Chapter 2

Linear conic programming

In linear conic programming, we maximize or minimize a linear function over the intersection of an affine space and a convex cone. A conic program is a convex problem and its feasible set is given by linear constraints and a conic constraint. Depending on how the cone is described, checking membership of this cone can be challenging itself and a difficult task. The cone may have an implicit description. In this sense, linear conic programs are closely related to linear semi-infinite programs (SIP). Yet conic programs are not typical SIPs, since the cone constraints provide a specific structure.

2.1 Formulating linear conic programs

In general, the bottleneck of a conic program is the cone in consideration. In the thesis, we consider so-called proper cones.

Definition 2.1. Let $S^k$ be the space of real symmetric $k \times k$ matrices. A proper cone $\mathcal{K}$ is a subset of $S^k$ which satisfies the following:

(i) If $\alpha, \beta \geq 0$ and $X, Y \in \mathcal{K}$, then $\alpha X + \beta Y \in \mathcal{K}$, i.e. $\mathcal{K}$ is convex,

(ii) $\mathcal{K}$ is closed,

(iii) $\mathcal{K}$ is pointed: $\mathcal{K} \cap -\mathcal{K} = \{0\}$,

(iv) $\mathcal{K}$ is full-dimensional: $\text{int} \mathcal{K} \neq \emptyset$. 

5
We denote the interior, relative interior, boundary, convex hull, convex conic hull, linear span and dimension of a set $S$ as $\text{int } S$, $\text{ri } S$, $\text{bd } S$, $\text{conv } S$, $\text{cone } S$, $\text{span } S$, and $\text{dim } S$, respectively.

Cones can also be defined in a real topological vector space, see e.g. [Bar81]. However, as the thesis is mainly concerned with the finite dimensional Euclidean space, we consider the space $\mathcal{S}^k$ directly. By defining $X \preceq Y$ if and only if $Y - X \in \mathcal{K}$, a proper convex cone induces a partial ordering on $\mathcal{S}^k$. Pointedness of the cone is necessary to show the antisymmetry property: if $X \preceq Y$ and $Y \preceq X$ then $X = Y$.

Note that we can simply identify $\mathcal{S}^k \equiv \mathbb{R}^m$ where $m := \frac{1}{2}k(k + 1)$.

**Assumption 2.2.** Throughout this thesis, we assume that $\mathcal{K} \subseteq \mathcal{S}^k$ is a proper cone.

Recall that $\text{tr}(M) = \sum_{i=1}^{k} M_{ii}$ denotes the trace of a matrix $M \in \mathcal{S}^k$. The standard inner product in the space $\mathcal{S}^k$ is given by $\langle X, Y \rangle := \text{tr}(XY)$ for $X, Y \in \mathcal{S}^k$. It is also called the Frobenius inner product on the space of matrices.

**Definition 2.3.** Consider a cone $\mathcal{K} \subset \mathcal{S}^k$. Its dual cone $\mathcal{K}^*$ with respect to the standard inner product $\langle \cdot, \cdot \rangle$ in $\mathcal{S}^k$ is defined as

$$\mathcal{K}^* := \{Y \in \mathcal{S}^k \mid \langle Y, X \rangle \geq 0 \text{ for all } X \in \mathcal{K}\}.$$

It follows easily from the definition that the dual cone $\mathcal{K}^*$ is a closed, convex cone. It is well known that if the primal cone $\mathcal{K}$ is proper then so is the dual cone.

**Proposition 2.4.** Under Assumption 2.2, the dual cone $\mathcal{K}^*$ is a proper cone as well.

**Proof.** We sketch the proof. We need to show that $\mathcal{K}^*$ is pointed and full-dimensional. By contradiction, suppose that $\mathcal{K}^*$ was not pointed. Then there exists a nonzero matrix $Z \in \mathcal{K}^* \cap -\mathcal{K}^*$. This implies that $\langle Z, X \rangle \geq 0$ and $\langle -Z, X \rangle \geq 0$ both hold for all $X \in \mathcal{K}$. By combining
2.1. Formulating linear conic programs

these two inequalities, we get \( \langle Z, X \rangle = 0 \) for all \( X \in \mathcal{K} \). Using that \( \mathcal{K} \) is full dimensional, we have \( Z = 0 \), a contradiction.

Let us sketch that \( \mathcal{K}^* \) is full-dimensional. Since \( \mathcal{K} \) is pointed, no line is included in the cone. This implies that there exists a supporting hyperplane at 0 such that no other element from \( \mathcal{K} \) is included in the hyperplane. The normal matrix \( C \) of this separating hyperplane fulfills \( \langle C, X \rangle > 0 \) for all \( X \in \mathcal{K} \setminus \{0\} \). This implies that \( C \in \text{int} \mathcal{K}^* \).

We list some proper cones which have been widely used in applications:

- Second order cone \( \mathcal{SOC}^m := \{ v \in \mathbb{R}^m \mid v_1 \geq \sqrt{v_2^2 + \ldots v_m^2} \} \)
- Nonnegative matrix cone \( \mathcal{NN}^k := \{ M \in \mathcal{S}^k \mid M_{ij} \geq 0 \text{ for all } i, j \} \).
- Positive semidefinite matrix cone
  \[ \mathcal{S}^k_+ := \{ M \in \mathcal{S}^k \mid y^T M y \geq 0 \text{ for all } y \in \mathbb{R}^k \} \]
- Copositive matrix cone
  \[ \mathcal{COP}^k := \{ M \in \mathcal{S}^k \mid y^T M y \geq 0 \text{ for all } y \in \mathbb{R}^k_+ \} \]

We say that a cone \( \mathcal{K} \) is self-dual if \( \mathcal{K} = \mathcal{K}^* \). For example, the second order cone, the positive semidefinite cone and the nonnegative cone are self-dual [AG03, Hel00]. On the other hand, the copositive cone is not self dual (see e.g. [BSM03]). Its dual cone is the following cone:

- Completely positive cone
  \[ \mathcal{CP}^k := \{ A \in \mathcal{S}^k \mid A = \sum_{i=1}^s b^i (b^i)^T \text{ with } b^i \in \mathbb{R}^k_+, s \in \mathbb{N} \} \]

**Definition 2.5.** Given a proper cone \( \mathcal{K} \), a face \( F \) of \( \mathcal{K} \) is a convex subset of \( \mathcal{K} \) with the following property: Consider any segment \([X, Y] := \{ \lambda X + (1 - \lambda)Y \mid \lambda \in [0, 1] \}\) for \( X, Y \in \mathcal{K} \). If a relatively interior point \( X^\lambda := \lambda X + (1 - \lambda)Y \) for some \( \lambda \in (0, 1) \) lies in \( F \), then the whole segment \([X, Y]\) is contained in \( F \).
Chapter 2. Linear conic programming

Trivial faces are \{0\} and \(\mathcal{K}\). We say that a face \(F\) is proper if \(F \neq \mathcal{K}\) and \(F \neq \emptyset\). Note that any intersection of a supporting hyperplane and a cone \(\mathcal{K}\) defines a face. On the other hand, if a proper face \(F\) can be given as an intersection of a hyperplane and the cone \(\mathcal{K}\), then we say that \(F\) is an exposed face. A cone is called facially exposed if all proper faces are exposed. It is known that \(\mathcal{N}\mathcal{N}^k, \mathcal{S}^k_+\) and \(\mathcal{SOC}^m\) are facially exposed while \(\mathcal{COP}^k\) is not, for the proofs see e.g. [Pat00, Dic11].

**Standard formulation:** Linear conic problems can be given in different equivalent forms. First, let us consider the so-called “standard form” of linear conic programs:

\[
\begin{align*}
\text{max} & \quad c^T x \\
\text{s.t.} & \quad X := B - \sum_{i=1}^{n} x_i A_i \in \mathcal{K} \quad (P)
\end{align*}
\]

where \(c \in \mathbb{R}^n\), \(B, A_i \in \mathcal{S}^k\) \((i = 1, \ldots, n)\) and \(\mathcal{K} \subseteq \mathcal{S}^k\) is a proper cone.

The dual problem derived via the Lagrangian approach is

\[
\begin{align*}
\text{min} & \quad \langle B, Y \rangle \\
\text{s.t.} & \quad \langle A_i, Y \rangle = c_i, \quad i = 1, \ldots, n, \\
& \quad Y \in \mathcal{K}^*. \quad (D)
\end{align*}
\]

In the thesis, we often denote matrices by capital letters and vectors by lowercase letters. We reserve subscripts for the components of a vector \(x\), denoted as \(x_i\), and superscripts for indexing vectors. As mentioned before, we regard \(\mathcal{S}^k \equiv \mathbb{R}^m\) where \(m := \frac{1}{2}k(k + 1)\).

Throughout the thesis, we assume the following

**Assumption 2.6.** Consider problem \((P)\). The matrices \(A_1, \ldots, A_n\) are linearly independent.

Under Assumption 2.6, there is a one-to-one correspondence between a vector variable \(x\) and a matrix variable \(X\). Thus, we may refer to \(x\) or \(X\) as a feasible solution of problem \((P)\).

