Formation control in the port-Hamiltonian framework
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Chapter 4

Formation control of nonholonomic wheeled robots
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Formation control of nonholonomic wheeled robots

This chapter deals with formation control of a network of nonholonomic wheeled robots, as opposed to the fully actuated point masses considered in the previous Chapter 3. First an introduction to the three problems considered in this chapter is given in Section 4.1. Then, Section 4.2 continues with the derivation of the dynamical model of the wheeled robot in the port-Hamiltonian framework.

The three subsequent sections deal with three related problems: formation control (Section 4.3), formation control with velocity tracking (Section 4.4), and formation control in the presence of matched input disturbances (Section 4.5). Each section provides a formal problem formulation, controller design, and closed-loop analysis. The results are illustrated with simulation and experimental results. The highlights and concluding remarks of the chapter are given in Section 4.6.

The results in this chapter are published in [60, 113, 115, 116]. The results in Section 4.5 are based on a collaboration with Matin Jafarian and Claudio De Persis.

4.1 Introduction

This chapter considers three formation control problems for a network of wheeled robots. Starting point is the approach using virtual couplings presented in Chapter 3. Recall that the objective for formation control is to achieve a prescribed geometrical shape, using only local feedback rules. In addition to standard formation control the problems of velocity tracking and disturbance rejection are considered here.

Control of nonholonomic wheeled robots has received quite some attention in recent years [5, 17, 34, 49, 67, 91, 97]. Due to the nonholonomic constraint on the wheel axle, the wheeled robot does not satisfy Brockett’s necessary condition for continuous smooth feedback stabilization [13]. Hence discontinuous control laws [5, 17, 68] and time-varying control laws [91, 97] have been developed to stabilize the robot dynamics. While [5, 17, 91, 97] are devoted to a single robot, more recent work focused on the control of multiple wheeled robots [34, 49, 67].

Another reason why the wheeled robot is of interest, is the availability of many types of wheeled robots for implementation of algorithms in practice. One example of such a robot is the e-puck (Figure C.1), which is used throughout this thesis for
the experimental results. The e-puck robot is designed for engineering education at the university level [79] and provides an easy interface to test algorithms.

The algorithms in this chapter achieve formation control by assigning virtual couplings between the robots in the network. This approach is a flexible form of the more rigid virtual structure approach for formation control of mobile robots [67]. As in Chapter 3, each virtual coupling consists of a virtual spring and a virtual damper. The virtual springs determine the formation shape by shaping the energy function of the network, while the virtual dampers shape the transient response by injecting damping. The interconnection topology amongst robots and virtual couplings is as in Chapter 3 described by a tree graph (i.e., an undirected, connected, acyclic graph).

The problems considered in Chapter 3 require the network of robots to come to a hold once the formation is achieved. However, motivated by Section 1.2, the objective is often to track a prescribed reference velocity. Applications include sweeping [22, 66] and deployment for coverage [25, 82]. Velocity tracking for port-Hamiltonian systems is achieved using generalized canonical transformations [45] to derive the error dynamics with respect to the reference velocity. By stabilizing these error dynamics, the network of wheeled robots tracks the reference velocity thereby enabling the movement of the whole network.

Another, more practical challenge in the field of formation keeping is to reach and maintain a desired formation shape despite input disturbances. Two types of disturbances are considered in this chapter: constant and harmonic (or sinusoidal) disturbances. In applications constant and harmonic disturbances correspond to respectively offsets in actuators and sensors, and acoustic disturbances and vibrations in rotating equipment [90]. Output regulation techniques [16, 29] and proportional-integral controllers with quantized information and time-varying topologies [120] have been studied in this context. For port-Hamiltonian systems input disturbance rejection is achieved using an adaptive internal-model-based controller [47, 48]. To the authors best knowledge existing literature does not consider disturbance rejection for a network of nonholonomic port-Hamiltonian systems.

The contributions of this chapter include the derivation of a dynamical model of the wheeled robot in the port-Hamiltonian framework and the design and analysis of three formation control problems. Standard formation control is achieved by assigning virtual couplings between the front ends of the robots. Second, a velocity tracking controller is proposed using generalized canonical transformations, which guarantees that all robots in the network track a constant reference velocity. A local (nonlinear) heading controller ensures that all robots also move along the same desired heading, under a natural assumption on the initial heading of the robots. Third, matched input disturbances are counteracted using an internal-model-based controller. The controller provides stability for constant disturbances and complete
rejection of harmonic disturbances.

Utilization within the ROSE project

Wheeled robots do not comply with the legged robot design discussed in Section 1.2.2. However, there are two main reasons why studying wheeled robots aligns with the utilization within the ROSE project. First, the constraint on the wheel axle of the wheeled robot prevents the robot from moving sideward. The sideward motion of a legged robot is often less (energy-)efficient than the forward/backward motion. Chapter 3 incorporated this consideration by altering the friction coefficients accordingly, the present chapter does so by imposing a nonholonomic constraint on the sideward motion of the robots.

The second reason concerns the experimental validation of theoretical and simulation results. Since there is no prototype of the new robot design available at the time of writing (let alone a network of them), another type of robot is used for this purpose. The Discrete Technology & Production Automation (DTPA) laboratory of the University of Groningen provides several e-puck robots, which are used for the experimental results in this chapter.

The next section continues with the dynamical modeling of the wheeled robot in the port-Hamiltonian framework.

4.2 Dynamical model of the wheeled robot

A schematic figure of wheeled robot $i$ is depicted in Figure 4.1. Let the point $(x_{A,i}, y_{A,i})$ denote the center of the wheel axle and $(x_{B,i}, y_{B,i})$ a point in front of the robot. Here, the center of mass is assumed to be at the center of the wheel axle [67, 68]. The point $(x_{B,i}, x_{B,i})$ is at a distance $d_{AB,i} > 0$ from the axle center and is used in the controller design later on. Furthermore, let $\phi_i$ denote the heading of the robot.

