at \( x \). We say that \( f \) is transversal to \( V \) if the transversality condition is satisfied for every \( x \in U \).

Consider an SDP problem in self-dual form \((P_0)\) and maps

\[ \theta : \mathbb{R}^n \times S^k \times V_{m,n} \to S^k \text{ given by } \theta(x, B, A_1, \ldots, A_n) = B - \sum_{i=1}^n x_i A_i \]

and for fixed \( t = (B, A_1, \ldots, A_n) \)

\[ f_t : \mathbb{R}^n \to S^k \text{ given by } f_t(x) = B - \sum_{i=1}^n x_i A_i. \]

Clearly, we have \( df_t(x)(\mathbb{R}^n) = \text{span}\{A_i\} \). Suppose that the rank of the matrix \( X := B - \sum_{i=1}^n x_i A_i \) is \( r \) and take submanifold \( D_r \). The transversality condition (3.5) is equivalent to the following:

\[ \text{span}\{A_i\} + \mathcal{T}_X D_r = S^k. \]  

(3.6)

As the partial derivative \( \nabla_B \theta(x, B, A_1, \ldots, A_n) \) of \( \theta \) wrt. \( B \) is equal to the identity map, the mapping \( \theta \) is transversal to any submanifold of \( S^k \) and consequently \( f_t \) is also transversal to any submanifold of \( S^k \) for
almost all $t$, for further details see [GP74, p.68] or [AHO97]. In other words, equation (3.6) holds for almost all parameters $(B, A_1, \ldots, A_n)$.

It is clear that $T_X D_r$ is the tangent space $\text{tan}(X, \mathcal{S}_k)$, so
\begin{equation}
\mathcal{S}_k = \text{span}\{A_i\} + \text{tan}(X, \mathcal{S}_k^k).
\end{equation}

Let us denote $\mathcal{L} = \text{span}\{A_i\}$ as usual. The positive semidefinite cone is nice (cf. Definition 2.26) and thus using equations (2.22) and (3.7), we obtain that the above transversality condition is equivalent to
\begin{equation}
\mathcal{L}^\perp \cap \text{span } J^\Delta(X) = \{0\},
\end{equation}
which is exactly the primal nondegeneracy condition. Therefore, primal nondegeneracy holds for almost all parameters $(B, A_1, \ldots, A_n)$.

Using a similar technique, one can show that strict complementarity holds for almost all parameters $(B, A_1, \ldots, A_n)$, for the proof see [AHO97, Theorem 15].

### 3.5 Stability issues

We have seen in the previous sections that uniqueness, nondegeneracy and strict complementarity hold for almost all problem instances of conic programs. Now in this section, let us discuss stability of these properties.

In smooth nonlinear optimization (see e.g. [JJT00]), the stability of properties such as uniqueness, nondegeneracy and strict complementarity is typically proven by applying the (smooth) Implicit Function Theorem to an appropriate system of optimality conditions. However, a general linear conic program is not a smooth problem due to the cone constraint, but rather a problem described by Lipschitz continuous functions. In general, the stability of these properties in linear conic programming is still an open question. Using extra information on the cone, stability of these properties can be proven for the case of linear programming or semidefinite programming. Let us consider the linear programming case first.
3.5. Stability issues

**Stability analysis for LP:** Consider the pair of primal-dual LP’s

\[(LP) \quad \max \ c^T x \quad \text{s.t.} \quad X := B - Ax \in \mathbb{R}^m_+; \]

\[(LD) \quad \min \ B^T Y \quad \text{s.t.} \quad A^T Y = c, \quad Y \in \mathbb{R}^m_+; \]

for instances \((A, B, c)\) near \((\bar{A}, \bar{B}, \bar{c})\). We assume that the LP given by parameter \((A, B, c)\) is a generic instance. In other words, \(\bar{A}\) has full rank \(n\) and there are complementary solutions \(X, Y \in \mathbb{R}^m\) satisfying uniqueness, nondegeneracy and strict complementarity. Let us consider the problem given by parameter \((A, B, c)\) and denote the active index set and its complement as

\[\mathcal{I} = \{i \in \{1, \ldots, m\} \mid X_i = 0\} \quad \text{and} \quad \mathcal{I}^C = \{i \in \{1, \ldots, m\} \mid X_i > 0\}.\]

Let \(\mathcal{L} := \text{span}\{A_j \mid j = 1, \ldots n\}\) where \(A_j\) is the \(j\)th column of \(A\).

