Formation control in the port-Hamiltonian framework
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Document Version
Publisher's PDF, also known as Version of record

Publication date:
2015

Link to publication in University of Groningen/UMCG research database

Citation for published version (APA):

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Chapter 5

Orbital phasing of satellites on circular orbits
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Orbital phasing of satellites on circular orbits

This chapter considers a port-Hamiltonian approach to the problem of orbital phasing of satellite constellations on circular orbits. The problem setting and a theoretical background for the problem are discussed in Section 5.1. Section 5.2 continues with the dynamical model for each satellite and using generalized canonical transformations the error system with respect to a circular orbit is derived. A formal problem definition of the orbital phasing problem is given in Section 5.3, followed by the controller design and closed-loop analysis. The highlights and concluding remarks of the chapter are given in Section 5.4.

The results in this chapter are published in [111, 112, 114].

5.1 Introduction

Formation flying of satellite constellations has received quite some attention in recent years. Different definitions of the terms satellite constellation and formation flying have been used in literature, for example based on the fact whether the states of the satellites are coupled (formation flying [101]) or not (constellation [102]).

In this chapter the word satellite constellation refers to a group of satellites which collaborate in order to achieve a higher level goal. The term formation flying refers to this higher level goal, which in this chapter corresponds to achieving an equal distribution of the satellites in the constellation on a circular orbit.

Using satellite constellations opens up possibilities for new types of missions, which are not possible with the traditional one satellite setup [26]. For example, the OLFAR mission aims at exploring the below 30MHz frequency bandwidth radio signals. To achieve sufficient spatial resolution, such a low frequency telescope in space needs an aperture diameter of 10 to 100 kilometers.\(^1\) Clearly, these types of applications are not feasible with a single satellite.

The dynamic environment where constellation operates can be divided into two regimes [101], namely deep space and planetary orbits. In deep space the relative dynamics of a constellation are usually approximated, often by a double

\(^1\)www.lr.tudelft.nl/en/organisation/departments/space-engineering/space-systems-engineering/projects/olfar-orbiting-low-frequency
integrator. For planetary orbits on the other hand the constellation dynamics are considered explicitly, including gravitational forces and disturbances such as drag. In this chapter the focus is on one type of planetary orbits, namely circular orbits.

Much formation flying research has focused on the case where only one point in the formation (e.g. the center of mass or the leader satellite) is on the planetary orbit [23, 65], while the individual satellites themselves do not need to be on the orbit. In contrast, [76, 108] and this chapter address the problem where each satellite is on the orbit, while the satellites equally distribute spatially on the orbit. This problem is also known as orbital phasing, since the control goal is to keep spacecrafts phased on the orbit [102].

Orbital phasing on circular orbits is of special interest to Global Navigation Satellite Systems (GNSS) like the Global Positioning System (GPS) and more recently Galileo. GPS requires 24 satellites to phase on six circular orbits, while Galileo requires 30 satellites to phase on three orbits. Other applications may be found in meteorological, environmental and military applications.

In Section 5.3 a controller is presented, based on energy considerations to solve the orbital phasing problem. Recently there has been an increasing interest in these so-called energy-based models [86], which allow for analysis and control design of nonlinear, multi-domain systems such as satellite constellations. The energy function of a system determines not only the static, but also the transient behavior [86] thereby enabling stabilization and performance studies. Furthermore, practitioners are familiar with energy concepts and therefore energy-based models may serve as a lingua franca amongst (control) engineers [86]. Two common energy-based modeling frameworks are the Lagrangian [23] and port-Hamiltonian framework [41, 45].

This chapter provides a theoretical framework for the orbital phasing of satellites on circular orbits. The control systems consists of virtual couplings and is divided into two parts. The internal control system is a local controller which assigns virtual couplings between each satellite and the target orbit. The external control system assigns virtual couplings between the satellites in the constellation in a similar way as Chapters 3 and 4. Stability is proven using the energy function of the closed-loop system as a Lyapunov function candidate. The main differences with respect to Chapters 3 and 4 is that satellites are lossless systems subject to a nonlinear gravitational potential and that the graph modeling the interaction topology is cyclic.

For satellite constellations the limited availability of propellant asks for energy-efficient control schemes. Insight into the controller’s power requirements and energy consumption is inherent to the port-Hamiltonian framework making use of the energy functions (see Remark 5.5).
5.2. Dynamical model

Utilization within the ROSE project

The algorithms in this chapter achieve orbital phasing for satellite constellations on circular target orbits, the main application area being Global Navigation Satellite Systems. The algorithms presented in this chapter provide a first step for the application of virtual couplings in space applications on planetary orbits. The next step is to generalize these results to the formation flying of satellites in deep space. This is of particular interest for the Far-InfraRed Interferometry (FIRI) mission of ESA and SRON, where the goal is to achieve a high angular resolutions (in the order of a few arcseconds) to investigate wavelengths between 25 and 300 micron. Achieving such high resolutions requires huge baselines, which can only be achieved using tight formation control of the satellites in the constellation.

The next section provides the dynamical modeling of the satellite constellation in the port-Hamiltonian framework.