Without loss of generality, we can assume that \(\dim \mathcal{S}^k = \frac{k(k+1)}{2} \geq n\). In the sequel, the feasible sets and optimal values of the conic programs \((P)\) and \((D)\) are denoted by \(\mathcal{F}_P, \mathcal{F}_D\) and \(v_P, v_D\), respectively.
2.1. Formulating linear conic programs

Linear conic programming represents an important class of convex problems with a multitude of applications. It contains linear programming (LP) with $\mathcal{K} := \mathbb{N}^{\mathbb{N}^k}$, semidefinite programming with $\mathcal{K} := \mathcal{S}_k^k$ and copositive programming with $\mathcal{K} := \mathcal{COP}_k$ as special cases. For surveys, we refer to [Nem07, Sha01, Pat00, Dür10].

We can rewrite a pair of conic problems $(P)$ and $(D)$ through a linear operator $A \in \mathbb{R}^{m \times n}$ consisting of columns $A_i \in \mathcal{S}_k$. Thus, we sometimes refer to the following formulation

$$\begin{align*}
\max \ c^T x & \quad \text{s.t.} \quad X := B - Ax \in \mathcal{K} \\
\min \ \langle B, Y \rangle & \quad \text{s.t.} \quad A^T Y = c, \ Y \in \mathcal{K}^* 
\end{align*}$$

with given vectors $c \in \mathbb{R}^n$, $B \in \mathbb{R}^m$, a matrix $A \in \mathbb{R}^{m \times n}$ and variables $x \in \mathbb{R}^n, Y \in \mathbb{R}^m$.

**Self-dual formulation:** Let us introduce the self-dual formulation of $(P)$ and $(D)$ for later use as well. We can rewrite $(P)$ and $(D)$ in so-called “self-dual” form, see e.g. [PT01],

$$\begin{align*}
\max \ \langle C, B \rangle - \langle C, X \rangle & \quad \text{s.t.} \quad X \in (B + \mathcal{L}) \cap \mathcal{K} \\
\min \ \langle B, Y \rangle & \quad \text{s.t.} \quad Y \in (\mathcal{L}^\perp + C) \cap \mathcal{K}^* 
\end{align*}$$

where $C, B \in \mathcal{S}_k$, $\mathcal{L}$ is a linear subspace in $\mathcal{S}_k$.

Below, we will see that the problems $(P)$ and $(D)$ are equivalent to $(P_0)$ and $(D_0)$, respectively. Let us identify $\mathcal{L} = \text{span}\{A_1, \ldots, A_n\}$ and choose some $C \in \mathcal{S}_k$ satisfying $\langle A_i, C \rangle = c_i$ for $i = 1, \ldots, n$. Then it is straightforward to see that the feasible sets of $(P_0)$ and $(P)$ coincide. For $X = B - \sum_{i=1}^n x_i A_i$, we obtain

$$\langle C, B \rangle - \langle C, X \rangle = \langle C, B - X \rangle = \langle C, \sum_{i=1}^n x_i A_i \rangle = \sum_{i=1}^n x_i \langle C, A_i \rangle = c^T x.$$

Thus, the objective function values of $(P)$ and $(P_0)$ are the same. The dual problems $(D_0)$ and $(D)$ have the same objective function, and in view of the relation

$$Y - C \in \mathcal{L}^\perp \iff \langle Y - C, A_i \rangle = 0 \text{ for all } i \iff \langle Y, A_i \rangle = c_i \text{ for all } i$$
the feasible sets coincide, so \((D_0)\) and \((D)\) are equivalent as well.

**SIP formulation:** Linear semi-infinite programs (SIP) can be viewed as an extension of linear programming. They are optimization problems with a linear objective and possibly infinitely many linear constraints. The representation of the feasible set is given by the intersection of infinitely many closed half spaces. Let us introduce the standard form of a SIP.

\[
\max_{x \in \mathbb{R}^n} c^T x \quad \text{s.t.} \quad b(Y) - a(Y)^T x \geq 0 \quad \text{for all } Y \in \mathcal{Z} \quad (\text{SIP}_P)
\]

with a possibly infinite index set \(\mathcal{Z} \subset \mathbb{R}^m\) and continuous functions \(a : \mathcal{Z} \to \mathbb{R}^n\) and \(b : \mathcal{Z} \to \mathbb{R}\). The Haar dual reads:

\[
\min \sum_{Y_j \in \mathcal{Z}} y_j b(Y_j) \quad \text{s.t.} \quad \sum_{Y_j \in \mathcal{Z}} y_j a(Y_j) = c, \quad y_j \geq 0, \quad (\text{SIP}_D)
\]

where the min is taken over all finite sums. For an introduction to SIP we refer to [GL98a, GG83, HK93].

As we mentioned earlier, linear conic programs can be seen as a special case of (SIP). By identifying \(m = \frac{1}{2} k(k+1)\) as before, the conic condition \(X \in \mathcal{K}\) can be expressed as

\[
\langle X, Y \rangle \geq 0 \quad \text{for all } Y \in \mathcal{Z} = \mathcal{K}^*,
\]

or equivalently, with a compact index set:

\[
\langle X, Y \rangle \geq 0 \quad \text{for all } Y \in \mathcal{Z} = \mathcal{K}^*_0 := \{Y \in \mathcal{K}^* \mid \|Y\| = 1\}.
\]

Here \(\|\cdot\|\) denotes a norm on \(S^k\) (e.g. the Frobenius norm \(\|Y\| = \sqrt{\text{tr}(YY^T)}\)). Let us consider conic programs in the form \((\tilde{P})\) and \((\tilde{D})\). The primal program \((\tilde{P})\) can be written as (\(\text{SIP}_P\)) with

\[
b(Y) := \langle B, Y \rangle, \quad a(Y) := A^T Y \quad \text{and} \quad \mathcal{Z} := \mathcal{K}^* \quad (2.1)
\]

where \(A \in \mathbb{R}^{m \times n}\) with columns \(A_i \in S^k\), i.e.,

\[
a(Y) := (\langle A_1, Y \rangle, \ldots, \langle A_n, Y \rangle)^T.
\]
The feasibility condition for \( (\text{SIP}_D) \) then becomes
\[
c = \sum_j y_j A^T Y_j, \quad y_j \geq 0
\]
and by putting \( Y := \sum_j y_j Y_j \in \mathcal{K}^* \), this coincides with the feasibility condition \( c = A^T Y \) of \( (\tilde{D}) \). Moreover, in view of
\[
\sum_j y_j b(Y_j) = \sum_j y_j \langle Y_j, B \rangle = \langle Y, B \rangle,
\]
the dual \( (\text{SIP}_D) \) is equivalent to \( (\tilde{D}) \). Thus, the dual problem obtained via the Lagrangian is equivalent to the Haar dual of SIP in the conic case.

2.2 Duality theory of conic problems

It is known that weak duality holds in general optimization problems, see e.g. [GG83, Sha01]. Consider a pair \((P)\) and \((D)\) of conic problems in standard form. For a primal feasible \( x \in \mathcal{F}_P \) and a dual feasible \( Y \in \mathcal{F}_D \), we have
\[
\langle B, Y \rangle - c^T x = \langle X, Y \rangle + \left( \sum_{i=1}^n x_i A_i, Y \right) - c^T x = \langle X, Y \rangle \geq 0 \quad (2.2)
\]
as \( X \in \mathcal{K} \) and \( Y \in \mathcal{K}^* \). By convention let us put that if \((P)\) or \((D)\) is infeasible then \( v_P = -\infty \) or \( v_D = +\infty \), respectively. From this convention and (2.2), the weak duality holds
\[
v_D \geq v_P. \quad (2.3)
\]

A primal feasible problem \((P)\) is said to be bounded if \( v_P \) is finite, otherwise \((P)\) is unbounded \((v_P = +\infty )\). Similarly, a dual feasible problem can be bounded or unbounded. Note that our feasible sets are closed.

Table 2.1 illustrates the possible states for \((P)\) and \((D)\), see[GG83]. The states \( x \) in Table 2.1 are excluded due to the weak duality (2.3). In
linear programming, if a primal or dual problem is feasible and bounded, then so is the other. Thus, the cases 2 and 3 do not occur in linear programming and we can easily construct examples for the other cases in LP. However, in conic programming, the cases 2 and 3 occur, see Example 2.7.