The strict complementarity condition means that \(Y_i = 0\) holds if and only if \(i \in \mathcal{I}^C\). It follows that

\[J(X) = \text{cone}\{e_i \mid i \in \mathcal{I}^C\} = G^\Delta(Y) \quad \text{(3.8)}\]

and

\[G(Y) = \text{cone}\{e_i \mid i \in \mathcal{I}\} = J^\Delta(X). \quad \text{(3.9)}\]

The nondegeneracy conditions for \(X\) and \(Y\) are

\[\mathcal{L}^\perp \cap \text{span} \ J^\Delta(X) = \{0\} \quad \text{and} \quad \mathcal{L} \cap \text{span} \ G^\Delta(Y) = \{0\}. \quad \text{(3.10)}\]

From (3.10) we deduce \(|\mathcal{I}| \leq n\), resp. \(|\mathcal{I}^C| \leq m - n\) and thus, using \(m = |\mathcal{I}| + |\mathcal{I}^C| \leq m - n + n = m\), we find \(|\mathcal{I}| = n\). Moreover, the condition \(\mathcal{L} \cap \text{span} G^\Delta(Y) = \mathcal{L} \cap \text{span}\{e_i \mid i \in \mathcal{I}^C\} = \{0\}\) implies that the matrix

\[\begin{pmatrix}
A^T \\
e_i^T, i \in \mathcal{I}^C
\end{pmatrix}
\]

is nonsingular. Then the \(n \times n\)-matrix

\[\overline{A}_\mathcal{I} := ([\overline{A}_1]_\mathcal{I}, \ldots, [\overline{A}_n]_\mathcal{I}) \quad \text{with} \quad [\overline{A}_i]_\mathcal{I} := ([\overline{A}_i]_j, j \in \mathcal{I})^T\]
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is nonsingular as well. Therefore, under a small perturbation, the matrix $A_T$ remains nonsingular as the determinant is a continuous function.

The optimal solutions of the problem instance given by $(\bar{A}, \bar{B}, \bar{c})$ satisfy the following system of linear equations.

$$B_T - A_T x = 0 \quad \text{and} \quad A_T^T Y_T - \bar{c} = 0$$

with $\bar{Y}_i = 0$ for $i \notin \mathcal{I}$.

For a parameter $(A, B, c)$ near $(\bar{A}, \bar{B}, \bar{c})$, consider solutions $x$ (resp. $X$) and $Y$ given as the solutions of the systems

$$B_T - A_T x = 0 \quad \text{and} \quad A_T^T Y_T - c = 0$$

with $Y$ defined by $Y_i = [Y_T^2]_i$ for $i \in \mathcal{I}$ and $Y_i = 0$ otherwise.

One can check directly that the obtained $x$ and $Y$ are strictly complementary, nondegenerate, unique optimal solutions of $(P)$ and $(D)$ with parameter $(A, B, c)$. Therefore, these “nice” properties are stable in linear programming.

**Stability analysis for SDP:** We now study the stability of uniqueness, nondegeneracy and strict complementarity for SDP. Consider

$$(SDP_P) \quad \max \langle C, B \rangle - \langle C, X \rangle \quad \text{s.t.} \quad X := B - \sum_{i=1}^{n} x_i A_i \in \mathcal{S}_+^k$$

$$(SDP_D) \quad \min \langle B, Y \rangle \quad \text{s.t.} \quad Y := \sum_{j=1}^{m-n} y_j A_j^\perp + C \in \mathcal{S}_+^k$$

as programs depending on the parameter $Q := (B, C, \{A_i\}_{i=1}^n) \in (\mathcal{S}_+^k)^{n+2}$. We again assume that the matrices $A_i, i = 1, \ldots, n$, are linearly independent and that $A_j^\perp, j = 1, \ldots, m - n$, is a basis of the orthogonal complement of span$\{A_i\}_{i=1}^n$.