5.2 Dynamical model

Consider an earth-centered inertial frame of reference in Cartesian coordinates, where the origin of the reference frame is at the center of planet earth. The $z$-axis of the reference frame is assumed to be normal to the orbital plane of interest. The interest here is merely on the satellite dynamics in the orbital plane, hence the dynamics along the $z$-axis are omitted. Each satellite is modeled as a point mass which is subject to the gravitational field of planet earth. Let $q^c_{i} = (q^c_{x,i}, q^c_{y,i})^T$ denote the position and $p^c_{i} = (p^c_{x,i}, p^c_{y,i})^T$ the momentum in Cartesian coordinates of satellite $i$ in the earth-centered inertial frame of reference (see Figure 5.1). Each satellite is modeled as a point $m_i$ moving in the orbital plane. Setting $n = 2$ and $D^c_i = 0$ the dynamics of satellite $i$ are equal to the fully actuated agent dynamics (3.1) and are given by

$$\begin{pmatrix}
\dot{q}^c_{i} \\
\dot{p}^c_{i}
\end{pmatrix} = \begin{pmatrix}
0 & I_2 \\
-I_2 & 0
\end{pmatrix} \begin{pmatrix}
\frac{\partial H^c_i}{\partial q^c_{i}} \\
\frac{\partial H^c_i}{\partial p^c_{i}}
\end{pmatrix} + \begin{pmatrix}
0 \\
I_2
\end{pmatrix} f^c_{i},$$

$$v^c_i = \frac{\partial H^c_i}{\partial p^c_{i}},$$

with input force $f^c_{i} = (f^c_{x,i}, f^c_{y,i})^T$, and output velocity $v^c_i = (v^c_{x,i}, v^c_{y,i})^T$. The Hamiltonian $H^c_i(q^c_{i}, p^c_{i})$ is the sum of the kinetic energy stored in satellite mass $m_i$ and the gravitational potential energy. Here, planet earth is assumed to be a perfect sphere (i.e., deviations like the $J_2$-perturbation are neglected) [2] and its kinetic
and gravitational potential energy is modeled as

$$H^c_i(q^c_i, p^c_i) = \frac{(p^c_{x,i})^2}{2m_i} + \frac{(p^c_{y,i})^2}{2m_i} - \frac{\mu_em_i}{\|q^c_i\|},$$

(5.2)

with $\|q^c_i\|$ the distance with respect to the center of planet earth, which is defined as $\|q^c_i\| = \sqrt{(q^c_{x,i})^2 + (q^c_{y,i})^2}$.

In contrast with most energy functions, Hamiltonian (5.2) is not bounded from below and therefore $H^c_i(q^c_i, p^c_i)$ can not be directly used as a Lyapunov function candidate. However, from the physics it follows that $\|q^c_i\| > R_E > 0$, with $R_E$ the radius of planet earth, (see Assumption 5.1). Thus $H^c_i(q^c_i, p^c_i)$ is in fact bounded from below. Moreover, $H^c_i(q^c_i, p^c_i)$ has multiple critical points which might give rise to undesired equilibria. The critical points for $H^c_i(q^c_i, p^c_i)$ are defined as those $q^c_i, p^c_i$ for which $\frac{\partial H^c_i}{\partial q^c_i} = \frac{\partial H^c_i}{\partial p^c_i} = 0$.

In order to facilitate the control design of equal distribution on a circular orbit polar coordinates are employed. Let $r_i \in \mathbb{R}$ denote the radial distance and $\phi_i \in [0, 2\pi]$ the azimuthal angle (see Figure 5.1), and let $p_i \in \mathbb{R}$, $h_i \in \mathbb{R}$ denote the corresponding (angular) momenta. Transforming (5.1) from Cartesian to polar coordinates is achieved by a canonical coordinate transformation $(q^c_{x,i}, q^c_{y,i}, p^c_{x,i}, p^c_{y,i}) \mapsto (r_i, \phi_i, p_i, h_i)$ which is given by [98]

$$\begin{pmatrix} q^c_{x,i} \\ q^c_{y,i} \end{pmatrix} = \mathcal{L}_i(r_i, \phi_i) = \begin{pmatrix} r_i \cos \phi_i \\ r_i \sin \phi_i \end{pmatrix},$$

$$\begin{pmatrix} p^c_{x,i} \\ p^c_{y,i} \end{pmatrix} = \left( \frac{\partial \mathcal{L}_i(r_i, \phi_i)}{\partial (r_i, \phi_i)} \right)^{-T} \begin{pmatrix} p_i \\ h_i \end{pmatrix},$$

where

![Figure 5.1: Position of satellite i in Cartesian coordinates (q^c_{x,i}, q^c_{y,i}) and polar coordinates (r_i, \phi_i) in an earth-centered inertial frame.](image-url)
5.2. Dynamical model

where \( \frac{\partial L_i(r_i, \phi_i)}{\partial (r_i, \phi_i)} \) denotes the Jacobian matrix, which is given by

\[
\frac{\partial L_i(r_i, \phi_i)}{\partial (r_i, \phi_i)} = \begin{pmatrix}
\cos \phi_i & -r_i \sin \phi_i \\
\sin \phi_i & r_i \cos \phi_i
\end{pmatrix}.
\]

The dynamics (5.1) of satellite \( i \) in polar coordinates\(^2\) are now given by

\[
\begin{pmatrix}
\dot{r}_i \\
\dot{\phi}_i \\
\dot{p}_i \\
\dot{h}_i
\end{pmatrix} =
\begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\frac{\partial H_i}{\partial r_i} \\
\frac{\partial H_i}{\partial \phi_i} \\
\frac{\partial H_i}{\partial p_i} \\
\frac{\partial H_i}{\partial h_i}
\end{pmatrix} +
\begin{pmatrix}
0 \\
0 \\
1 \\
0
\end{pmatrix}
f_i,
\]

\( v_i = \begin{pmatrix}
\frac{\partial H_i}{\partial p_i} \\
\frac{\partial H_i}{\partial h_i}
\end{pmatrix}, \)

with input force \( f_i = (f_{r,i}, f_{\phi,i})^T \), along the radial distance \( r_i \) and azimuthal angle \( \phi_i \), and corresponding output velocity \( v_i = (v_{r,i}, v_{\phi,i})^T \). The Hamiltonian (5.2) in polar coordinates is given by

\[
H_i(r_i, p_i, h_i) = \frac{p_i^2}{2m_i} + \frac{h_i^2}{2m_i r_i^2} - \frac{\mu_e m_i}{r_i}.
\]

Physics impose the following natural assumption on the radial distance \( r_i \).