<table>
<thead>
<tr>
<th></th>
<th>Infeasible</th>
<th>Bounded</th>
<th>Unbounded</th>
</tr>
</thead>
<tbody>
<tr>
<td>(D)</td>
<td>1</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>(P)</td>
<td>3</td>
<td>5</td>
<td>x</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>x</td>
<td>x</td>
</tr>
</tbody>
</table>

Table 2.1: Duality states

**Example 2.7.** Consider the following pair of conic problems: \(^{1}\)

\[
\begin{align*}
\text{max} & \quad x_1 \\
\text{s.t.} & \quad X = \begin{pmatrix} x_1 & 1 & 0 \\ 1 & x_2 & 0 \\ 0 & 0 & -x_1 \end{pmatrix} \in \mathcal{CP} \\
\end{align*}
\]

\[
\begin{align*}
\text{min} & \quad 2Y_{12} \\
\text{s.t.} & \quad -Y_{11} + Y_{33} = 1 \\
& \quad -Y_{22} = 0 \\
& \quad Y \in \mathcal{COP}^3
\end{align*}
\]

It is known in [Dia62] that \(\mathcal{COP}^k = S^k_+ + N^k N^k\) and \(\mathcal{CP}^k = S^k_+ \cap N^k \cap N^k\) for \(k \leq 4\). If a copositive matrix has a zero on the diagonal, then the corresponding row and column have to be nonnegative. As \(Y_{22} = 0\), the component \(Y_{12}\) has to be nonnegative to be feasible. By taking \(Y_{33} = 1\) and all other components zero, the dual optimal solution is attained and the optimal value is zero, so the dual problem is bounded. The diagonal components of \(X\) have to nonnegative, so \(x_1 = 0\) and hence the corresponding row or column have to be zero, contradicting to \(X_{12} = X_{21} = 1\). Thus, the primal problem is infeasible.

Let us look at an overview of feasibility states. We consider the conic problem in linear semi-infinite programming (SIP) form. Note that

\(^{1}\)The author would like to thank P.J.C.Dickinson for providing Examples 2.7 and 2.11.
the representation of the feasible set \( \mathcal{F}_P \) can be given in different ways. However, in the thesis we assume that the system

\[
\sigma_P := \{ b(Y) \geq a(Y)^T x \text{ for all } Y \in \mathcal{K}^* \}
\]

is the linear representation of the feasible set \( \mathcal{F}_P \).

In SIP, the geometry of the feasible set \( \mathcal{F}_P \) is closely related to the following so-called moment cones:

- 1st moment cone \( \mathcal{N}_1 := \text{cone}(a(Y), Y \in \mathcal{K}^*) \)
- 2nd moment cone \( \mathcal{N}_2 := \text{cone}\left( \left\{ \begin{pmatrix} a(Y) \\ b(Y) \end{pmatrix}, Y \in \mathcal{K}^* \right\} \right) \)

Consider a finite set of indices \( I \subset \mathcal{K}^* \) with \( |I| < \infty \), and the corresponding finite subsystem

\[
\sigma_P(I) = \{ b(Y_i) \geq a(Y_i)^T x \text{ with } Y_i \in I \}.
\]

The finite subsystem results in an outer polyhedral approximation of \( \mathcal{F}_P \). In contrast to LP, an interesting situation occurs in SIP. Consider an infeasible problem \( (P) \). The linear system \( \sigma_P \) is called asymptotically inconsistent if every finite subsystem \( \sigma_P(I) \) of \( \sigma_P \) is feasible. Otherwise, \( \sigma_P \) is called strongly inconsistent.

Let us “illustrate” the asymptotically inconsistent case for a general SIP problem, see [GL98a, Example 4.1.]. Consider the system \( \sigma = \{ a(t)^T x \geq b(t) \text{ for all } t \in \mathcal{Z} \} \) with \( \mathcal{Z} = \mathbb{R} \cup \{s\} \) and \( s \notin \mathbb{R} \). Let \( a(t) = (- \exp(t), 1)^T \) and \( b(t) = (1 - t) \exp(t) \) for all \( t \in \mathbb{R} \), and let the remaining inequality for index \( s \) be \( x_2 \leq 0 \).

Figure 2.1: Asymptotically inconsistent case
Chapter 2. Linear conic programming

In fact, the linear equations given by $a(t)^T x = b(t)$ corresponding to indices $t \in \mathbb{R}$ are the tangent lines of the exponential function $x_2 = \exp x_1$. Thus, the problem is not feasible. But every finite subset of $\sigma$ is feasible, see Figure 2.1.

Consider a conic problem in SIP form $(\text{SIP}_P)$ and its feasible set $\mathcal{F}_P$. The following Table 2.2 from [GL98b] gives equivalent conditions for feasibility, asymptotical and strong inconsistency of problem $(\text{SIP}_P)$.

<table>
<thead>
<tr>
<th>Feasible</th>
<th>Asymptotically inconsistent</th>
<th>Strongly inconsistent</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\begin{pmatrix} 0_n \ -1 \end{pmatrix} \notin \text{cl}(\mathcal{N}_2)$</td>
<td>$\begin{pmatrix} 0_n \ -1 \end{pmatrix} \notin \text{cl}(\mathcal{N}_2) \setminus \mathcal{N}_2$</td>
<td>$\begin{pmatrix} 0_n \ -1 \end{pmatrix} \in \mathcal{N}_2$</td>
</tr>
</tbody>
</table>

Table 2.2: States of a linear system

In the following, we explain further each case in Table 2.2. We say that a linear inequality $b_0 \geq a_0^T x$ is a consequence of $\sigma_P$ if the inequality holds for all $x \in \mathcal{F}_P$. Utilizing a separation theorem for $\mathcal{K}$ and $(a_0^T, b_0)^T$, one can show the following:

**Lemma 2.8.** [GL98a] Let a feasible problem $(\text{SIP}_P)$ and the corresponding linear system $\sigma_P = \{b(Y) \geq a(Y)^T x \text{ for all } Y \in \mathcal{K}^*\}$ be given. The inequality $b_0 \geq a_0^T x$ is a consequence of $\sigma_P$ if and only if

$$\begin{pmatrix} a_0 \\ b_0 \end{pmatrix} \in \text{cl cone} \left\{ \begin{pmatrix} a(Y) \\ b(Y) \end{pmatrix}, Y \in \mathcal{K}^* \right\}.$$

The equivalent condition for feasibility in Table 2.2 is a direct result of the generalized Gale theorem. Using Lemma 2.8, let us demonstrate the generalized Gale theorem.

**Theorem 2.9** ([GL98a] Theorem of alternatives 1). Given a linear system $\sigma = \{b(Y) \geq a(Y)^T x \text{ for all } Y \in \mathcal{K}^*\}$. Then exactly one of the following alternatives holds:

- either $\mathcal{F} = \{x \mid b(Y) \geq a(Y)^T x \text{ for all } Y \in \mathcal{K}^*\} \neq \emptyset$
- or $\begin{pmatrix} 0_n \\ -1 \end{pmatrix} \in \text{cl cone} \left\{ \begin{pmatrix} a(Y) \\ b(Y) \end{pmatrix}, Y \in \mathcal{K}^* \right\}$. 

14
2.2. Duality theory of conic problems

Proof. First assume \( F = \{ x \in \mathbb{R}^n \mid b(Y) \geq a(Y)^T x \text{ for all } Y \in \mathcal{K}^* \} = \emptyset \). Consider the following system in \( \mathbb{R}^{n+1} \):

\[
\begin{cases}
b(Y)x_{n+1} - a(Y)^T x \geq 0 \text{ for all } Y \in \mathcal{K}^* \nn_{n+1} > 0
\end{cases}
\]

By construction, the above system is also infeasible. However, the subsystem \( \{ b(Y)x_{n+1} - a(Y)^T x \geq 0 \text{ for all } Y \in \mathcal{K}^* \} \) is homogeneous and thus feasible. Moreover, every feasible solution of this subsystem must satisfy \( x_{n+1} \leq 0 \) as the linear system \((*)\) is infeasible. Thus, by Lemma 2.8, we have the following for system \((*)\):

\[
\begin{pmatrix} 0_n \\ -1 \\ 0 \end{pmatrix} \in \text{cl cone } \left\{ \begin{pmatrix} a(Y) \\ b(Y) \\ 0 \end{pmatrix}, Y \in \mathcal{K}^* \right\} \text{ and consequently } \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \in \text{cl } \mathcal{N}_2.
\]

Now, let \( (0_n^T, -1)^T \in \text{cl } \mathcal{N}_2 \). We need to show that \( F \) is infeasible. There exists a sequence \( (a(Y_k)^T, b(Y_k))^T \in \mathcal{N}_2 \) with \( Y_k \in \mathcal{K}^* \) such that

\[
\lim_{k \to \infty} \begin{pmatrix} a(Y_k) \\ b(Y_k) \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.
\]

Furthermore, every feasible solution \( x \in F \) satisfies \( b(Y_k) \geq a(Y_k)^T x \) for all \( k \). Then in the limit, \( -1 \geq 0^T x \) has to hold for every feasible \( x \in F \). Thus, \( F \) is empty. \( \square \)

The second case of Table 2.2 does not occur in linear programming, as one can show that a linear image of a polyhedral cone is closed, for a proof see e.g. [Pat07]. In general, a linear image of a proper cone does not have to be closed (for a counterexample, see e.g. [Roc70, p.73] ) and asymptotically inconsistent cases do occur in conic programming.