Let $\overline{Q} := (\bar{B}, \bar{C}, \{\bar{A}_i\}_{i=1}^n)$ be a given parameter and assume that $\bar{x}, \bar{y}$ (or $\bar{X}, \bar{Y}$) are complementary solutions of the corresponding generic SDP pair $(P_0)$ and $(D_0)$ with rank $\bar{X} = r$ (where $\bar{X} := \bar{B} - \sum_{i=1}^{n} \bar{x}_i \bar{A}_i$), and rank $\bar{Y} = s$ (where $\bar{Y} := \sum_{j=1}^{m-n} \bar{y}_j \bar{A}_j^\perp + \bar{C}$). This means that the solutions are unique, nondegenerate and strictly complementary, i.e. $k = s + r$.  

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For parameters \( \{ A_i \}_{i=1}^n \) in a sufficiently small neighborhood of \( \{ A_i^\perp \}_{j=1}^{m-n} \) we can assume that the orthogonal complement \( \{ A_i^\perp \}_{j=1}^{m-n} \) depends at least \( C^1 \)-smoothly on the parameters \( \{ A_i \}_{i=1}^n \). Indeed, we can compute the \( \{ A_i^\perp \} \)’s by a smooth Gram-Schmidt orthogonalization process. So for \( Q = (B, C, \{ A_i \}_{i=1}^n) \) near \( \bar{Q} \) we can define the functions

\[
\theta(x, Q) := B - \sum_{i=1}^n x_i A_i, \quad \psi(y, Q) := \sum_{j=1}^{m-n} y_j A_j^\perp + C
\]

which smoothly depend on \( x, y \) and on the parameter \( Q \). In [AHO97, Lemma 22] it has been proven that the set

\[
W_{r,s} := \{ (X, Y) \in S^k \times S^k | \text{rank } X = r, \text{rank } Y = s, \langle X, Y \rangle = 0 \}
\]

with \( k = r+s \) is a smooth \( C^\infty \)-submanifold of \( S^k \times S^k \) with \( \text{dim } W_{r,s} = m \) and thus with codimension \( m \). This means that in some neighborhood of \( (\bar{X}, \bar{Y}) \in W_{r,s} \) there are smooth functions \( H_\ell(X, Y) \) with \( \ell = 1, \ldots, m \) such that for \( (X, Y) \approx (\bar{X}, \bar{Y}) \), \( (X, Y) \in S^k \times S^k \) we have

\[
(X, Y) \in W_{r,s} \iff H_\ell(X, Y) = 0, \quad \ell = 1, \ldots, m.
\]

From [AHO97, proof of Theorem 15] it follows that for the weakly generic parameter \( \bar{Q} \) the mapping \( \left( \theta(x, \bar{Q}), \psi(y, \bar{Q}) \right) : \mathbb{R}^m \to S^k \times S^k \), with \( (x, y) \mapsto (\theta(x, \bar{Q}), \psi(y, \bar{Q})) \), intersects the manifold \( W_{r,s} \) transversally at \( (\bar{x}, \bar{y}) \). By standard results in differential topology (see e.g. [JJT00, Remark 7.3.1]), this means that the gradients

\[
\nabla_{(x,y)} H_\ell(\theta(\bar{x}, \bar{Q}), \psi(\bar{y}, \bar{Q})), \quad \ell = 1, \ldots, m \quad \text{are linearly independent}
\]

(3.11)

at the solution \( (\theta(\bar{x}, \bar{Q}), \psi(\bar{y}, \bar{Q})) \) of \( H_\ell(\theta(\bar{x}, \bar{Q}), \psi(\bar{y}, \bar{Q})) = 0, \quad \ell = 1, \ldots, m \).