**Assumption 5.1.** The radial distance \( r_i \) is bounded from below by \( r_i > R_E > 0 \), with \( R_E \approx 6.371 \cdot 10^6 \) m the radius of planet earth, for all \( i = 1, \ldots, N \) satellites.

Note that Assumption 5.1 corresponds to the physical fact that satellites always fly above the surface of planet earth. It ensures that the Hamiltonian \( H_i(p_i, h_i) \) is lower bounded and that it may be used as a Lyapunov function candidate. Furthermore, it guarantees that the Jacobian \( \frac{\partial L_i(r_i, \phi_i)}{\partial (r_i, \phi_i)} \) is nonsingular.

Next, the error system of (5.3) with respect to a circular target orbit is presented.

### 5.2.1 Derivation of the error dynamics with respect to a circular orbit using generalized canonical transformations

To facilitate the controller design the error dynamics of (5.3) with respect to a circular orbit are derived. A circular orbit is characterized by its radius \( R_0 \) and the corresponding angular velocity \( \omega_0 = \sqrt{\frac{\mu_e}{R_0^3}} \), with \( \mu_e = 398.6004418 \cdot 10^{12} \) m\(^3\)/s\(^2\) the gravitational constant [23]. Next a generalized canonical transformation of

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\(^2\)The remainder of this chapter uses polar coordinates. For easy of notation a \( P \) superscript is omitted in the remainder.
the form (2.11) is applied to derive the error dynamics with respect to the circular orbit.

Let $\vec{r}_i, \vec{\phi}_i, \vec{p}_i, \vec{h}_i$ denote the error state variables, which are defined by the following time-dependent coordinate transformation function $\Phi(r_i, \phi_i, p_i, h_i)$

$$
\begin{pmatrix}
\vec{r}_i \\
\vec{\phi}_i \\
\vec{p}_i \\
\vec{h}_i
\end{pmatrix} =
\begin{pmatrix}
r_i - R_0 \\
\phi_i - \omega_0 t \\
p_i \\
h_i - h_0(r_i)
\end{pmatrix} =: \Phi(r_i, \phi_i, p_i, h_i),
$$

(5.5)

with desired angular momentum $h_0_i(r_i) = m_i \omega_0 r_i^2$. Furthermore define the fictitious potential $U(h_i) = -\omega_0 h_i + c_i$, with $c_i$ some arbitrary constant, and set $\beta = 0, K = S = 0$. Substituting $\Phi, U, \beta$ into the partial differential equation (2.13) results in

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -\omega_0 \\
0 & 0 & 1 & 0 & 0 \\
m_i \omega_0 r_i & 0 & 0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
-1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & -\omega_0 \\
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
0 \\
-1
\end{pmatrix} = 0.
$$

Hence by Theorem 2.7 this choice of $\Phi(r_i, \phi_i, p_i, h_i, t), U_i(h_i), \beta_i$ yields a canonical transformation such that the error dynamics with respect to a circular orbit follow from (2.12) and are given by

$$
\begin{pmatrix}
\dot{\vec{r}}_i \\
\dot{\vec{\phi}}_i \\
\dot{\vec{p}}_i \\
\dot{\vec{h}}_i
\end{pmatrix} =
\begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
-1 & 0 & 0 & 0 & c_i(\vec{r}_i) \\
0 & -1 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\frac{\partial H_i}{\partial \vec{r}_i} \\
\frac{\partial H_i}{\partial \vec{\phi}_i} \\
\frac{\partial H_i}{\partial \vec{p}_i} \\
\frac{\partial H_i}{\partial \vec{h}_i}
\end{pmatrix}
+ \begin{pmatrix}
0 \\
0 \\
1 \\
0
\end{pmatrix}
f_i,
$$

(5.6)

with $c_i(\vec{r}_i) = 2m_i \omega_0 (\vec{r}_i + R_0)$. The new Hamiltonian $\tilde{H}_i(\vec{r}_i, \vec{\phi}_i, \vec{p}_i, \vec{h}_i)$ is calculated as

$$
\tilde{H}_i(\vec{r}_i, \vec{\phi}_i, \vec{p}_i, \vec{h}_i) = H_i \left( \Phi^{-1}(\vec{r}_i, \vec{\phi}_i, \vec{p}_i, \vec{h}_i, t) \right) + U_i \left( \Phi^{-1}(\vec{h}_i) \right)
= \frac{\vec{p}_i^2}{2m_i} + \frac{(\vec{h}_i + h_0_i(r_i))^2}{2m_i(\vec{r}_i + R_0)^2} - \frac{\mu_em_i}{\vec{r}_i + R_0} - \omega_0 (\vec{h}_i + h_0_i(r_i)) + c_i,
$$
which, using \( h_{0i}(\vec{r}_i) = m_i \omega_0 (\vec{r}_i + R_0)^2 \) and \( \mu_e = \omega_0^2 R_0^3 \), can be simplified to