As we see from Table 2.2, asymptotically inconsistent cases arise only when the cone \( \mathcal{N}_2 \) is not closed. The second moment cone \( \mathcal{N}_2 \) is a linear image of the proper cone \( \mathcal{K}^* \). A thorough study regarding the closedness of linear images of a proper cone is done in [Pat07]. This article gives necessary and sufficient conditions for the closedness of the linear image.
of so-called “nice” cones, see Definition 2.26. A detailed examination of
the asymptotically inconsistent case for the semidefinite cone is studied
in [LMT14]. Let us give an example of a second moment cone which is
not closed.

Example 2.10. [Pat07, Example 4.3] Let us take \( \mathcal{K} = \mathcal{K}^* = \mathcal{S\mathcal{O}\mathcal{C}^3} \) with
\( a = (1, 1, 0) \) and \( b = (0, 0, 1) \). We defined the second moment cone as
\[
\mathcal{N}_2 = \text{cone} \left( \left\{ \begin{pmatrix} a(Y) \\ b(Y) \end{pmatrix}, Y \in \mathcal{K}^* \right\} \right).
\]
We show that \( \mathcal{N}_2 \) is not closed by checking that \( (0, -1)^T \in \text{cl} \mathcal{N}_2 \setminus \mathcal{N}_2 \).
First, it is easy to check that the system
\[
\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \text{ and } y_1 \geq \sqrt{y_2^2 + y_3^2}
\]
has no solution. So
\[
\begin{pmatrix} 0 \\ -1 \end{pmatrix} \notin \mathcal{N}_2 = \text{cone} \left( \left\{ \begin{pmatrix} a(Y) \\ b(Y) \end{pmatrix}, Y \in \mathcal{K}^* \right\} \right).
\]
Consider the sequence \( v_k = (\mu_k, -\mu_k + \varepsilon_k, -1)^T \) with \( \mu_k \geq \frac{\varepsilon_k}{2} + \frac{1}{2\varepsilon_k} \) and
\( \varepsilon_k > 0 \). By construction, it is straightforward to check that \( v_k \in \mathcal{S\mathcal{O}\mathcal{C}^3} \) for all \( k \). We have
\[
\begin{pmatrix} a(v_k) \\ b(v_k) \end{pmatrix} = \begin{pmatrix} \varepsilon_k \\ -1 \end{pmatrix} \to \begin{pmatrix} 0 \\ -1 \end{pmatrix} \text{ as } \varepsilon_k \to 0.
\]
Thus, \( (0, -1)^T \in \text{cl} \mathcal{N}_2 \) and we have shown that \( \mathcal{N}_2 \) is not closed.

In applications, one is mostly concerned with the case 5 of Table 2.1
where both primal and dual problems are feasible and bounded. Recall
that the difference \( \delta := v_D - v_P \) is called the duality gap for a pair
of primal and dual problems. When the duality gap is zero, we say that
the strong duality holds. One needs some extra conditions to ensure
2.2. Duality theory of conic problems

a zero duality gap. The following Table 2.3 from [GL98b] summarizes strong duality and its relation to the 1st moment cone $\mathcal{N}_1$. Here we assume that $\mathcal{F}_P \neq \emptyset$ and denote $C_1 := \text{cl} \left( \{(c) \times \mathbb{R} \} \cap \mathcal{N}_2 \right)$ and $C_2 := \{(c) \times \mathbb{R} \} \cap (\text{cl} \mathcal{N}_2)$. Note that for the linear map $a(Y) = A^T Y$ in (2.1) and under Assumption 2.6, the corresponding 1st moment cone $\mathcal{N}_1$ is full-dimensional.

<table>
<thead>
<tr>
<th>Duality gap based on $c$</th>
<th>( \delta = 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c \in \text{ri} \mathcal{N}_1$</td>
<td>( \delta = 0 )</td>
</tr>
<tr>
<td>$c \in \text{rbd} \mathcal{N}_1$</td>
<td>( \delta = 0 ) \iff ( C_1 = C_2 )</td>
</tr>
<tr>
<td>$c \notin \text{cl} \mathcal{N}_1$</td>
<td>( \delta = 0 )</td>
</tr>
</tbody>
</table>

Table 2.3: Duality gap states under the assumption $\mathcal{F}_P \neq \emptyset$

By definition of the 1st moment cone $\mathcal{N}_1$, it is clear that the dual problem $(SIP_D)$ is feasible if and only if $c \in \mathcal{N}_1$. In other words, the condition $c \notin \text{cl} \mathcal{N}_1$ implies that the dual problem is infeasible and the corresponding feasible problem $(SIP_P)$ is unbounded.

In both cases $c \in \text{int} \mathcal{N}_1$ and $c \notin \text{cl} \mathcal{N}_1$, the duality gap is zero. If $c \in \text{bd} \mathcal{N}_1$ and the condition $C_1 = C_2$ does not hold, any positive duality gap can arise, see for example [SS00, Example 4.1.2] for SDP or the following example for COP.

Example 2.11. Let $a > 0$ and consider the problems:

$$\begin{align*}
\text{max} \quad & -x_1 \\
\text{s.t.} \quad & \begin{pmatrix} x_1 + a & 0 & 0 \\ 0 & x_2 & x_1 \\ 0 & x_1 & 0 \end{pmatrix} \in \mathcal{COP}^3
\end{align*}$$

The corresponding feasible sets for these problem are

$$\mathcal{F}_{\text{COP}} = \{(x_1, x_2) | x_1, x_2 \geq 0\},$$

$$\mathcal{F}_{\text{CP}} = \{Y \in \mathcal{S}^3 | Y_{11} = 1, Y_{12} = Y_{22} = Y_{13} = 0, Y_{33} \geq 0, \sqrt{Y_{33}} \geq Y_{13} \geq 0\}.$$ 

Therefore, both primal and dual problems are feasible and the duality gap is $a$. Let us illustrate that this is the case $c \in \text{bd} \mathcal{N}_1$ of Table 2.3.
Rewriting the primal copositive problem, we obtain
\[
\begin{align*}
\max & \quad -x_1 \\
\text{s.t.} & \quad (z_1^2 + 2z_2z_3)x_1 + z_2^2x_2 + z_3^2a \geq 0 \text{ for all } z_1, z_2, z_3 \in \mathbb{R}_+.
\end{align*}
\]
Then we have
\[
\mathcal{N}_1 = -\left\{ \text{cone} \left( \frac{z_1^2 + 2z_2z_3}{z_2^2} \right) \mid z_1, z_2, z_3 \in \mathbb{R}_+ \right\}.
\]
Since \( z_2^2 \geq 0 \), the coefficient vector \( c = (-1, 0)^T \) of objective function is an element of \( \text{bd} \mathcal{N}_1 \).

Now we show that \( C_1 \neq C_2 \) holds (see Table 2.3). It is straightforward to check that \( C_1 = \{ -(1, 0, a)^T \} \) in this case.

Let us choose \( z_1 = 0 \), \( z_2(\beta) = \sin \beta \), \( z_3(\beta) = 1/(2\beta) \) for \( \beta > 0 \) small enough. By construction, with the choice of \( z_1, z_2, z_3 \), it is clear that
\[
-\begin{pmatrix}
\frac{\sin \beta}{\beta} \\
\sin^2 \beta \\
0
\end{pmatrix} \in \mathcal{N}_2 = \left\{ \text{cone} \begin{pmatrix}
-z_1^2 - 2z_2z_3 \\
-z_2^2 \\
z_1^2a
\end{pmatrix} \mid z_1, z_2, z_3 \in \mathbb{R}_+ \right\}
\]
and the sequence converges to \( -(1, 0, 0)^T \in \text{cl} \mathcal{N}_2 \) as \( \beta \to 0 \). Therefore, \( -(1, 0, 0)^T \in C_2 \) and \( C_1 \neq C_2 \).

Before turning to the condition \( c \in \text{int} \mathcal{N}_1 \), let us introduce Slater’s condition.

**Definition 2.12.** [Slater’s condition] We say that Slater’s condition holds for (\( \tilde{P} \)), if there exists a feasible solution \( x \) such that
\[
X := B - Ax \in \text{int} \mathcal{K}.
\]
Analogously, we say that Slater’s condition holds for (\( \tilde{D} \)), if there exists a feasible solution \( Y \), i.e., \( A^T Y = c \), such that \( Y \in \text{int} \mathcal{K}^* \).

In linear continuous SIP, we say that the feasible set \( \mathcal{F}_{SIP} \) of (SIP\(_P\)) satisfies the Slater condition (see e.g. [GL98a]) if there exists \( x_0 \) such that
\[
b(Y) - a(Y)^Tx_0 > 0 \quad \text{for all } Y \in \mathcal{Z} = \mathcal{K}_0^*.
\]
2.2. Duality theory of conic problems

It is not difficult to show (see [ADS13]) that for the SIP formulation of conic programming with $a(Y)$ and $b(Y)$ as in (2.1) this is equivalent to the primal Slater condition in Definition 2.12.