The dynamics of the wheeled robot are derived from the rigid body dynamics with a nonholonomic constraint on the wheel axle. Then, this constraint is solved for, thereby obtaining the dynamics on the constrained state space, which is used in the analysis of Sections 4.3, 4.4, and 4.5. In this section a superscript $rb$ is added to differentiate between the rigid body dynamics and the dynamics on the constrained state space.

Let $q_i \in \mathbb{R}^3$ denote the position, and let $p^{rb}_i \in \mathbb{R}^3$ denote the corresponding momentum of robot $i$, where $q_i = (x_{A,i}, y_{A,i}, \phi_i)^T$ and $p^{rb}_i = (p^{rb}_x, p^{rb}_y, h^{rb}_i)^T$ (see Figure 4.1). Note that the position vector $q_i$ includes the heading of the robot, since it is modeled as a rigid body rather than a point mass (see Section 3.2). The momentum is related to the position by $p^{rb}_i = M^{rb}_i q_i$, with constant mass-inertia
matrix $M_{rb,i} = \text{diag}(m_i, m_i, I_{cm,i})$ where $m_i$ denotes the robot's mass and $I_{cm,i}$ denotes the moment of inertia around the center of mass $(x_{A,i}, y_{A,i})$. The rigid body dynamics are of the form (2.8) and are given by

$$
\begin{pmatrix}
\dot{q}_i \\
\dot{p}_{rb,i}
\end{pmatrix} = 
\begin{pmatrix}
0 & I_3 \\
-I_3 & -D_{rb,i}
\end{pmatrix}
\begin{pmatrix}
\frac{\partial H_{rb,i}}{\partial q_i} \\
\frac{\partial H_{rb,i}}{\partial p_{rb,i}}
\end{pmatrix} + 
\begin{pmatrix}
0 \\
G_i(q_i)
\end{pmatrix} u_i,
$$

with input $u_i = (u_{f,i}, u_{\phi,i})^T \in \mathbb{R}^2$, output $y_i = (y_{f,i}, y_{\phi,i})^T \in \mathbb{R}^2$. Here $u_{f,i}, u_{\phi,i}$ correspond to a forward force and torque acting at $(x_{A,i}, y_{A,i})$ respectively, while $y_{f,i}, y_{\phi,i}$ refer to the corresponding (angular) velocities. The dissipation matrix $D_{rb,i}$ and input matrix $G_i(q_i)$ are given by

$$
D_{rb,i} = 
\begin{pmatrix}
d_{f,i} & 0 & 0 \\
0 & d_{f,i} & 0 \\
0 & 0 & d_{\phi,i}
\end{pmatrix},
$$

$$
G_i(q_i) = 
\begin{pmatrix}
\cos \phi_i & 0 \\
\sin \phi_i & 0 \\
0 & 1
\end{pmatrix},
$$

with $d_{f,i}$ and $d_{\phi,i}$ respectively the forward and angular friction coefficient of the rigid body. The Hamiltonian is the kinetic energy stored in the rigid body and is given by

$$
H_{rb,i}(p_{rb,i}) = \frac{1}{2} p_{rb,i}^T M_{rb,i}^{-1} p_{rb,i}.
$$

In addition to the rigid body dynamics (4.1), physics impose a nonholonomic constraint on the wheel axle, which is not able to move sideward. This nonholonomic
constraint is expressed in terms of the axle’s center \((x_{A,i}, y_{A,i})\) as
\[
\sin \phi_i \dot{x}_{A,i} - \cos \phi_i \dot{y}_{A,i} = 0,
\] (4.2)
which may be rewritten as
\[
\begin{pmatrix}
\sin \phi_i & -\cos \phi_i & 0
\end{pmatrix}
A^T_i(q_i)
\begin{pmatrix}
\frac{\partial H^b_r}{\partial p^b_i}(p^b_i)
\end{pmatrix} = 0.
\] (4.3)
Since (4.3) is of the form (2.9) one may solve for (4.3) to obtain the dynamics on the constrained state space (Section 2.3.2). First note that 
\(\text{rank } A^T_i(q_i) = 1\) and define the following matrix
\[
S_i(q_i) = \begin{pmatrix}
\cos \phi_i & 0 \\
\sin \phi_i & 0 \\
0 & 1
\end{pmatrix},
\]
such that \(A^T_i(q_i)S_i(q_i) = 0\) and \(\text{rank } S_i(q_i) = 2\). Now define the new momenta coordinates \(p_i = (p_{f,i}, h_i)^T, p_{s,i}\) as
\[
\begin{align*}
p_i &:= S^T_i(q_i)p^b_i, \\
p_{s,i} &:= A^T_i(q_i)p^b_i,
\end{align*}
\] (4.4)
Here \(p_{f,i}, p_{s,i}\) refer to the forward and sideward momenta of robot \(i\). Clearly \((q_i, p^b_i) \mapsto (q_i, p_i, p_{s,i})\) defines a coordinate transformation, since the rows of \(S_i(q_i)\) are orthogonal to the rows of \(A_i(q_i)\). In the new coordinates, the \(p_{s,i}\) dynamics may be eliminated to obtain the dynamics on the constrained state space, which are given by
\[
\begin{pmatrix}
\dot{q}_i \\
\dot{p}_i \\
y_i
\end{pmatrix} = \begin{pmatrix}
0 & S_i(q_i) & -D^r_i \\
-S^T_i(q_i) & \frac{\partial H^r_r}{\partial q_i} & \frac{\partial H^r_r}{\partial p_i}
\end{pmatrix} + \begin{pmatrix}
0 \\
0
\end{pmatrix} u_i,
\] (4.5)
with Hamiltonian \(H^r_r = \frac{1}{2}p^T_i M^{-1}_i p_i\), where \(M^r_i = \text{diag}(m_i, I_{cm,i})\). The new dissipation matrix \(D_i\) is defined as
\[
D^r_i = S^T_i(q_i)D_i^b S_i(q_i) = \text{diag}(d_{f,i}, d_{\phi,i}).
\]