\[
\bar{H}_i(\vec{r}_i, \vec{\phi}_i, \vec{p}_i, \vec{h}_i) = \frac{\vec{p}_i^2}{2m_i} + \frac{\vec{h}_i^2}{2m_i(\vec{r}_i + R_0)^2} + \frac{h_{0i}(\vec{r}_i)\vec{h}_i}{m_i(\vec{r}_i + R_0)^2} + \frac{h_{0i}(\vec{r}_i)^2}{2m_i(\vec{r}_i + R_0)^2} - \frac{\omega_0^2 R_0^3 m_i}{\vec{r}_i + R_0} - \omega_0 \vec{h}_i - \omega_0 h_{0i}(\vec{r}_i) + c_i
\]

where \( \vec{p}_i = (\vec{v}_i, \vec{m}_i) \) and \( \vec{h}_i = (\vec{\phi}_i, \vec{p}_i, \vec{r}_i, \vec{n}_i) \). The error dynamics for the whole constellation follows from directly

\[
(\frac{\partial}{\partial \vec{r}}) \vec{f}, (\frac{\partial}{\partial \vec{\phi}}) \vec{f}, (\frac{\partial}{\partial \vec{p}}) \vec{f}, (\frac{\partial}{\partial \vec{h}}) \vec{f} = \vec{v} = \left( \begin{array}{c} \frac{\partial H}{\partial \vec{r}} \\ \frac{\partial H}{\partial \vec{\phi}} \\ \frac{\partial H}{\partial \vec{p}} \\ \frac{\partial H}{\partial \vec{h}} \end{array} \right),
\]

(5.7)

To compactly rewrite (5.6) for all \( N \) satellites denote the collocated vectors \( \vec{r} = (\vec{r}_1, \ldots, \vec{r}_N)^T \), \( \vec{\phi} = (\vec{\phi}_1, \ldots, \vec{\phi}_N)^T \), \( \vec{p} = (\vec{p}_1, \ldots, \vec{p}_N)^T \), \( \vec{h} = (\vec{h}_1, \ldots, \vec{h}_N)^T \), \( f_r = (\vec{f}_r, \ldots, \vec{f}_N)^T \), \( f_{\phi} = (\vec{f}_{\phi}, \ldots, \vec{f}_{\phi,N})^T \), \( \vec{v}_r = (\vec{v}_r, \ldots, \vec{v}_{r,N})^T \), \( \vec{v}_{\phi} = (\vec{v}_{\phi}, \ldots, \vec{v}_{\phi,N})^T \), \( f_r = (\vec{f}_r, \vec{f}_{\phi})^T \), \( \vec{v} = (\vec{v}_r, \vec{v}_{\phi})^T \), \( \bar{\omega}_0 = 1_N \omega_0 \) and system matrices \( M = \text{diag}(m_1, \ldots, m_N) \), \( \bar{R}(\vec{r}) = \text{diag}(\vec{r}_1 + R_0, \ldots, \vec{r}_N + R_0) \), \( \bar{C}(\vec{r}) = \text{diag}(c_1, \ldots, c_N(\vec{r}_N)) \). The error dynamics for the \( N \) satellites (5.3) in the constellation are given by

\[
\vec{\dot{v}} = \begin{pmatrix} \frac{\partial H}{\partial \vec{r}} \\ \frac{\partial H}{\partial \vec{\phi}} \\ \frac{\partial H}{\partial \vec{p}} \\ \frac{\partial H}{\partial \vec{h}} \end{pmatrix},
\]

(5.8)

The Hamiltonian \( \bar{H}(\vec{r}, \vec{\phi}, \vec{p}, \vec{h}) \) for the whole constellation follows from directly from (5.7) and is given by

\[
\bar{H}(\vec{r}, \vec{\phi}, \vec{p}, \vec{h}) = \sum_{i=1}^{N} \bar{H}_i(\vec{r}_i, \vec{\phi}_i, \vec{p}_i, \vec{h}_i)
\]

(5.9)

with \( c = \sum_{i=1}^{N} c_i \).
Having derived the error dynamics (5.8) of the satellite constellation, the next section continues with the design of the control system to achieve the orbital phasing objectives on a circular orbit.

5.3 Orbital phasing on circular orbits

The control objective is to make all satellites in the constellation converge to a circular orbit, while phasing them on the orbit. The main idea behind this approach is to use virtual springs and dampers to control the satellites in the constellation: An internal control system stabilizes each satellite with respect to the circular orbit by assigning a virtual spring to $\bar{r}$ and a virtual damper to $\dot{r}$ and $\dot{\phi}$, while an external control system distributes the constellation on the orbit by assigning virtual springs and dampers to the relative azimuthal angle $\varphi$ and relative angular velocity $\dot{\varphi}$.

The interconnection topology captures which satellites are interconnected using a virtual coupling of the external control system. In the present approach the interaction topology is modeled as a ring graph $G(V, E)$, where the $N$ nodes in $V$ refer to the internally controlled satellites in the constellation and the $E = N$ edges in $E$ refers to the the external control systems.

In terms of (5.8) and the relative azimuthal angle $\varphi = (\varphi_1, \ldots, \varphi_N)^T$ the orbital phasing objectives are given by to

\[
\begin{aligned}
\bar{r} &\to 0, \\
\dot{\bar{r}} &\to 0, \\
\bar{h} &\to 0, \\
\varphi &\to \varphi^*,
\end{aligned}
\quad \text{as } t \to \infty,
\]

with $\varphi^* = (\varphi^*_1, \ldots, \varphi^*_N)^T$ the desired relative azimuthal angle for the external control systems. Note that there is no requirement on the absolute azimuthal angle $\phi$.