The primal Slater condition enforces the closure of the 2nd moment cone $\mathcal{N}_2$.

**Proposition 2.13.** [GG83, Theorem 7] Suppose that problem $(SIP_P)$ satisfies the primal Slater condition. Then the 2nd moment cone $\mathcal{N}_2$ is closed.

**Proof.** Consider

$$\mathcal{N}_0 := \text{conv} \left( \left\{ \begin{pmatrix} a(Y) \\ b(Y) \end{pmatrix} \mid Y \in \mathcal{K}^*, \|Y\| = 1 \right\} \right).$$

Note that $\text{cone}(\mathcal{N}_0) = \mathcal{N}_2$ and the set $\mathcal{N}_0$ is compact as $a(Y)$ and $b(Y)$ are continuous. Take a vector $z \in \text{cl}\mathcal{N}_2$, then there exist sequences $\alpha_i \geq 0$ and $\{z^i\}$ with $z^i \in \mathcal{N}_0$ such that $z = \lim_{i \to \infty} \alpha_i z^i$. There exists a subsequence of $\{z^i\}$ which converges to some $\tilde{z} \in \mathcal{N}_0$ as the set $\mathcal{N}_0$ is compact. If the corresponding subsequence $\{\alpha_i\}$ is bounded, we obtain $z = \lim_{i \to \infty} \alpha_i z^i = \alpha \tilde{z}$ with $\alpha \geq 0$ and $\tilde{z} \in \mathcal{N}_0$. Thus, in this case, $z \in \mathcal{N}_2$.

It remains to consider the case when $\{\alpha_i\}$ is unbounded. In this case, we can assume $\alpha_i > 0$ and $\lim_{i \to \infty} 1/\alpha_i = 0$. Therefore,

$$\tilde{z} = \lim_{i \to \infty} z^i = \lim_{i \to \infty} \frac{1}{\alpha_i} \alpha_i z^i = \lim_{i \to \infty} \lim_{i \to \infty} \alpha_i z^i = 0 z = 0.$$

This means that $0 \in \mathcal{N}_0$ and so there exist $Y_1, \ldots, Y_q \in \mathcal{K}^*$ with $q < \infty$ and $\beta_1, \ldots, \beta_q \geq 0$ with $\sum_{i=1}^{q} \beta_i = 1$ such that

$$0 = \sum_{i=1}^{q} \beta_i b(Y_i) \quad \text{and} \quad 0 = \sum_{i=1}^{q} \beta_i a(Y_i).$$

Combining the above two equations, the following holds for any $x \in \mathbb{R}^n$

$$0 = \sum_{i=1}^{q} \beta_i \left( b(Y_i) - a(Y_i)^T x \right).$$
Chapter 2. Linear conic programming

This is a contradiction to the primal Slater condition. Therefore, the sequence $\{\alpha_i\}$ cannot be unbounded. 

It is shown in [ADS13, Lemma 3.1] that the condition $c \in \text{int } \mathcal{N}_1$ is equivalent to the dual Slater condition, i.e.

$$c \in \text{int } \mathcal{N}_1 \iff \text{there exists a feasible } Y \in \text{int } \mathcal{K}^*.$$  \hspace{1cm} (2.4)

Furthermore, this implies that the optimal solution set of $(SIP_P)$ is compact, for the proof see e.g. [GL88, Theorem 2.1] or [ADS13]. More precisely for feasible $(SIP_P)$ (see [ADS13, Theorem 3.1]), the following is known

$$c \in \text{int } \mathcal{N}_1 \iff \text{the primal optimal solution set is nonempty and compact.}$$  \hspace{1cm} (2.5)

Similarly, it is known that if the primal problem satisfies Slater’s condition, then the dual optimal solution is attained, see e.g. [ADS13, Lemma 3.1]. By combining the above results, we have the following standard result regarding solvability.

**Corollary 2.14.** Consider a pair of conic problems $(P)$ and $(D)$. Under the primal and dual Slater conditions, the primal and dual optimal solutions $\overline{X}$ and $\overline{Y}$ are attained and the duality gap is zero.

### 2.3 Optimality conditions

Now let us turn to optimality conditions. Consider a conic problem $(P)$. Given $X \in \mathcal{K}$, we denote the minimal face of the cone $\mathcal{K}$ containing $X$ by $\text{face}(X, \mathcal{K})$, the minimal face of $\mathcal{K}^*$ containing $Y \in \mathcal{K}^*$ by $\text{face}(Y, \mathcal{K}^*)$, and we define

$$J(X) := \text{face}(X, \mathcal{K}) \quad \text{and} \quad G(Y) := \text{face}(Y, \mathcal{K}^*).$$  \hspace{1cm} (2.6)

Clearly, we have $X \in \text{ri } J(X)$ for each $X \in \mathcal{K}$. For a face $F$ of $\mathcal{K}$, we define the complementary face as

$$F^\Delta := \{Q \in \mathcal{K}^* \mid \langle Q, S \rangle = 0 \text{ for all } S \in F\}.$$
2.3. Optimality conditions

Clearly, \( F^\triangle \subseteq \mathcal{K}^* \) is a closed convex cone. Moreover, it is not difficult to see that if \( X \in \text{ri} F \), then \( F^\triangle = \{ Q \in \mathcal{K}^* \mid \langle Q, X \rangle = 0 \} \). Thus, the complementary face of \( J(X) \) is equivalently described as

\[
J^\triangle(X) = \{ Y \in \mathcal{K}^* \mid \langle Y, X \rangle = 0 \}.
\]

(2.7)

The complementary face \( G^\triangle(Y) \) of \( G(Y) \) is denoted analogously.

Primal and dual feasible solutions \( X \in \mathcal{F}_P \), \( Y \in \mathcal{F}_D \) are called complementary if \( \langle X, Y \rangle = 0 \), i.e., \( Y \in J^\triangle(X) \). By weak duality, \( X \) and \( Y \) must then be optimal solutions of \((P)\) and \((D)\).

In SIP, one considers the so-called set of active indices which corresponds to the complementary face in our description. The complementary face \( J^\triangle(X) \) of \( X \in \mathcal{K} \) and the corresponding moment cone \( \mathcal{M}(X) \) play a crucial role in optimality conditions. We define

\[
\mathcal{M}(X) := \text{cone}\{ a(Y_j) \mid Y_j \in J^\triangle(X) \},
\]

(2.8)

where \( a(Y) \) is given by (2.1).

As usual, we say with respect to the SIP formulation that \( \overline{x} \in \mathcal{F}_P \) or \( \overline{X} \in \mathcal{F}_P \) satisfies the Karush-Kuhn-Tucker condition (KKT), if there exist \( k \in \mathbb{N} \), \( Y_j \in J^\triangle(\overline{X}) \) and multipliers \( y_j > 0 \), \( j = 1, \ldots, k \), such that

\[
c = \sum_{j=1}^{k} y_j a(Y_j), \quad \text{or equivalently} \quad c \in \mathcal{M}(\overline{X}).
\]

(2.9)

It is well known that the KKT condition is sufficient for optimality in conic programming. Let us shortly discuss the sufficiency. Suppose that \( \overline{X} \in \mathcal{F}_P \) satisfies the KKT condition. From (2.9), we have that the matrix

\[
\overline{Y} := \sum_{j=1}^{k} y_j Y_j \in \mathcal{F}_D
\]

is a dual feasible solution. As \( \langle \overline{X}, \overline{Y} \rangle = 0 \), the primal and dual feasible solutions \( \overline{X}, \overline{Y} \) are complementary and so \( \overline{X} \) is a maximizer of \((P)\) and \( \overline{Y} \) is a minimizer of \((D)\) by weak duality. Thus, the KKT condition is sufficient for optimality of \( \overline{X} \in \mathcal{F}_P \) in conic programming.
Next, let us consider necessary conditions for optimality. As in linear programming, an extended Farkas lemma can be applied to show the necessity of the KKT condition under a constraint qualification. The following extended Farkas theorem follows directly from Lemma 2.8 by considering the convex cone $J^\Delta(\overline{X})$ instead of $\mathcal{K}^*$ and by choosing $a_0 := c$.