Remark 4.1 (Brockett’s condition). The well-known paper by Brockett [13] provides a necessary condition under which systems can be asymptotically stabilized using continuous feedback. From Proposition 4.2.14 in [98] it follows that for (4.5) this
condition boils down to
\[ \bigcup \{ x_i : \| x_i - x_{0,i} \| < \epsilon \} \left( \begin{pmatrix} 0 & S_i(q_i) \\ -S_i^T(q_i) & -D_i^r \end{pmatrix} + \begin{pmatrix} 0 \\ I_2 \end{pmatrix} \right) = \mathbb{R}^5, \] 
(4.6)

with \( x_i = (q_i, p_i) \) and thus
\[ \text{im} \left( \begin{pmatrix} 0 & S_i(q_i) \\ -S_i^T(q_i) & -D_i^r \end{pmatrix} + \begin{pmatrix} 0 \\ I_2 \end{pmatrix} \right) \subset \text{im} \left( \begin{pmatrix} S_i(q_i) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ I_2 \end{pmatrix} \right). \]

By an appropriate change of configuration coordinates \( q_i \rightarrow \tilde{q}_i \), \( S_i(\tilde{q}_i) \) takes the form
\[ S_i(\tilde{q}_i) = \begin{pmatrix} \bar{S}_i(\tilde{q}_i) \\ I_2 \end{pmatrix} \]

and vectors of the form \( \begin{pmatrix} * \\ 0 \end{pmatrix} \) can not be in the image of \( S_i(\tilde{q}_i) \) and hence not in \( \text{im} [J_i(q_i) - R_i] + \text{im} g_i \). Thus (4.5) can not be asymptotically stabilized using continuous feedback. However, the formation control objective is defined in terms of the relative displacement between front ends of the robots. Therefore the formation control objective is not hindered by (4.6).

To compactly write the dynamics for all \( N \) robots denote the stacked vectors \( q = (q_1, \ldots, q_N)^T, p = (p_1, \ldots, p_N)^T, u = (u_1, \ldots, u_N)^T, y = (y_1, \ldots, y_N)^T \) and the system matrices \( S(q) = \text{block.diag}(S_1(q_1), \ldots, S_N(q_N)), D^r = \text{block.diag}(D^r_1, \ldots, D^r_N), M^r = \text{block.diag}(M^r_1, \ldots, M^r_N) \). The dynamics (4.5) for all \( i = 1, \ldots, N \) robots is now compactly denoted by
\[ \begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0 & S(q) \\ -S^T(q) & -D^r \end{pmatrix} \begin{pmatrix} \frac{\partial H^r}{\partial q} \\ \frac{\partial H^r}{\partial p} \end{pmatrix} + \begin{pmatrix} 0 \\ I_{2N} \end{pmatrix} \bar{u}, \] 
(4.7)

with Hamiltonian \( H^r = \sum_{i=1}^N H^r_i(p_i) = \frac{1}{2} p^T M^{-1} p \).

**Change of inputs**

For control purposes, Sections 4.3, 4.4.1 and 4.5 require a change of inputs for the wheeled robots. By assigning the virtual couplings to the front end of the robot (point \((x_{B,i}, y_{B,i})\) in Figure 4.1) rather than the center of the wheel axle (point \((x_{A,i}, y_{A,i})\) in Figure 4.1), the formation control is not hindered by the constraint on the wheel axle. Furthermore, sensors and end effectors are usually positioned
4.2. Dynamical model of the wheeled robot

For each robot $i$ its input $u_i = (u_{f,i}, u_{\phi,i})^T$ is transformed into a new input $ar{u}_i = (u_{x,i}, u_{y,i})^T$ where $u_{x,i} (u_{y,i})$ denotes a force along the $x$ direction ($y$ direction) acting on the point $(x_{B,i}, y_{B,i})$ (see Figure 4.2). The output $y_i$ is transformed accordingly into the new output $\bar{y}_i = (y_{x,i}, y_{y,i})^T$. The transformation $(u_i, y_i) \mapsto (\bar{u}_i, \bar{y}_i)$ is given by

$$u_i = \bar{G}_i(\phi_i)\bar{u}_i, \quad \bar{y}_i = \bar{G}_i^T(\phi_i)y_i,$$

where

$$\bar{G}_i(\phi_i) = \begin{pmatrix} \cos \phi_i & \sin \phi_i \\ -d_{AB,i} \sin \phi_i & d_{AB,i} \cos \phi_i \end{pmatrix}.$$ 

Note that controlling the point $(x_{B,i}, y_{B,i})$ is also quite natural from a practical point of view, since on-board sensors and end-effectors are usually positioned at the front end of the robot. The dynamics of robot $i$ with new input $\bar{u}_i$ and new output $\bar{y}_i$ follow from (4.5) and (4.8) and are given by

$$\begin{pmatrix} \dot{q}_i \\ \dot{p}_i \end{pmatrix} = \begin{pmatrix} 0 & S_i(q_i) \\ -S_i^T(q_i) & -D_i^T \end{pmatrix} \begin{pmatrix} \frac{\partial H_i}{\partial q_i} \\ \frac{\partial H_i}{\partial p_i} \end{pmatrix} + \begin{pmatrix} 0 \\ \bar{G}_i(q_i) \end{pmatrix} \bar{u}_i,$$

$$\bar{y}_i = \bar{G}_i^T(q_i) \frac{\partial H_i}{\partial p_i},$$

where the Hamiltonian $H_i^T$ remains the same as in (4.5). To compactly write the dynamics of $N$ robots with new input $\bar{u} = (\bar{u}_1, \ldots, \bar{u}_N)^T$ and output $\bar{y} = (\bar{y}_1, \ldots, \bar{y}_N)^T$ define $\bar{G}(q) = \text{block.diag}(\bar{G}_1(q_1), \ldots, \bar{G}_N(q_N))$, such that the dynamics for $N$
robots are given by

\[
\begin{pmatrix}
\dot{q} \\
\dot{p}
\end{pmatrix} = \begin{pmatrix}
0 & S(q) \\
-S^T(q) & -D_r
\end{pmatrix} \begin{pmatrix}
\frac{\partial H^r}{\partial q} \\
\frac{\partial H^r}{\partial p}
\end{pmatrix} + \begin{pmatrix}
0 \\
\tilde{G}(q)
\end{pmatrix} \bar{u},
\]

with Hamiltonian

\[
H^r = \sum_{i=1}^{N} H_i^r(p_i) = \frac{1}{2} p^T M^r p.
\]

\[\text{4.3 Formation control}\]