5.3.1 Internal control system

The internal control system consists of a virtual spring on $\bar{r}$ and virtual dampers on $\dot{r}$ and $\dot{\phi}$. The dynamics of such spring-damper systems are well known (see Chapters 3 and 4) and given by

\[
\begin{aligned}
\dot{\bar{r}} &= ( I_N \ 0 ) v_{r\phi}, \\
f_{r\phi} &= ( I_N \ 0 ) \frac{\partial S^r}{\partial \bar{r}}(\bar{r}) + ( D_r \ 0 \ D_\phi ) v_{r\phi},
\end{aligned}
\]

(5.11)
5.3. Orbital phasing on circular orbits

with dissipation matrices $D_r, D_\phi \in \mathbb{R}^{N \times N}$, input $v_{r\phi} = (v_r, v_\phi)^T \in \mathbb{R}^{2N}$ and output $f_{r\phi} = (f_r, f_\phi)^T \in \mathbb{R}^{2N}$. Here $v_r$ is the radial velocity and $v_\phi$ is the angular velocity around the center of planet earth, while $f_r, f_\phi$ are the corresponding forces. The Hamiltonian $S^r(r)$ of the internal control system equals the potential energy stored in the virtual springs, which is given by

$$S^r(r) = \frac{1}{2} r^T K_r r,$$

with virtual spring constants matrix $K_r \in \mathbb{R}^{N \times N}$. Each internal control system (5.11) is interconnected with (5.6) via a negative feedback interconnection

$$\begin{cases} f = -f_{r\phi} + \bar{f}, \\ v_{r\phi} = \bar{v}. \end{cases} \quad (5.12)$$

5.3.2 External control system

Let $\varphi \in \mathbb{R}^N$ denote the relative azimuthal angle (not to be confused with the absolute azimuthal angle $\phi$). The external control system assigns a virtual spring on $\bar{\varphi} = \varphi - \varphi^*$, with $\varphi^*$ the desired relative azimuthal angle$^3$ and a virtual damper on $\dot{\bar{\varphi}}$. The dynamics of the external control system are similar to (5.11) and are given by

$$\begin{align*}
\dot{\bar{\varphi}} &= v_\varphi, \\
 f_\varphi &= \frac{\partial S^\varphi}{\partial \bar{\varphi}}(\bar{\varphi}) + D_\varphi v_\varphi. \quad (5.13)
\end{align*}$$

with dissipation matrix $D_\varphi \in \mathbb{R}^{N \times N}$, input $v_\varphi \in \mathbb{R}^N$, and output $f_\varphi \in \mathbb{R}^N$.

The Hamiltonian $S^\varphi(\bar{\varphi})$ of the internal control system equals the potential energy stored in the virtual springs, which is given by

$$S^\varphi(\bar{\varphi}) = \frac{1}{2} \bar{\varphi}^T K_\varphi \bar{\varphi},$$

with diagonal spring constants matrix $K_\varphi \in \mathbb{R}^{N \times N}$.

The interconnection of the external control system with the internally controlled constellation is described by the incidence matrix $B$ of the graph, such that the coupling of satellites on the nodes and virtual couplings at the edges is given by

$^3$In order for the desired relative azimuthal angles to be feasible, they need to satisfy $\sum_{j=1}^N \varphi_j^* = 0$. By choosing the azimuthal angle domain as $\bar{\varphi} \in [0, 2\pi]$, it follows that $\varphi_j^* = \frac{2\pi}{N}$ for $j = 1, \ldots, N-1$ and $\varphi_N^* = -\frac{2\pi(N-1)}{N}$. 


\[
\begin{aligned}
\bar{f} &= \begin{pmatrix} 0 \\ -B \end{pmatrix} f_\phi, \\
\bar{v}_\phi &= \begin{pmatrix} 0 \\ B^T \end{pmatrix} \bar{v}.
\end{aligned}
\] (5.14)

Note that (5.14) results in an autonomous system. If required, an external control port may be added (see [100]). The next section presents the closed-loop analysis and a sufficient condition for the origin \((\bar{r}, \bar{p}, \bar{h}, \bar{\phi}) = 0\) to be locally asymptotically stable.

### 5.3.3 Closed-loop analysis

Combining the error system (5.8) with the internal control system dynamics (5.11) and the external control system dynamics (5.13) using couplings (5.12) and (5.14) gives the closed-loop satellite constellation dynamics

\[
\begin{pmatrix}
\bar{r} \\
\bar{p} \\
\bar{h} \\
\bar{\phi}
\end{pmatrix} =
\begin{pmatrix}
0 & I_N & 0 & 0 \\
-I_N & -D_r & \bar{C}(\bar{r}) & 0 \\
0 & -\bar{C}(\bar{r})^T & -(D_\phi + B D_\phi B^T) & -B \\
0 & 0 & B^T & 0
\end{pmatrix}
\begin{pmatrix}
\frac{\partial \hat{H}}{\partial \bar{r}} \\
\frac{\partial \hat{H}}{\partial \bar{p}} \\
\frac{\partial \hat{H}}{\partial \bar{h}} \\
\frac{\partial \hat{H}}{\partial \bar{\phi}}
\end{pmatrix}.
\] (5.15)

Note that the azimuthal angle \(\bar{\phi}\) is eliminated, since the distribution objective (5.10) merely considers the relative azimuthal angle \(\bar{\phi}\). Eliminating \(\bar{\phi}\) is straightforward, since it is a cyclic coordinate (i.e., \(\frac{\partial \hat{H}}{\partial \bar{\phi}} = 0\)).

The Hamiltonian of the closed-loop satellite constellation is simply the sum of the subsystems Hamiltonians and is given by

\[
\hat{H}(\bar{r}, \bar{p}, \bar{h}, \bar{\phi}) = H(\bar{r}, \bar{p}, \bar{h}) + S^r(\bar{r}) + S^p(\bar{p}).
\] (5.16)

Note that the origin \((\bar{r}, \bar{p}, \bar{h}, \bar{\phi}) = 0\) of (5.15) corresponds to objectives (5.10) being achieved.