**Theorem 2.15 (Theorem of alternatives 2).** Consider a conic problem $(P)$ and a feasible $\overline{X}$. For every $c \in \mathbb{R}^n$, exactly one of the following alternatives is true:

- $c \in \text{cl} \mathcal{M}(\overline{X})$
- there is a solution $v \in \mathbb{R}^n$ such that \[
\begin{align*}
&c^T v > 0, \quad \text{and} \\
&a(Y)^T v \leq 0 \quad \text{for all } Y \in J^\Delta(\overline{X}).
\end{align*}
\]

Suppose that $\overline{X} \in \mathcal{F}_P$ is an optimal solution of $(P)$. This implies that there is no feasible ascent direction. Thus, considering the SIP formulation of $(P)$, the following linear system is infeasible.

\[
\begin{align*}
&b(Y) \geq a(Y)^T (\overline{x} + v) \quad \text{for all } Y \in J^\Delta(\overline{X}) \\
&c^T (\overline{x} + v) > c^T \overline{x}.
\end{align*}
\]

The above statement is equivalent to the following: There exists no $v \in \mathbb{R}^n$ such that

\[
\begin{align*}
&a(Y)^T v \leq 0 \quad \text{for all } Y \in J^\Delta(\overline{X}) \\
&c^T v > 0.
\end{align*}
\]

Thus, applying the extended Farkas theorem (Theorem 2.15) to statement (2.10), if $\overline{X}$ is an optimal solution, then $c \in \text{cl} \mathcal{M}(\overline{X})$ holds. So we need a constraint qualification to impose closedness of the cone $\mathcal{M}(\overline{X})$.

Consider $(SIP_P)$ satisfying the Slater condition with feasible set $\mathcal{F}_P := \{x \mid b(Y) \geq a(Y)^T x \text{ for all } Y \in \mathcal{K}^*\}$. Instead of the index set $\mathcal{Z} := \mathcal{K}^*$, let us take $\mathcal{Z} := J^\Delta(X)$. Similarly, as shown in Proposition 2.13, we have that the corresponding moment cone $\mathcal{M}(\overline{X})$ is closed under the primal Slater condition. Therefore, we derive the standard result that under the
primal Slater condition, the KKT condition is necessary for optimality of \( \overline{X} \in \mathcal{F}_P \).

**Theorem 2.16.** Consider conic problem \((P)\) in SIP form satisfying the primal Slater condition. Let \( \overline{X} \) be an optimal solution of \((P)\), then KKT condition \((2.9)\) is satisfied at \( \overline{X} \).

### 2.4 Properties related to cones

In this section, we consider conic problems in self-dual form and introduce some notions which are generalizations of terms well-known in linear programming.

**Definition 2.17.** The extreme points of \( \mathcal{F}_P \) (resp. \( \mathcal{F}_D \)) are called primal (resp. dual) basic feasible solutions.

The following characterization of basic solutions is given in [PT01, Theorem 1]:

**Lemma 2.18.** Let \( X \) be feasible for a conic problem \((P_0)\) in self dual form. Then \( X \) is a basic feasible solution if and only if

\[
\text{span}(J(X)) \cap \mathcal{L} = \{0\}.
\]  

(2.11)

A similar condition for the complementary face in the corresponding dual problem leads to the concept of (primal) nondegeneracy:

**Definition 2.19.** A primal feasible solution \( X \) is called nondegenerate, if

\[
\text{span}(J^\triangle(X)) \cap \mathcal{L}^\perp = \{0\}.
\]  

(2.12)

Nondegeneracy of a dual feasible solution \( Y \) is defined analogously.

As in linear programming, for general conic programming it is shown in [Pat00] that primal nondegeneracy implies uniqueness of the dual optimal solution if it exists. If the dual problem is feasible, then the primal Slater condition implies the existence of a dual optimal solution, see e.g.
[ADS13, Theorem 3.1]. In the following, we demonstrate that nondegeneracy implies the Slater condition. Thus, if the primal problem has a nondegenerate optimal solution and the dual problem is feasible, then there exists a unique dual optimal solution.

Now let us show a lemma before proving the statement that the existence of a nondegenerate solution implies the Slater condition.

**Lemma 2.20.** Let $X$ be a nondegenerate feasible solution of $(P)$, i.e., $X \in \mathcal{F}_P$ and $\mathcal{L}^\perp \cap \text{span}(J^\wedge(X)) = \{0\}$. Then there exists $L \in \mathcal{L}$ such that
\[
\langle S, L \rangle > 0 \quad \text{for all } S \in J^\wedge(X) \cap \mathcal{B}_1,
\] (2.13)
where $\mathcal{B}_1 := \{S \in \mathcal{S}^k \mid \|S\| = 1\}$ is the unit sphere.

**Proof.** The statement is shown using another version of a theorem of the alternative (see [GL98a, p.68]): Let $\emptyset \neq I, J$ be (possibly infinite) index sets, and let $b^i, a^j \in \mathbb{R}^m$ for $i \in I, j \in J$. Suppose the set $\text{conv}\{a^j \mid j \in J\}$ is closed. Then precisely one of the following alternatives is true:

(I) \[
\begin{cases}
\langle a^j, L \rangle > 0 & \text{for all } j \in J \\
\langle b^i, L \rangle = 0 & \text{for all } i \in I
\end{cases}
\]

has a solution $L$

(II) $0 \in \text{conv}\{a^j \mid j \in J\} + \text{span}\{b^i \mid i \in I\}$

For our purposes, let $J := J^\wedge(X) \cap \mathcal{B}_1$, and let $a^S := S$ for $S \in J$. Then the set $\text{conv}\{S \mid S \in J\}$ is compact and $0 \notin \text{conv}\{S \mid S \in J\}$. Let further $\{b^i \mid i \in I\}$ be a basis of $\mathcal{L}^\perp$. Then obviously the nondegeneracy assumption for $X$ implies $\mathcal{L}^\perp \cap \text{conv}(J) = \emptyset$, and thus
\[
0 \notin \text{span}\{b^i \mid i \in I\} + \text{conv}\{S \mid S \in J\}.
\]

Therefore, system (I) must be true. Hence, there exist some $L$ such that $\langle S, L \rangle > 0$ for all $S \in J$ and $\langle b^i, L \rangle = 0$ for all $i \in I$, i.e., $L \in \mathcal{L}$, as desired. 

**Proposition 2.21.** Let $X$ be a nondegenerate feasible solution of $(P_0)$. Then Slater’s condition holds for $(P_0)$. An analogous result is true for the problem $(D_0)$. 

24
2.4. Properties related to cones

**Proof.** From Lemma 2.20, the primal nondegeneracy condition

\[ \mathcal{L}^\perp \cap \text{span}(J^\Delta(X)) = \{0\} \]

implies the existence of a vector \( L \in \mathcal{L} \) such that

\[ \langle S, L \rangle > 0 \quad \text{for all } S \in J^\Delta(X) \cap \mathcal{B}_1 \quad (2.14) \]

with \( \mathcal{B}_1 := \{ S \in \mathcal{S}^k \mid \|S\| = 1 \} \). It is sufficient to show that for some \( \alpha > 0 \) small enough we have

\[ (X + \alpha L) \in (B + \mathcal{L}) \cap \text{int } \mathcal{K}. \]

Indeed, \( X \in (B + \mathcal{L}) \) implies \((X + \alpha L) \in (B + \mathcal{L})\), and to prove \((X + \alpha L) \in \text{int } \mathcal{K}\), one needs to show that

\[ \langle X + \alpha L, S \rangle > 0 \quad \text{for all } S \in \mathcal{K}^* \cap \mathcal{B}_1 \]

is valid for some \( \alpha > 0 \).

It follows from (2.14) and the compactness of the set \( J^\Delta(X) \cap \mathcal{B}_1 \) that there exists some \( \varepsilon > 0 \) such that \( \langle L, S \rangle \geq 2\varepsilon > 0 \) for all \( S \in J^\Delta(X) \cap \mathcal{B}_1 \). By continuity of the linear function \( \langle L, \cdot \rangle \), there exists some \( \delta > 0 \) such that

\[ \langle L, S \rangle \geq \varepsilon \quad \text{for all } S \in \mathcal{J}^\Delta_\delta(X) \cap \mathcal{B}_1, \quad (2.15) \]

where \( \mathcal{J}^\Delta_\delta(X) := \{ S \in \mathcal{K}^* \mid \|S - \overline{S}\| < \delta \text{ for some } \overline{S} \in \mathcal{J}^\Delta(X) \} \). Since \( X \in \mathcal{K} \), we have \( \langle X, S \rangle \geq 0 \) for all \( S \in \mathcal{K}^* \), and by the definition of \( \mathcal{J}^\Delta(X) \) in (2.7) we have that \( \langle X, S \rangle > 0 \) for all \( S \in (\mathcal{K}^* \setminus \mathcal{J}^\Delta_\delta(X)) \cap \mathcal{B}_1 \). By compactness of this set, there exists some \( T \) such that

\[ \langle X, S \rangle \geq T > 0 \quad \text{for all } S \in (\mathcal{K}^* \setminus \mathcal{J}^\Delta_\delta(X)) \cap \mathcal{B}_1. \quad (2.16) \]

Let \( m := \min \{ \langle L, S \rangle \mid S \in (\mathcal{K}^* \setminus \mathcal{J}^\Delta_\delta(X)) \cap \mathcal{B}_1 \} \). We claim that \((X + \alpha L) \in \text{int } \mathcal{K}\) for all \( 0 < \alpha < \frac{T}{|m|} \). We have the following two cases:

- \( S \in (\mathcal{K}^* \setminus \mathcal{J}^\Delta_\delta(X)) \cap \mathcal{B}_1 \): then
  \[ \langle X + \alpha L, S \rangle = \langle X, S \rangle + \langle \alpha L, S \rangle \geq T + \alpha m > 0. \]
Chapter 2. Linear conic programming

- \( S \in J_\delta^\Delta(X) \cap \mathcal{B}_1 \): using \( \langle X, S \rangle \geq 0 \) and (2.15), we have
  \[
  \langle X + \alpha L, S \rangle = \langle X, S \rangle + \langle \alpha L, S \rangle \geq \alpha \varepsilon > 0.
  \]
  By combining these two cases, we have \( (X + \alpha L) \in (B + L) \cap \text{int} \mathcal{K} \). □

Recall that the optimal solutions \( \bar{X} \) of \( (P_0) \) and \( \bar{Y} \) of \( (D_0) \) are called complementary, if \( \bar{Y} \in J \triangle (\bar{X}) \).