The formation control problem was already introduced in Section 3.4. The main objective in formation control is to achieve a prescribed geometrical shape for the network of agents. The main difference with the results presented in Section 3.4 is the type of systems considered. In Section 3.4 formation control for a network of fully actuated agents in the presence of ideal Coulomb friction is considered, while here the focus is on a network of nonholonomic wheeled robots.

Formation control for a network of wheeled robots is achieved by assigning virtual couplings between the front ends (point \((x_{B,i}, y_{B,i})\) in Figure 4.1) of the robots. This differs from the previous chapter, where the springs were assigned between the centers of mass of the agents. This change of point of action is required to deal with the nonholonomic constraint on the wheel axle of the robot.

To formally define the formation control objective, let \(z_j, z_j^* \in \mathbb{R}^2\) denote respectively the relative displacement between two wheeled robots and the desired relative displacement. For \(E\) virtual couplings define the collocated vectors \(z = (z_1, \ldots, z_E)^T\) and \(z^* = (z_1^*, \ldots, z_E^*)^T\). Then, the formation control objective is formally defined as

\[
\begin{cases}
 p \to 0, \\
z \to z^*
\end{cases}, \quad \text{as } t \to \infty.
\]

Note that the formation control objective (4.11) is exactly the same as the objective stated in (3.3). The results presented here to achieve (4.11) are published in [113, 115].

\[\text{4.3.1 Formation control of wheeled robots using virtual springs}\]

To achieve formation control the \(N\) wheeled robots are interconnected using \(E\) virtual couplings (i.e., virtual springs and dampers) in a similar way as Chapter 3. A tree graph (i.e., an undirected connected acyclic graph) is used to describe the interconnection topology: the nodes of the graph correspond to robots, while the edges of the graph correspond to virtual couplings.
Let $z_j \in \mathbb{R}^2$ denote the length along the $x$ and $y$ direction of the virtual coupling. The input to the control system $w_j \in \mathbb{R}^2$ is a velocity, while the corresponding output $\tau_j \in \mathbb{R}^2$ is a force. Furthermore let $D_j^c \in \mathbb{R}^{2 \times 2}$ denote the corresponding virtual dissipation matrix, defined as $D_j^c = \text{diag}(d^c_{x,j}, d^c_{y,j})$. The dynamics of virtual coupling $j$ are given by 

$$
\dot{z}_j = w_j,
\tau_j = \frac{\partial H_j^c}{\partial z_j}(z_j) + D_j^c w_j,
$$

(4.12)

with Hamiltonian $H_j^c(z_j)$. The Hamiltonian $H_j^c(z_j)$ equals the potential energy stored in virtual spring $j$ which is given by

$$
H_j^c(z_j) = \frac{1}{2} (z_j - z_j^*)^T K_j^c (z_j - z_j^*),
$$

where $z_j^* = (z_{x,j}^*, z_{y,j}^*)^T$ denotes the nominal spring length and $K_j^c = \text{diag}(k_{x,j}^c, k_{y,j}^c)$ denotes the virtual spring constant matrix. Defining the springs in this way corresponds to position-based control in terms of [3], which implies that not only the inter-robot displacement, but also the inter-robot heading is controlled.

To compactly write the dynamics of $E$ virtual couplings of the form (4.12), define the collocated vectors $z = (z_1, \ldots, z_E)^T$, $z^* = (z_{1}^*, \ldots, z_{E}^*)^T$, $w = (w_1, \ldots, w_E)^T$, $\tau = (\tau_1, \ldots, \tau_E)^T$, and system matrices $K^c = \text{block.diag}(K_1^c, \ldots, K_E^c)$, $D^c = \text{block.diag}(D_1^c, \ldots, D_E^c)$. Then the dynamics of $E$ virtual couplings of the form (4.12) are given by

$$
\dot{z} = w,
\tau = \frac{\partial H^v}{\partial z}(z) + D^c w,
$$

(4.13)

with Hamiltonian $H^c(z) = \sum_{j=1}^E H_j^c(z_j) = \frac{1}{2} (z - z^*)^T K^c (z - z^*)$.

The incidence matrix $B$ of the graph describes which robots (nodes) are interconnected by virtual couplings (edges). The corresponding coupling of robots on the nodes and virtual couplings at the edges is given by [3, 100]

$$
\left\{
\begin{array}{l}
\bar{u} = -(B \otimes I_2) \tau, \\
w = (B^T \otimes I_2)^T \bar{y}.
\end{array}
\right.
$$

(4.14)

The closed-loop network dynamics are obtained by eliminating the interconnection constraint (4.14) using the wheeled robot dynamics (4.10) and virtual
coupling dynamics (4.13) and are given by

\[
\begin{pmatrix}
\dot{q} \\
\dot{p} \\
\dot{z}
\end{pmatrix} =
\begin{pmatrix}
0 & S(q) & 0 \\
-S^T(q) & -(D^r + BD^cB^T) & -\tilde{G}(q)(B \otimes I_2) \\
0 & (B \otimes I_2)^T \tilde{G}^T(q) & 0
\end{pmatrix}
\begin{pmatrix}
\partial H \\
\partial H \\
\partial H
\end{pmatrix},
\tag{4.15}
\]

with closed-loop Hamiltonian \(H(p, z) = H^r(p) + H^c(z)\).