Locally near \( (\bar{x}, \bar{y}, \bar{Q}) \) we consider the system of equations

\[
\hat{H}_\ell(x, y, Q) := H_\ell(\theta(x, Q), \psi(y, Q)) = 0, \quad \ell = 1, \ldots, m \quad (3.12)
\]

in the variables \( (x, y, Q) \). By applying the Implicit Function Theorem to (3.12), and taking into account (3.11), we see that for \( Q \approx \bar{Q} \) there exists a unique \( C^\infty \)-solution function \( x(Q), y(Q) \) of the system

\[
\hat{H}_\ell(x(Q), y(Q), Q) = 0, \quad \ell = 1, \ldots, m.
\]
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By continuity arguments, using the fact that \( \theta(x(Q), Q) = X \in S^k_+ \) and \( \psi(y(Q), Q) = Y \in S^k_+ \), it follows that

\[
(\theta(x(Q), Q), \psi(y(Q), Q)) \in S^k_+ \times S^k_+ \cap W_{r,s}
\]

and the solutions \( x(Q), y(Q) \) define strict complementary solutions of the programs \((P_0), (D_0)\) wrt. the data \( Q \approx Q \). So we have proven the following stability result.

**Proposition 3.20.** Let \( Q \) be an instance from the generic set of SDP programs such that the solutions \( x, y \) (or \( X, Y \) with \( \text{rank } X = r, \text{rank } Y = s \)) of the corresponding programs \((P_0), (D_0)\) satisfy uniqueness, nondegeneracy and strict complementarity. Then there exists a (nonempty open) neighborhood \( U \) of \( Q \) such that for any problem data \( Q \in U \) the corresponding SDP programs \((P_0)\) and \((D_0)\) have solutions \( (x(Q), y(Q)) \approx (x, y) \) (or \( (X(Q), Y(Q)) \approx (X, Y) \) ) that are unique, nondegenerate and strictly complementary with the same ranks, \( \text{rank } X(Q) = r \) and \( \text{rank } Y(Q) = s \).

3.6 Genericity results for semi-algebraic cones

In the previous section for the stability of SDP, we made use of the fact that the set of certain positive semidefinite matrices can locally be described by smooth manifolds. This is generally possible if the cones \( K, K^* \) are so-called semi-algebraic sets: it is well-known that semi-algebraic sets allow a complete partition (stratification) of the set into smooth manifolds, see e.g. [BR90].

**Definition 3.21.** A semi-algebraic subset of \( \mathbb{R}^n \) is a set of points satisfying a finite boolean combination of polynomial equations and inequalities with real coefficients.

Even though there exist convex sets which are not semi-algebraic, most problems in applications have semi-algebraic feasible sets.
3.6. Genercity results for semi-algebraic cones

**Lemma 3.22.** Let the cone $\mathcal{K} \subset \mathcal{S}^k$ be semi-algebraic. Then the dual cone is also semi-algebraic.

**Proof.** Since $\mathcal{K}$ is semi-algebraic, there exist polynomials $p_i$ and $q_j$ with real coefficients such that

$$\mathcal{K} = \{ A \in \mathcal{S}^k \mid p_i(A) \leq 0, q_j(A) = 0 \text{ for all } i = 1, \ldots, t, j = 1, \ldots, l \}.$$  

Using the definition of the dual cone, we have

$$\mathcal{K}^* = \{ B \in \mathcal{S}^k \mid \langle A, B \rangle \geq 0 \text{ for all } A \in \mathcal{K} \}.$$  

By reformulating, we derive a so-called first order formula for $\mathcal{K}^*$,

$$\mathcal{K}^* = \{ B \in \mathcal{S}^k \mid \forall A \in \mathcal{S}^k (\langle A, B \rangle \geq 0) \lor (p_1(A) > 0) \lor \ldots \lor (p_t(A) > 0) \lor (q_1(A) \neq 0) \lor \ldots \lor (q_l(A) \neq 0) \}.$$  

Now the Tarski-Seidenberg-Theorem [Che66, Theorem 2.6] implies that the dual set $\mathcal{K}^*$ is semi-algebraic. □

It is straightforward to check that the copositive and completely positive cones are semi-algebraic.