Now consider an open set \(D\) in which the origin \((\bar{r}, \bar{p}, \bar{h}, \bar{\phi}) = 0\) is the unique critical point for \(\hat{H}\) (i.e., \((\bar{r}, \bar{p}, \bar{h}, \bar{\phi}) = 0\) is the only point in \(D\) such that \(\frac{\partial \hat{H}}{\partial(\bar{r}, \bar{p}, \bar{h}, \bar{\phi})} = 0\)). In the remainder of the stability analysis this set \(D\) is used to prove local asymptotic stability of the origin. First, calculate the column vector of partial
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derivatives \( \frac{\partial \hat{H}}{\partial (\bar{r}, \bar{p}, \bar{h}, \bar{\varphi})} \) as

\[
\frac{\partial \hat{H}}{\partial \bar{r}} = -\bar{h}^T M^{-1} \bar{R}(\bar{r})^{-3} \bar{h} + K_r \bar{r} \\
- \bar{\omega}_0^T \left[ M (\bar{R}(\bar{r})^3 - \bar{R}_0^3) \bar{R}(\bar{r})^{-2} \right] \bar{\omega}_0,
\]

\[
\frac{\partial \hat{H}}{\partial \bar{p}} = M^{-1} \bar{p},
\]

\[
\frac{\partial \hat{H}}{\partial \bar{h}} = M^{-1} \bar{R}(\bar{r})^{-2} \bar{h},
\]

\[
\frac{\partial \hat{H}}{\partial \bar{\varphi}} = K_{\varphi} \bar{\varphi}.
\]

(5.17)

Now continue with the Hessians \( \frac{\partial^2 \hat{H}}{\partial (\bar{r}, \bar{p}, \bar{h}, \bar{\varphi})^T} \), where the nonzero terms are given by

\[
\frac{\partial^2 \hat{H}}{\partial \bar{r}^2} = 3\bar{h}^T M^{-1} \bar{R}(\bar{r})^{-3} \bar{h} \\
- \bar{\omega}_0^2 \left[ M (\bar{R}(\bar{r})^3 + 2\bar{R}_0^3) \bar{R}(\bar{r})^{-3} \right] + K_r,
\]

\[
\frac{\partial^2 \hat{H}}{\partial \bar{p}^2} = M^{-1},
\]

\[
\frac{\partial^2 \hat{H}}{\partial \bar{h}^2} = M^{-1} \bar{R}(\bar{r})^{-2},
\]

\[
\frac{\partial^2 \hat{H}}{\partial \bar{\varphi}^2} = K_{\varphi},
\]

\[
\frac{\partial^2 \hat{H}}{\partial \bar{r} \partial \bar{h}} = -2M^{-1} \bar{R}(\bar{r})^{-3} \bar{W}(\bar{h}),
\]

(5.18)

with \( \bar{W}(\bar{h}) = \text{diag}(\bar{h}_1, \ldots, \bar{h}_N) \).

From Assumption 5.1 and substituting \((\bar{r}, \bar{p}, \bar{h}, \bar{\varphi}) = 0\) into (5.18) it follows that

\( \frac{\partial^2 \hat{H}}{\partial \bar{r}^2} (0) = -3\omega_0^2 M + K_r \), \( \frac{\partial^2 \hat{H}}{\partial \bar{p}^2} (0) = M^{-1} > 0 \), \( \frac{\partial^2 \hat{H}}{\partial \bar{h}^2} (0) = R_0^{-2} M^{-1} > 0 \), \( \frac{\partial^2 \hat{H}}{\partial \bar{\varphi}^2} (0) = K_{\varphi} > 0 \), \( \frac{\partial^2 \hat{H}}{\partial \bar{r} \partial \bar{h}} (0) = 0 \), which implies that the origin \((\bar{r}, \bar{p}, \bar{h}, \bar{\varphi}) = 0\) is a local minimum for \( \hat{H}(\bar{r}, \bar{p}, \bar{h}, \bar{\varphi}) \) if \( K_r > 3\omega_0^2 M \).

To find the critical point nearest to the origin, one should solve for \( \frac{\partial \hat{H}}{\partial \bar{r}} = 0 \). After some rewriting and noting that all matrices in (5.17) are diagonal matrices, the critical point nearest to the origin is given by

\[
r_i^* = \min \left| \frac{-R_0(2k_{r,i} - 3m_i\omega_0^2) \pm \sqrt{m_i\omega_0^2 R_0^2 (4k_{r,i} - 3m_i\omega_0^2)}}{2(k_{r,i} - m_i\omega_0^2)} \right|, \quad (5.19)
\]

for \( i = 1, \ldots, N \). The open set \( \mathcal{D} \), where the origin is a unique minimum is therefore
given by
\[ D = \{ (\vec{r}, \vec{p}, h, \varphi) \in \mathbb{R}^{5n} \mid |\vec{r}_i| < r^*_i \text{ for all } i = 1, \ldots, N \} \tag{5.20} \]

Now the main result of this chapter is stated in the following theorem.

**Theorem 5.2.** For \( K_r > 3\omega_0^2 M, D_r > 0, D_\phi > 0, K_\varphi > 0, D_\varphi > 0 \) the solutions of the closed-loop system (5.15) converge to \( \vec{r} = 0, \vec{p} = 0, h = 0, \varphi = 0 \). Hence objectives (5.10) are achieved for (5.3) on the set \( \Omega = \{ (\vec{r}, \vec{p}, h, \varphi) \in D \mid \hat{H}(\vec{r}, \vec{p}, h, \varphi) \leq H^* \} \), for some \( H^* > 0 \).