The main result of this section is now presented in the following theorem.

**Theorem 4.2.** Interconnect wheeled robots (4.10) with virtual couplings (4.13) via coupling (4.14) using a tree graph topology. Then, the solutions to the closed-loop system (4.15) converge to \(p = 0, z = z^*\), thereby achieving the control objectives (4.11).

**Proof.** Take the closed-loop Hamiltonian \(H(p, z)\) as a candidate Lyapunov function. Since \(H(p, z)\) is quadratic in \(p\) and \(z\), it follows that \(H(x) \geq 0\) for all \(p \in \mathbb{R}^{2N}, z \in \mathbb{R}^{2E}\). The time derivative \(\dot{H}(p, z)\) follows from (4.15) and is given by

\[
\dot{H}(p, z) = -\frac{\partial^T H (p)}{\partial p} (D^r + BD^cB^T) \frac{\partial H}{\partial p} (p) \leq 0.
\]

Invoking LaSalle’s invariance principle (Theorem 2.4) gives that (4.15) converges to the largest invariant set where \(\dot{H}(p, z) = 0\). On this set \(\frac{\partial H}{\partial p} (p) = 0\), which implies that \(p = 0\) and \(\dot{p} = 0\). Substituting \(p = 0, \dot{p} = 0\) into (4.15) and rearranging the terms along the \(x\) and \(y\) direction gives

\[
- \cos \phi B K^c_x (z_x - z^*_x) - \sin \phi B K^c_y (z_y - z^*_y) = 0, \tag{4.16}
\]

\[
D_{AB} \sin \phi B K^c_x (z_x - z^*_x) - D_{AB} \cos \phi B K^c_y (z_y - z^*_y) = 0, \tag{4.17}
\]

where \(z_x = (z_{x,1}, \ldots, z_{x,E})^T\), \(z_y = (z_{y,1}, \ldots, z_{y,E})^T\) and \(K^c_x = \text{diag}(k^c_{x,1}, \ldots, k^c_{x,E})\), \(K^c_y = \text{diag}(k^c_{y,1}, \ldots, k^c_{y,E})\), \(\sin \phi = \text{diag}(\sin \phi_1, \ldots, \sin \phi_N)\), \(\cos \phi = \text{diag}(\cos \phi_1, \ldots, \cos \phi_N)\), \(D_{AB} = \text{diag}(d_{AB,1}, \ldots, d_{AB,N})\).

Multiplying (4.16) from the left by \(D_{AB} \cos \phi\), (4.17) by \(\sin \phi\), and summing the result gives

\[
D_{AB} B K^c_x (z_x - z^*_x) = 0. \tag{4.18}
\]

Since \(D_{AB}\) is a positive definite diagonal matrix, it follows from (4.18) that \(K^c_x(z_x - z^*_x) \in \ker B\). Noting that \(K^c_x\) is also a positive definite diagonal matrix and \(\ker B = 0\) it follows from (4.18) \(z_x = z^*_x\). In a similar fashion, by multiplying (4.16) from the left by \(D_{AB} \sin \phi\), (4.17) by \(\cos \phi\), and summing the result it follows that \(z_y = z^*_y\).

This completes the proof. \(\square\)

**Remark 4.3** (Cyclic graphs). In this approach, the graph topology is a design freedom and considering only acyclic graphs is not restrictive. Cyclic graphs might
give rise to undesired equilibria [9]. A work around to prevent undesired equilibria is to have an a priori condition of the form $K_x(z_x - z_{x}^*) \in \ker B, K_y(z_y - z_{y}^*) \in \ker B$ (see [3, 100]). In contrast, Chapter 5 does consider cyclic graphs for modeling the interaction amongst satellites on a (circular) orbit.

Remark 4.4 (Control law formation control). The resulting control law $u$ for the robots is easily obtained from (4.8) and (4.14) as

$$ u = \bar{G}(q)\bar{u} = \bar{G}(q) (B \otimes I_2) \frac{\partial H}{\partial z}(z) - \bar{G}(q) (B \otimes I_2) D^c (B \otimes I_2)^T \frac{\partial H}{\partial p}(p) $$

$$ = -\bar{G}(q) (B \otimes I_2) K^c(z - z^*) - \bar{G}(q) (B \otimes I_2) D^c (B \otimes I_2)^T M^{-1} p. $$

Due the use of virtual couplings in the port-Hamiltonian framework, the two terms in (4.19) have a clear physical interpretation. The virtual spring forces along the $x$ and $y$ direction ensures that the formation control objectives (4.11) are achieved, while the virtual damping forces can be used to shape the transient response. The input matrix $\bar{G}(q)$ transforms these virtual spring and damper forces into inputs for the wheeled robots (see equation (4.8)).

Furthermore, note that (4.19) is a distributed control law. Each agent only requires measurements on the relative displacement $z$ and relative velocity $\dot{z}$ with respect to its two neighbors, which are characterized by the graph topology.

Simulation and experimental results are presented in the next section to illustrate the effectiveness of Theorem 4.2.