**Definition 2.22.** The solutions \( \bar{X} \) and \( \bar{Y} \) are called strictly complementary, if

\[
\bar{X} \in \text{ri} J(\bar{X}) \quad \text{and} \quad \bar{Y} \in \text{ri} J^\Delta(\bar{X}). \tag{2.17}
\]

**Remark 2.23.** By considering the dual problem, strict complementarity can similarly be defined as

\[
\bar{Y} \in \text{ri} G(\bar{Y}) \quad \text{and} \quad \bar{X} \in \text{ri} G^\Delta(\bar{Y}). \tag{2.18}
\]

Neither of the conditions (2.17) or (2.18) implies the other unless \( \mathcal{K} \) or \( \mathcal{K}^* \) are facially exposed, as noted in [Pat00, Remark 3.3.2]. For an illustrative example for these “asymmetric” definitions of strict complementarity we refer to [DJ14, Example 1]. Throughout the thesis, strict complementarity condition refers to the primal cone \( \mathcal{K} \) unless explicitly stated.

As we mentioned earlier not all cones appearing in optimization are facially exposed: it is well known that the cone of semidefinite matrices is facially exposed, but the cone of copositive matrices is not, see [Dic11, Theorem 8.22].

**Remark 2.24.** In [PT01], in order to describe the set of problem instances which have strict complementary solutions, a slightly different definition of strict complementarity is given: feasible solutions \( \bar{X} \) and \( \bar{Y} \) are called strictly complementary if

\[
\bar{X} \in \text{ri} F \quad \text{and} \quad \bar{Y} \in \text{ri} F^\Delta \quad \text{holds for some face } F \text{ of } \mathcal{K}. \tag{2.19}
\]

It is clear that (2.17) implies (2.19). Conversely, let (2.19) be satisfied. We always have \( \bar{X} \in \text{ri} J(\bar{X}) \). So \( \bar{X} \in \text{ri} F \) implies \( F^\Delta = J^\Delta(\bar{X}) \) by (2.7). Therefore, (2.17) and (2.19) are equivalent.
2.4. Properties related to cones

In [Pat00], strict complementarity for $X, Y$ is defined by

$$J^\Delta(X) = G(Y). \quad (2.20)$$

It can be shown that (2.20) and (2.17) are equivalent, see the proof of [PT01, Theorem 2].

The following lemma collects some relations between nondegeneracy, strict complementarity, basic solutions and uniqueness.

**Lemma 2.25.** (see [Pat00], [PT01, Theorem 2]) Let $X$ be an optimal solution of $(P_0)$. Then the following hold.

(a) If $X$ is a unique solution, then $X$ is a basic solution.

(b) If $X$ is nondegenerate, then any complementary solution $Y$ of $(D_0)$ must be basic. Moreover, if there is a complementary solution $Y$, it must be unique.

(c) Suppose that $Y$ is a dual feasible solution and $X$ and $Y$ are strictly complementary. Then $Y$ is basic if and only if $X$ is nondegenerate.

Next, we define so-called nice cones and tangent spaces, see [Pat13, Pat07].

**Definition 2.26.** A closed convex cone $K$ is called nice if

$$K^* + F^\perp$$

is closed for any face $F$ of $K$.

For example, it can be shown that polyhedral cones and the positive semidefinite cone are nice, but the copositive cone is not (see [Pat13, Theorem 3], [Pat07]). Every nice cones are facially exposed, see [Pat13]. But it is shown in [Ros14] that the reverse implication is not true.

As usual, the tangent space of a convex closed cone is defined (see e.g. [Pat00]) as follows:

**Definition 2.27.** Let $K \subseteq S^k$ be a closed convex cone. The tangent space at $X \in K$ is

$$\text{tan}(X, K) := \{Z \in S^k \mid \text{dist}(X \pm tZ, K) = o(t)\}. \quad (2.21)$$
For nice cones, it has been shown in [Pat00, Lemma 3.2.1 or Remark 3.3.4] that tangent spaces and complementary faces are closely related:

\textbf{Proposition 2.28.} Let $\mathcal{K}$ be a nice cone. Then for $\overline{X} \in \mathcal{K}$

\[ [J^\triangle(\overline{X})]^\perp = \tan(\overline{X}, \mathcal{K}). \] (2.22)

Let us specify these notions for linear programming and semidefinite programming.

\textbf{Linear programming:}

\begin{align*}
(L_P) \max & \quad c^T x \\
\text{s.t.} & \quad X := B - Ax \in \mathbb{R}^m_+ \quad \quad (L_D) \min & \quad B^T Y \\
\text{s.t.} & \quad A^T Y = c
\end{align*}

where $A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^n$, $B \in \mathbb{R}^m$. Suppose the optimal solutions $\overline{X} := B - A\overline{x}$ and $\overline{Y}$ have exactly $r$ resp. $s$ nonzero components. After permuting components, we can assume without loss of generality that

\[ \overline{X}_{r+1} = \cdots = \overline{X}_m = 0, \quad \overline{X}_1, \ldots, \overline{X}_r > 0 \]

and

\[ \overline{Y}_1 = \cdots = \overline{Y}_{m-s} = 0, \quad \overline{Y}_{m-s+1}, \ldots, \overline{Y}_m > 0. \]

Complementarity with $\overline{X}_i \overline{Y}_i = 0$, $i = 1, \ldots m$, implies $r + s \leq m$. Faces and complementary faces can be obtained directly:

\[ J(\overline{X}) := \text{face}(\overline{X}, \mathbb{R}^m_+) = \{ X \in \mathbb{R}^m_+ \mid X_i = 0 \text{ for all } i = (r + 1), \ldots, m \} \]

and

\[ J^\triangle(\overline{X}) = \{ Y \in \mathbb{R}^m_+ \mid Y_i = 0 \text{ for all } i = 1, \ldots, r \}. \]

Furthermore, it is straightforward to check the following

- If $r + s = m$, then $\overline{X}$ and $\overline{Y}$ are strictly complementary.
2.4. Properties related to cones

- Let \( T \in \mathbb{R}^{(m-r) \times n} \) be the matrix consisting of the \((m-r)\) rows of \( A \) corresponding to zero components of \( X \). It is not difficult to show that the nondegeneracy condition (2.12) for \( X \) is equivalent to the standard definition of a nondegenerate feasible point \( X \) in LP, that is the rows of \( T \) are linearly independent.

**Semidefinite programming:** Let us consider an SDP satisfying the primal and dual Slater condition.

\[
\begin{align*}
\text{(SDP}_P\text{)} & \quad \max \ c^T x \quad \text{s.t.} \quad X := B - \sum_{i=1}^{n} x_i A_i \in S_k^+ \\
\text{(SDP}_D\text{)} & \quad \min \ \langle B, Y \rangle \quad \text{s.t.} \quad \langle A_i, Y \rangle = c_i \quad Y \in S_k^+
\end{align*}
\]

Consider primal and dual solutions \( X \in F_P \) and \( Y \in F_D \) with zero duality gap. As \( X, Y \) are positive semidefinite, there exist unique positive semidefinite square root matrices \( X^{\frac{1}{2}}, Y^{\frac{1}{2}} \). Then using properties of \( \text{tr}(X) := \sum_{i=1}^{k} X_{ii} \), we derive the following:

\[
0 = \langle X, Y \rangle = \text{tr}(XY) = \text{tr}(Y^{\frac{1}{2}} X^{\frac{1}{2}} Y^{\frac{1}{2}} X^{\frac{1}{2}} Y^{\frac{1}{2}}) = \|Y^{\frac{1}{2}} X^{\frac{1}{2}}\|^2.
\]

Thus, \( Y^{\frac{1}{2}} X^{\frac{1}{2}} = 0 \) and so \( YX = 0 \). So the complementarity condition \( \langle X, Y \rangle = 0 \) in SDP implies that \( XY = 0 \) and \( YX = 0 \). As \( X, Y \) commute, they share a common system of eigenvectors. Using this observation, one can obtain the following results

**Theorem 2.29.** \([AHO97, \text{Lemma 3}]\) Consider a pair of primal and dual problems \((\text{SDP}_P\text{)}\) and \((\text{SDP}_D\text{)}\). The feasible solutions \( X \in F_P \) and \( Y \in F_D \) are optimal if and only if there exists an orthonormal matrix \( Q \) and numbers \( \lambda_i, \omega_i \ (i = 1, \ldots, k) \) such that

\[
\begin{align*}
X &= Q \text{Diag}(\lambda_1, \ldots, \lambda_k) Q^T \\
Y &= Q \text{Diag}(\omega_1, \ldots, \omega_k) Q^T \\
\lambda_i \omega_i &= 0, \ i = 1, \ldots, k
\end{align*}
\]

where \( \text{Diag}(d_1, \ldots, d_k) \) is a diagonal matrix whose \( i \)th diagonal entry is \( d_i \).
Chapter 2. Linear conic programming

The faces and the complementary faces of the positive semidefinite cone are well known. Understanding facial structures and faces leads to many results in semidefinite programming. To be self-contained, let us demonstrate how a face and the complementary face are described for the positive semidefinite cone, see e.g. [Pat00].