**Proof.** Note that achieving objectives (5.10) corresponds to asymptotic stability of the origin \( (\vec{r}, \vec{p}, h, \varphi) = 0 \) of (5.15). The proof of Theorem 5.2 is obtained by invoking LaSalle’s invariance principle (Theorem 2.4). Substituting \( (\vec{r}, \vec{p}, h, \varphi) = 0 \) into (5.17) shows directly that \( (\vec{r}, \vec{p}, h, \varphi) = 0 \) is an equilibrium for (5.15). Calculating the time derivative \( \dot{\hat{H}}(\vec{r}, \vec{p}, h, \varphi) \) it is easily seen that
\[
\dot{\hat{H}}(\vec{r}, \vec{p}, h, \varphi) = -\frac{\partial^T \hat{H} \partial \hat{H}}{\partial p} D_r \frac{\partial \hat{H}}{\partial \vec{p}} - \frac{\partial^T \hat{H} \partial \hat{H}}{\partial \vec{p}} (D_\phi + BD_\varphi \bar{B}^T) \frac{\partial \hat{H}}{\partial \vec{p}}.
\]

Let \( S = \{ (\vec{r}, \vec{p}, h, \varphi) \in D \mid \hat{H}(\vec{r}, \vec{p}, h, \varphi) = 0 \} \). To find \( S \), note that from (5.17) it follows that \( \vec{p} = 0, h = 0 \). Hence \( S = \{ (\vec{r}, \vec{p}, h, \varphi) \in D \mid \vec{p} = 0, h = 0 \} \). Let \( (\vec{r}(t), \vec{p}(t), h(t), \varphi(t)) \) be a solution that belongs to \( S \):
\[
\vec{p}(t) \equiv 0 \Rightarrow \dot{\vec{p}}(t) \equiv 0 \Rightarrow \frac{\partial \hat{H}}{\partial \vec{p}} \equiv 0 \Rightarrow \vec{r}(t) \equiv 0 \text{ in } \Omega,
\]
where the last part follows from the fact that \( (\vec{r}, \vec{p}, h, \varphi) = 0 \) is the only critical point for \( \hat{H}(\vec{r}, \vec{p}, h, \varphi) \) in \( \Omega \) by definition. Furthermore
\[
\dot{h}(t) \equiv 0 \Rightarrow \dot{h}(t) \equiv 0 \Rightarrow B \frac{\partial \hat{H}}{\partial \varphi} \equiv 0 \Rightarrow BK_\varphi \varphi \equiv 0 \text{ in } \Omega. \tag{5.21}
\]

From (5.21) it follows that \( \varphi \in \ker B \), which implies for a undirected ring graph that \( \varphi \in \text{im} \bar{B} \). Furthermore, since \( \sum_{j=1}^{N} \varphi_j = 0 \) it follows that within the set \( S \) to should hold that \( \varphi(t) \equiv 0 \). Hence the smallest invariant set is given by \( (\vec{r}, \vec{p}, h, \varphi) = 0 \) and by LaSalle’s invariance principle (Theorem 2.4) the origin of (5.15) is asymptotically stable and hence objectives (5.10) are achieved, thereby completing the proof.

**Remark 5.3** (Tuning controller gains). The ratio between the spring constants \( K_r \) and \( K_\phi \) determines whether internal stabilization w.r.t. the orbit or spatial distribution gets a higher priority. Careful tuning of \( K_r \) and \( K_\phi \) is required to prevent large energy consumption due to satellites moving far away from their orbit (see Remark 5.5).
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**Remark 5.4 (Control input).** The satellites’ control inputs follow directly from (5.8)-(5.14).

\[
\begin{align*}
    f_r &= -K_r(r - R_0) - D_r M^{-1} p, \\
    f_\phi &= -BK_\varphi(\varphi - \varphi^*) - (D_\phi + BD_\varphi B^T)H(r, h),
\end{align*}
\]

(5.22)

where \( H(r, h) = M^{-1}R(r)^{-2}(h - \omega_0 MR(r)r) \), with \( R(r) = \text{diag}(r_1, \ldots, r_N) \). Note that (5.22) has a clear physical interpretation in terms of spring and damping forces.

**Remark 5.5 (Energy consumption).** The energy the controller consumes follows directly from the power supplied to each satellite, which is simply the product of the inputs \( f_{r,i}, f_{\phi,i} \) and output \( v_{r,i}, v_{\phi,i} \). Hence the total power supplied at time \( t \) is given by

\[
P(t) = \sum_{i=1}^{N} |f_{r,i}^T(t)v_{r,i}(t)| + \sum_{i=1}^{N} |f_{\phi,i}^T(t)v_{\phi,i}(t)|. 
\]

(5.23)

The total energy consumption equals the time-integral of the supplied power (5.23) and is given by \( E(t) = \int_0^t |P(\tau)|d\tau \). Using the power (5.23) and the related energy consumption allows for an easy comparison between different controller settings (see the simulation results in the next section for a worked example).

### 5.3.4 Simulation results

To illustrate the effectiveness of Theorem 5.2, simulation results are presented here. For the simulation, a circular orbit with an altitude of 20,000 km is considered (i.e., \( R_0 = 20 \cdot 10^6 + R_E = 20 \cdot 10^6 + 6.371 \cdot 10^6 = 26.371 \cdot 10^6 m \)). An orbit with this altitude is called a Medium Earth orbit (MEO) and is most commonly used for the navigation satellite systems mentioned in Section 5.1.