### 4.3.2 Simulation and experimental results

Consider a network of $N = 3$ wheeled robots of the form (4.5), with model parameters $m_i = 0.167 \, \text{kg}$, $I_{cm,i} = 9.69 \cdot 10^{-5} \, \text{kg} \, \text{m}^2$, $d_{f,i} = 2 \, \text{kg} / \text{s}$, $d_{\phi,i} = 0.2 \, \text{kg} \, \text{m}^2 / \text{s}$, $d_{AB,i} = 0.06 \, \text{m}$ for $i = 1, 2, 3$. The model parameters are chosen in accordance with the e-puck wheeled robot (Figure C.1) which are given in Table C.1. The three robots are interconnected using the incidence matrix

$$ B = \begin{pmatrix} -1 & 0 \\ +1 & -1 \\ 0 & +1 \end{pmatrix}, $$

which is associated to a so-called path graph. Since each robot is strictly passive (i.e., $d_{f,i} > 0, d_{\phi,i} > 0$) no virtual dampers are required for asymptotic stability ($d^c_{x,j} = d^c_{y,j} = 0 \, \text{kg} / \text{s}$ for $j = 1, 2$). Hence, each virtual coupling only corresponds to a virtual spring with spring parameters $z^*_{x,j} = 0.4 \, \text{m}$, $z^*_{y,j} = 0 \, \text{m}$, $k^c_{x,j} = k^c_{y,j} =$
4. Formation control of nonholonomic wheeled robots

![Figure 4.3: Trajectories of the e-puck wheeled robots for the simulation (dashed) and the experiment (solid). The crosses (×) and circles (○) denote the initial and final positions respectively.](image)

Two kg/s², for \( j = 1, 2 \). This choice for the graph topology and \( z^*_x, z^*_y \) yields a line formation. Note that this formation might look similar to the experimental results in Chapter 3 for formation control and deployment. However, other formation shapes (e.g. star and zig-zag formation) are also possible within the current setup and here the line formation was chosen because of its simplicity.

The simulations are performed using MATLAB and Simulink, while the experiments make use of the experimental setup discussed in Appendix A. The experimental setup consists of a 2.6 x 2.0 m table with an overhead camera for localization. Each robot is identified and localized using a data-matrix, attached on top of the e-puck (see Figure C.1 (right)). A vision algorithm runs in parallel to MATLAB and provides the ID, position, and heading of each e-puck. MATLAB calculates the corresponding control inputs (4.19) and sends them to the e-pucks via a Bluetooth connection.

Both the simulation and the experiment were run for \( t = 50 \) s, starting from the same initial conditions: \( x_B(0) = (0.87, 0.99, 1.02) \) m, \( y_B(0) = (0.39, 0.82, 1.30) \) m, \( \phi(0) = (6.23, 3.05, 3.02) \) rad, \( p_f(0) = (0, 0, 0) \) kg m/s, \( h(0) = (0, 0, 0) \) kg m²/s. The results are shown in Figures 4.3, 4.4, and 4.5.

Figure 4.3 shows the trajectories of the e-pucks for the simulation (dashed) and the experiment (solid). The dotted black lines represent the virtual springs at the final position. Even though there is a difference between the trajectories, the
4.4. Formation control with velocity tracking

The goal of this approach is to make the wheeled robot (4.5) track a reference velocity along a prescribed heading angle, while converging to a desired formation shape. In addition to the formation control design in Section 4.3, this section consider an additional velocity tracking and heading controller. The velocity tracking controller is a local controller which makes each wheeled robot track a predefined forward velocity \( v^*_i \) and builds upon generalized canonical transformations [45].

To ensure that the movement of the robots is along the desired heading \( \phi^*_i \), an additional heading controller is proposed. In this work the desired heading is assumed to be along the \( y \)-axis (see Figure 4.1), such that \( \phi^*_i = \frac{\pi}{2} \) (see Figure 4.6).

To achieve a desired formation shape virtual couplings are assigned between the agents. Each virtual coupling consists of a virtual spring, which steers robots to the desired formation, and a virtual damper, which may be used to shape the transient response. The interconnection topology (i.e., which robots are interconnected to each other by a virtual coupling) is modeled by a tree graph (i.e., an undirected connected acyclic graph). Each edge is labeled with a positive and a negative end arbitrarily. Consider two robots \( i \) and \( j \) which are interconnected using virtual coupling \( k \), where robot \( j \) is at the positive end of the corresponding edge. The relative displacement \( z_k \) of robot \( j \) with respect to robot \( i \) is defined as \( z_k = (x_{B,j},y_{B,j})^T - (x_{B,i},y_{B,i})^T \). Let \( z = (z_1,\ldots,z_E)^T \) and let \( z^* = (z^*_1,\ldots,z^*_E)^T \) denote the desired relative displacement.

The objectives of formation control with velocity tracking are now formally stated in terms of (4.7) as

\[
\begin{align*}
\phi \rightarrow \phi^*, \\
p \rightarrow M^*v^*, \quad \text{as } t \rightarrow \infty, \\
z \rightarrow z^*,
\end{align*}
\] (4.20)
Figure 4.4: Time evolution of the forward velocity $v_f$ and relative position $z_x$ for formation controller (4.19) (simulation). The dotted lines show the reference values.

Figure 4.5: Time evolution of the forward velocity $v_f$ and relative position $z_x$ for formation controller (4.19) (experiment). The dotted lines show the reference values.
where \( v^* \) denotes the reference velocity to be defined later on. There are two differences with respect to the formation control objectives (4.11). In (4.11) the momentum \( p \) should converge zero, while in (4.20) the momentum \( p \) should converge to a nonzero constant \( M^r v^* \). Furthermore, (4.11) imposes no requirement on the robots’ heading, while in (4.20) the heading \( \phi \) of each robot should converge to a desired value \( \phi^* \).

The results presented here to achieve (4.20) are published in [113].

4.4.1 Velocity tracking control and formation control using virtual couplings

The heading and velocity tracking controller are local controllers. Therefore these two controllers are developed for a single robot \( i \). The formation controller is a distributed controller, where (some) robots exchange local information. Section 4.4.1 presents the design and analysis of the whole network of \( N \) robots.

The control input \( u \) designed in this section consists of three parts (i.e., \( u = u_1 + u_2 + u_3 \)). The three control input signals \( u_1, u_2, u_3 \) correspond to respectively the heading controller, velocity tracking controller, and formation controller which are designed below. Since the heading and velocity tracking controllers are local controllers, they provide control inputs \( u_{1i} \) and \( u_{2i} \) respectively.