**Theorem 2.30.** Consider a matrix \( \overline{X} \in S_+^k \). Then the corresponding minimal face and the complementary face are the following:

\[
\text{face}(\overline{X}, S_+^k) = \{ X \in S_+^k \mid \mathcal{R}(X) \subseteq \mathcal{R}(\overline{X}) \} \tag{2.23}
\]

and

\[
\text{face}^\Delta(\overline{X}, S_+^k) = \{ X \in S_+^k \mid \mathcal{R}(X) \subseteq \mathcal{R}(\overline{X})^\perp \} \tag{2.24}
\]

where \( \mathcal{R}(X) \) is the column space of matrix \( X \).

**Proof.** Suppose that \( X_1, X_2 \in S_+^k \) such that \( \overline{X} = \alpha X_1 + (1 - \alpha) X_2 \) for some \( 0 < \alpha < 1 \). We show that \( \mathcal{R}(X_1) \subseteq \mathcal{R}(\overline{X}) \). Let us denote the null space of \( \overline{X} \) as \( \text{Null}(\overline{X}) \). For any \( z \in \text{Null}(\overline{X}) \), we obtain

\[
\alpha z^T X_1 z + (1 - \alpha) z^T X_2 z = z^T \overline{X} z = 0.
\]

Using the above equality and \( X_1, X_2 \in S_+^k \), we derive

\[
0 = z^T X_1 z = z^T X_1^{\frac{1}{2}} X_1^{\frac{1}{2}} z = ||X_1^{\frac{1}{2}} z||.
\]

Therefore, \( X_1^{\frac{1}{2}} z = 0 \) and so \( X_1 z = 0 \). This means that \( \text{Null}(X_1) \supseteq \text{Null}(\overline{X}) \) and equivalently \( \mathcal{R}(X_1) \subseteq \mathcal{R}(\overline{X}) \). Similarly, we can show that \( \mathcal{R}(X_2) \subseteq \mathcal{R}(\overline{X}) \).

Now let us take any \( X_1 \in S_+^k \) with \( \mathcal{R}(X_1) \subseteq \mathcal{R}(\overline{X}) \). We show that \( X_1 \in \text{face}(\overline{X}, S_+^k) \) by finding \( 1 > \alpha > 0 \) and \( X_2 \in S_+^k \) such that \( \overline{X} = \alpha X_1 + (1 - \alpha) X_2 \). As \( \text{Null}(\overline{X}) \subseteq \text{Null}(X_1) \), there exists \( \alpha > 0 \) small enough such that \( z^T \overline{X} z \geq \alpha z^T X_1 z \) for all \( z \in \mathcal{R}(\overline{X}) \) with \( ||z|| = 1 \).

As scaling of \( z \) does not affect the inequality, we have

\[
0 \leq z^T \overline{X} z - \alpha z^T X_1 z \tag{2.25}
\]
2.4. Properties related to cones

for all \( z \in \mathcal{R}(\bar{X}) \). For this choice of \( \alpha \), we define

\[
X_2 = \frac{1}{1 - \alpha}(\bar{X} - \alpha X_1).
\]

Consider any \( a = a^1 \oplus a^2 \in \mathbb{R}^k \) with \( a^1 \in \text{Null}(\bar{X}) \) and \( a^2 \in \mathcal{R}(\bar{X}) \). Using \( \text{Null}(\bar{X}) \subseteq \text{Null}(X_1) \), we derive

\[
(1 - \alpha)a^T X_2 a = a^T \bar{X} a - \alpha a^T X_1 a = (a^2)^T \bar{X} a^2 - \alpha (a^2)^T X_1 a^2 \geq 0
\]

for any \( a \in \mathbb{R}^k \). The last inequality holds due to (2.25) as \( a^2 \in \mathcal{R}(\bar{X}) \).

Therefore, \( X_2 \) is positive semidefinite as desired.

Let us turn to (2.24). First, we show that \( \langle \bar{X}, Y \rangle = 0 \) holds for any \( Y \in \mathcal{S}_+^k \) with \( \mathcal{R}(Y) \subseteq \mathcal{R}(\bar{X})^\perp \), which means that \( Y \in \text{face}^\Delta(\bar{X}, \mathcal{S}_+^k) \). Let us denote the \( i \)th column of \( X \) as \( x^i \). As \( \bar{x}^i \in \mathcal{R}(\bar{X}) \) and \( y^i \in \mathcal{R}(\bar{X})^\perp \), we obtain \( \langle \bar{X}, Y \rangle = \sum_{i=1}^{k} (\bar{x}^i)^T y^i = 0 \).

It remains to show that \( \mathcal{R}(Y) \subseteq \mathcal{R}(\bar{X})^\perp = \text{Null}(\bar{X}) \) holds for any \( Y \in \text{face}^\Delta(\bar{X}, \mathcal{S}_+^k) = \{ Y \in \mathcal{S}_+^k \mid \langle \bar{X}, Y \rangle = 0 \} \). As \( \langle \bar{X}, Y \rangle = 0 \), we have \( \bar{X}Y = 0 \). So for any \( v = Yd \in \mathcal{R}(Y) \), we find \( \bar{X}v = \bar{X}Yd = 0 \), i.e., \( v \in \text{Null}(\bar{X}) \). \( \square \)

Consider \( \bar{X} \in \mathcal{S}_+^k \) with rank \( \bar{X} = r \), and an orthonormal matrix \( Q \) which diagonalizes \( \bar{X} \). In other words,

\[
Q^T \bar{X} Q = \begin{pmatrix} \Lambda & 0 \\ 0 & 0 \end{pmatrix}
\]

with an \( r \times r \) diagonal matrix \( \Lambda \) consisting of the nonzero eigenvalues of \( \bar{X} \). The transformation \( X \mapsto Q^T X Q \) is a one to one mapping of \( \mathcal{S}_+^k \) to itself. Therefore, we have that

\[
Q^T (\text{face}(\bar{X}, \mathcal{S}_+^k)) Q = \text{face}(Q^T \bar{X} Q, \mathcal{S}_+^k)
\]

\[
= \text{face}(\begin{pmatrix} \Lambda & 0 \\ 0 & 0 \end{pmatrix}, \mathcal{S}_+^k) = \text{face}(\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \mathcal{S}_+^k).
\]

This implies that

\[
\dim \text{face}(\bar{X}, \mathcal{S}_+^k) = \frac{r(r + 1)}{2}.
\] (2.26)
Next let us specify strict complementarity and nondegeneracy for the SDP case. For the proofs and further details see e.g. [AHO97, Pat00]:

**Proposition 2.31.** Consider a pair of primal and dual problems $(SDP_P)$ and $(SDP_D)$.

- Consider primal and dual optimal solutions $\bar{X} \in S^k_+$ and $\bar{Y} \in S^k_+$ with rank $\bar{X} = r$ and rank $\bar{Y} = s$. If $r + s = k$, then the optimal solutions $\bar{X}$ and $\bar{Y}$ are strictly complementary.

- Suppose $X \in F_P$ with rank $X = r$. Let the eigenvalue decomposition be
  
  $\bar{X} = Q \begin{pmatrix} \Lambda & 0 \\ 0 & 0 \end{pmatrix} Q^T$

  with a $r \times r$ diagonal matrix $\Lambda$ and an orthonormal matrix $Q$. If
  
  $\text{span}\{A_i\} + \left\{ Q \begin{pmatrix} W & V \\ VT & 0 \end{pmatrix} Q^T \right\} = S^k$

  then $X$ is nondegenerate

In other words, strict complementarity means that exactly one of the two conditions $\lambda_i = 0$ or $\omega_i = 0$ holds in Theorem 2.29.

In linear programming, if $(P)$ and $(D)$ are feasible, then a pair of strictly complementary optimal solutions always exists. However, this is not anymore the case in SDP even when both primal and dual optimal solutions are unique and nondegenerate. For an illustrating example, see [AHO97, Example page 117].