The simulation is run for \( t = 500 \, \text{hr} \), in which the satellites encircle planet Earth approximately 42 times on the chosen MEO orbit (the period of the orbit is \( T_0 = \frac{2\pi}{\omega_0} = 11.83 \, \text{hr} \)). The constellation consists of \( N = 8 \) satellites with mass \( m_i = 2 \, \text{kg} \). The controller gains are set at \( k_{r,i} = 10, \, d_{r,i} = 15, \, d_{\phi,i} = 10 \) for the \( i = 1, \ldots, 8 \) internal control systems and \( k_{\phi,j} = 5, \, d_{\phi,j} = 10 \) for the \( j = 1, \ldots, 8 \) external control systems. These values were found after some trial-and-error.

The initial conditions are given in Table 5.1. Note that these initial conditions were chosen to clearly illustrate the simulation results. From a practical point of view they are considered to be quite far away from the desired position on the orbit. However, in this simulation \( K_r \gg 3M\omega_0 \) \( (k_{r,i} = 10, \, 3m_i\omega_0 = 3m_i\sqrt{\mu_e/R_0^3} = \)
5. Orbital phasing of satellites on circular orbits

<table>
<thead>
<tr>
<th>satellite</th>
<th>r [10^3 km]</th>
<th>φ [rad]</th>
<th>p [kg km/hr]</th>
<th>h [10^6 kg km^2/hr]</th>
</tr>
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<td>0</td>
<td>-26.93</td>
</tr>
<tr>
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<td>0</td>
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</tr>
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<td>0</td>
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</tr>
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<td>5.27</td>
<td>0</td>
<td>-3.79</td>
</tr>
</tbody>
</table>

Table 5.1: Initial conditions for the simulation.

The following is extracted from Table 5.1:

8.8457 \cdot 10^{-4} for all \( i = 1, \ldots, 8 \), such that the convergence is still quite fast. The simulation results are shown in Figures 5.2, 5.3, 5.4, and 5.5.

Figure 5.2 shows four snapshots of the satellite positions at times \( t = \{0, 10, 100, 500\} \) hr. Note that in all four figures, the satellites have a counterclockwise movement on the orbit. Initially (top left) the satellites are neither on the orbit, nor are they equally distributed on the orbit. After \( t = 500 \) hr the satellites have converged to the circular orbit and are phased on the orbit.

Due to the scaling of Figure 5.2 it is not immediately seen that the orbital phasing objectives (5.10) are achieved. For clarification Figure 5.3 shows the time evolution of the radial distance \( \bar{r} \) (top), radial velocity \( \bar{v}_r \) (middle) and angular velocity \( \bar{v}_\phi \) (bottom), while Figure 5.4 shows the time evolution of the azimuthal angle \( \bar{\phi} \). Here, Figures 5.3 and 5.4 correspond to the first three objectives and the final objective in (5.10) respectively. Note that Figure 5.3 shows the error variables defined in (5.5), such that convergence to the origin corresponds to achievement of first three objectives in (5.10). For the azimuthal angle depicted in Figure 5.4 it is easily seen that the angle distributes evenly on the orbit (i.e., \( \phi_{i+1} - \phi_i \to \frac{\pi}{4} \) for \( i = 1, \ldots, 7 \) and \( \phi_1 - \phi_8 \to -\frac{14\pi}{8} \) as \( t \to \infty \)).

From Figures 5.3 and 5.4 and (5.22) it follows that the control inputs converge to zero and no control effort is needed once the satellites are phased on the orbit. Furthermore, Figure 5.5 shows the total power (5.23) supplied to each satellite by the internal and external control system. The \( y \)-axis is cut off at 1.6W for clarity. The maximal power usage ranges from 1.30 W for satellite 8 to 24.80 W for satellite 4. Looking at micro propulsion systems like the T3 Cold-Gas Micropropulsion system\(^4\), these control inputs are well within the achievable thrust levels. On the other hand, considering electrical propulsion systems the peak power (see Figure 5.5) is well within the range of modern power supplies (e.g. solar power).

\(^4\)www.lr.tudelft.nl/organisatie/afdelingen/space-engineering/space-systems-engineering/expertise-areas/space-propulsion/propulsion-research/t3-mps/
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Figure 5.2: Snapshots of the satellites’ position in the orbital plane at $t = \{0, 10, 100, 500\} \text{ hr}$. The blue circles represent the satellites, while the dashed line indicates the circular orbit with radius $R_0$. The orbital motion is counterclockwise. All axis are in $1000\text{ km}$. 
Figure 5.3: Time evolution of the radial distance $\bar{r}$, radial velocity $\bar{v}_r$, and angular velocity $\bar{v}_\phi$. The dotted lines show the reference values.

Figure 5.4: Time evolution of the azimuthal angle $\bar{\phi}$. 
5.4 Concluding remarks

This chapter presents a port-Hamiltonian approach to the orbital phasing problem of satellite constellations on circular orbits. A dynamical model is obtained and using generalized canonical transformations the error system with respect to the target orbit is derived. The control objective of orbital phasing is achieved by assigning virtual couplings in between each satellite and the orbit as well as virtual couplings in between the satellites. Under a mild condition on one of the virtual spring constants the orbital phasing objectives are achieved and a complete stability proof is given. Simulation results show that the control systems fits well within the specification of modern satellite systems.

The algorithms for orbital phasing in this chapter use a similar approach as the formation control algorithms in Chapters 3 and 4. All algorithms use virtual couplings to shape the total energy of the network and inject damping for stability, while the physical structure is exploited in the analysis and design.

Figure 5.5: Time evolution of the total power $|P_r| + |P_\phi|$.