**Heading control**

Each robot is required to track a forward reference velocity along the \( y \)-axis (see Figure 4.1). The desired heading for each robot is therefore set at \( \phi_i^* = \frac{\pi}{2} \) (see Figure 4.6). The heading controller assigns a nonlinear spring to the heading of each robot, which guarantees convergence to the desired heading under the following natural assumption on the initial heading:

**Assumption 4.5.** The initial heading \( \phi_i(0) \) of robot \( i \) is contained in the interval \( \phi_i(0) \in (0, \pi) \).

Let \( \hat{\phi}_i = \phi_i - \phi_i^* \) denote the error state of the heading controller, then the
heading control dynamics are given by
\[ \dot{\hat{\phi}}_i = u^h_i, \]  
\[ y^h_i = \frac{\partial H^h_i}{\partial \hat{\phi}_i}, \]  
(4.21)
with \( u^h_i \) the input and \( y^h_i \) the output of the control system. Hamiltonian \( H^h_i(\hat{\phi}_i) \) is defined as
\[ H^h_i(\hat{\phi}_i) := -k_\phi^i \ln |\cos \hat{\phi}_i|, \]
with \( k_\phi^i > 0 \) the controller gain. This nonlinear Hamiltonian guarantees that under Assumption 4.5 it follows that \( \phi_i(t) \in (0, \pi) \) for all \( t \) (see further on for the proof). On the interval \( \hat{\phi}_i \in (-\pi/2, \pi/2) \) it is easy to verify that \( H^h_i(\hat{\phi}_i) \geq 0 \) since \( |\cos \hat{\phi}_i| \leq 1 \) and hence \( \ln |\cos \hat{\phi}_i| < 0 \). Note that the partial derivative of \( H^h_i \) with respect to \( \hat{\phi}_i \) is simply \( \frac{\partial H^h_i}{\partial \hat{\phi}_i} = k_\phi^i \tan \hat{\phi}_i \).

The coupling of robot (4.5) to the heading controller (4.21) is given by
\[
\begin{align*}
\begin{cases}
u_{1i} = -L_i y^h_i, \\
u^h_i = L^T_i y_i,
\end{cases}
\end{align*}
\]  
(4.22)
where \( L^T_i = (0 \ 1) \). Before stating the proposition on the heading controller, first consider the following natural assumption on the initial heading \( \phi_i(0) \).

**Remark 4.6 (Relaxation of Assumption 4.5).** The interval \( (0, \pi) \) in Assumption 4.5 is used for simplicity of notation. This interval may be 'stretched' to e.g. the interval \( (-\pi/2, 3\pi/2) \) resulting however in a more complicated expression.

The following proposition provides the result for the heading controller.

**Proposition 4.7.** Consider the system (4.5). Using \( u_i = u_{1i} \), with \( u_{1i} \) defined in (4.21)-(4.22), the system (4.5) converges to \( \phi_i = \phi^*_i, p_i = 0 \).

**Proof.** Let \( \tilde{q}_i = (x_{A,i}, y_{A,i}, \hat{\phi}_i)^T \), then setting \( u_i = u_{1i} \) the closed-loop dynamics follow from (4.5), (4.21), (4.22)
\[
\begin{pmatrix}
\dot{\tilde{q}}_i \\
\dot{p}_i
\end{pmatrix} = \begin{pmatrix} 0 & \tilde{S}_i(\tilde{q}_i) \\
-\tilde{S}_i^T(\tilde{q}_i) & -D^r_i \end{pmatrix} \begin{pmatrix} \frac{\partial H^r_i}{\partial \tilde{q}_i} \mapsto \partial H^r_i \partial \tilde{p}_i \end{pmatrix},
\]  
(4.23)
with
\[
\tilde{S}_i(\tilde{q}_i) = \begin{pmatrix} \cos(\hat{\phi}_i + \phi^*_i) & 0 \\
\sin(\hat{\phi}_i + \phi^*_i) & 0 \\
0 & 1 \end{pmatrix},
\]
and Hamiltonian

\[ \tilde{H}^r_i(\hat{\phi}_i, p_i) = H^h_i(\hat{\phi}_i) + H^r_i(p_i) \]
\[ = -k^\phi_i \ln |\cos \hat{\phi}_i| + \frac{1}{2} p_i^T (M^r_i)^{-1} p_i. \]

Take the Hamiltonian \( \tilde{H}^r_i(\hat{\phi}_i, p_i) \) as a Lyapunov candidate function and calculate its time derivative as

\[ \dot{\tilde{H}}^r_i = -\frac{\partial^T \tilde{H}^r_i}{\partial p_i} D_i \frac{\partial \tilde{H}^r_i}{\partial p_i} \leq 0. \]

Invoking LaSalle’s invariance principle one obtains from \( \dot{\tilde{H}}^r_i = 0 \) that \( \frac{\partial \tilde{H}^r_i}{\partial p_i} = 0 \) and thus \( p_i = 0 \) and \( \dot{p}_i = 0 \). Substituting into (4.23) yields

\[ -\tilde{S}^T_i(\hat{q}_i) \frac{\partial \tilde{H}^r_i}{\partial \hat{q}_i} = \begin{pmatrix} \cos(\hat{\phi}_i + \phi^*_i) & \sin(\hat{\phi}_i + \phi^*_i) & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial \tilde{H}^r_i}{\partial x_{A,i}} \\ \frac{\partial \tilde{H}^r_i}{\partial y_{A,i}} \\ \frac{\partial \tilde{H}^r_i}{\partial \hat{\phi}_i} \\ \frac{\partial \tilde{H}^r_i}{\partial \phi_i} \end{pmatrix} = 0, \]

which implies that \( \frac{\partial \tilde{H}^r_i}{\partial \phi_i} = 0 \), implying that \( k^\phi_i \tan \hat{\phi}_i = 0 \).