**Proposition 3.23.** The copositive cone $\mathcal{COP}$ and the completely positive cone $\mathcal{CP}$ are semi-algebraic sets.\(^2\)

**Proof.** Let us consider the copositive cone first. Note that $\mathcal{COP}$ is represented by the following first order formula

$$\mathcal{COP} = \{ A \in \mathcal{S}^k \mid \forall x \in \mathbb{R}^k (x^T A x \geq 0) \lor (\min_i x_i < 0) \}.$$  

Using the Tarski-Seidenberg-Theorem [Che66, Theorem 2.6] again, we see that the copositive cone is semi-algebraic. From Lemma 3.22, it directly follows that the completely positive cone (the dual of the copositive cone) is semi-algebraic. □

\(^2\)The author would like to thank D. Drusvyatskiy for pointing out Proposition 3.23 (personal communication).
Without using Lemma 3.22, we can directly verify that the completely positive cone $\mathcal{CP}$ is semi-algebraic. Denote $m := \frac{1}{2}k(k + 1)$ and consider the set

$$\{(A, b^1, \ldots, b^m) \in S^k \times (\mathbb{R}^k)^m \mid A - \sum_{j=1}^m b^j (b^j)^T = 0 \land b^j \geq 0 \text{ for all } j\}.$$ 

The projection of the above set onto $S^k$ is exactly the completely positive cone. From Tarski-Seidenberg-Theorem it follows that a projection of a semi-algebraic set is semi-algebraic. Thus, $\mathcal{CP}$ is a semi-algebraic set.

In [BDL11], the authors consider the conic problem $(P)$ for a semi-algebraic cone $\mathcal{K}$ and fixed $(A, B)$. In other words, the objective vector $c$ is the parameter of the conic program and the following genericity result is obtained:

*BDL11, Theorem 5.1* Let $\mathcal{K}$ be a semi-algebraic cone and $A, B$ be given such that $\mathcal{F}_P$ is compact. Then there exists a generic set $S \subset \mathbb{R}^n$ such that for all $c \in S$ the corresponding programs $(P)$ and $(D)$ are strictly complementary and there exists a unique, primal nondegenerate maximizer.

As we have outlined above, this result holds in particular for copositive and completely positive problems.

### 3.7 Genericity results in linear semi-infinite optimization

In the preceding discussions we have seen that a conic program can be seen as a special case of a linear semi-infinite program SIP (cf. Section 3.2). There are many papers dealing with generic properties (in the sense of density and stability) of semi-infinite problems in the form $(SIP_P)$, $(SIP_D)$. We refer the reader to [JZ85] and [GLT03, GT08, GT09, GTVdS12].

One might expect that these genericity results for SIP can directly be transferred to conic programming, but unfortunately this is not the
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case. In this section, we comment why most genericity results from linear semi-infinite optimisation (SIP) cannot be directly applied to conic programming.

In the above articles, SIP programs are considered in the form \((SIP_P)\) with infinite, compact index set \(\mathcal{Z} \subset \mathbb{R}^m\). In [JZ85] the problem data \((c, a(Y), b(Y))\) are elements of the space \(\mathbb{R}^n \times C^2(\mathcal{Z})^n \times C^2(\mathcal{Z})\). In [GLT03, GT08, GT09, GTVdS12] the problem data \((c, a(Y), b(Y))\) are taken from \(\mathbb{R}^n \times C(\mathcal{Z})^n \times C(\mathcal{Z})\) endowed with the norm of uniform convergence

\[
\|(c, a, b)\| = \max \left\{ \|c\|_\infty, \max_{Y \in \mathcal{Z}} \|(a(Y), b(Y))\|_\infty \right\}.
\]

But as we have seen previously, for the conic problem in \((SIP_P)\) form, the data \((a(Y), b(Y))\) are given by (2.1) and are of the special form

\[
a(Y) = A^T Y, \quad b(Y) = \langle B, Y \rangle,
\]

linear in \(Y\). So the set of conic programs represents only a small subset of the set of SIP instances, for example given by \((a(Y), b(Y))\) \(\in C(\mathcal{Z})^n \times C(\mathcal{Z})\). Thus this subset of conic programs allows much less freedom for perturbations, so that roughly speaking we can say:

- The density results cannot be transferred from the general SIP theory to the special case of conic programming.
- Openness results remain valid in the following sense: the sufficient conditions for stability in SIP remain valid for conic programming, but not the necessary conditions. Typically the sufficient conditions for stability in SIP are too strong in conic programming.

We just remark that [GT09, Theorem 1] gives genericity results (density and openness) for the special case of (finite) linear programs. The above statements are explained further and exemplified in Section 4.2.