From Assumption 4.5 and \( \phi^*_i = \frac{\pi}{2} \) it follows that \( \hat{\phi}_i(0) \in (-\frac{\pi}{2}, \frac{\pi}{2}) \). Since \( \tilde{H}^r_i(t) \leq \tilde{H}^r_i(0) \leq M < \infty \) for all \( t \) and \( \tilde{H}^r_i \to \infty \) as \( \hat{\phi}_i \to \pm \frac{\pi}{2} \), it follows that \( \hat{\phi}_i(t) \in (-\frac{\pi}{2}, \frac{\pi}{2}) \) for all \( t \). Then, it immediately follows that \( \hat{\phi}_i = 0 \) since \( \hat{\phi}_i = 0 \) is the only value on the interval \( \hat{\phi}_i(t) \in (-\frac{\pi}{2}, \frac{\pi}{2}) \) for which \( k^\phi_i \tan \hat{\phi}_i = 0 \). Since \( \hat{\phi}_i = 0 \) implies \( \phi_i = \phi^*_i \), this completes the proof. \( \square \)

The next subsection continues with the velocity tracking control.

**Velocity tracking control**

The design of the velocity tracking controller is based on the use of generalized canonical coordinate transformations \([45]\). The main idea is to first derive the error dynamics w.r.t. the reference velocity and then to stabilize these dynamics. In this section, a constant forward velocity is considered, where the wheeled robots move along a straight line. Let \( v^*_i \in \mathbb{R}^2 \) denote the desired velocity and define the error variables

\[ \begin{pmatrix} \tilde{q}_i \\ \tilde{p}_i \end{pmatrix} = \begin{pmatrix} \tilde{q}_i - \tilde{S}_i(\hat{q}_i)v^*_i t \\ p_i - M^r_i v^*_i \end{pmatrix} =: \Phi_i(q_i, p_i, t), \]

(4.24)

where \( v^*_i = (v^*_{f,i}, 0)^T \) with forward reference velocity \( v^*_{f,i} \in \mathbb{R} \). Note that the heading \( \hat{\phi}_i \) and angular momentum \( h_i \) are not affected by (4.24) since \( h^*_i = 0 \) (i.e.,
\[ \ddot{\phi}_i = \hat{\phi}_i, \bar{h}_i = h_i. \] The corresponding error Hamiltonian is defined as

\[ \bar{H}^r_i(\bar{q}_i, \bar{p}_i) := -k_i^\phi \ln |\cos \bar{\phi}_i| + \frac{1}{2} \bar{p}_i^T (M_i^r)^{-1} \bar{p}_i = -k_i^\phi \ln |\cos \bar{\phi}_i| + \frac{1}{2} p_i^T (M_i^r)^{-1} p_i - p_i^T v_i^r + \frac{1}{2} v_i^r^T M_i^r v_i^r, \]  

(4.25)

where \( U_i(p_i) \) is a fictitious potential. Now consider the following proposition.

**Proposition 4.8.** Define \( u_{2i} = -\beta - D_i^l (y_i + \alpha_i) \) with \( \beta_i(p_i) = D_i^r v_i^r, \alpha_i = \frac{\partial U_i}{\partial p_i} = -(v_i^r, 0)^T, \) and \( D_i^l \geq \epsilon I_2 > 0 \in \mathbb{R}^{2 \times 2}. \) Using \( u_i = u_{1i} + u_{2i} \), the system (4.5) converges to \( \phi_i = \phi_i^*, p_i = M_i^r v_i^r. \)

**Proof.** To verify whether \( \Phi_i \) yields a generalized canonical transformation, substitute \( \Phi_i, U_i(p_i) \) and \( \beta_i = D_i^r v_i^r \) into PDE (2.13). It is easily checked that the PDE holds and the error dynamics of (4.23) w.r.t. the reference velocity are given by

\[ \begin{pmatrix} \dot{\bar{q}}_i \\ \dot{\bar{p}}_i \end{pmatrix} = \begin{pmatrix} 0 & \bar{S}^T_i(\bar{q}_i) \\ -\bar{S}^T_i(\bar{q}_i) & - (D_i^r + D_i^l) \end{pmatrix} \begin{pmatrix} \frac{\partial \bar{H}^r_i}{\partial \bar{q}_i} \\ \frac{\partial \bar{H}^r_i}{\partial \bar{p}_i} \end{pmatrix}, \]  

(4.26)

where

\[ \bar{S}_i(\bar{q}_i) = \begin{pmatrix} \cos(\bar{\phi}_i + \phi_i^*) & 0 \\ \sin(\bar{\phi}_i + \phi_i^*) & 0 \end{pmatrix}. \]

Note that matrix \( \bar{S}_i(\bar{q}_i) \) follows directly from (2.14). Take the error Hamiltonian \( \bar{H}^r_i(\bar{q}_i, \bar{p}_i) \) as a candidate Lyapunov function. It is easily verified that \( \bar{H}^r_i(\bar{q}_i, \bar{p}_i) \geq 0 \) and that the time derivative is given by

\[ \dot{\bar{H}}^r_i = -\frac{\partial^T \bar{H}^r_i}{\partial \bar{p}_i} (D_i^r + D_i^l) \frac{\partial^T \bar{H}^r_i}{\partial \bar{p}_i}. \]

Invoking LaSalle’s invariance principle (Theorem 2.4) gives that (4.26) converges to the largest invariant set where \( \dot{\bar{H}}^r_i = 0. \) On this set \( \frac{\partial^T \bar{H}^r_i}{\partial \bar{p}_i} = 0, \) which implies that \( \bar{p}_i = 0 \) and \( \dot{\bar{p}}_i = 0. \) Substituting \( \bar{p}_i = 0, \dot{\bar{p}}_i = 0 \) into (4.26) gives

\[ -\bar{S}^T_i(\bar{q}_i) \frac{\partial \bar{H}^r_i}{\partial \bar{q}_i} = 0. \]

Using similar arguments as in the proof of Proposition 4.7 it immediately follows that \( \ddot{\phi} = 0, \) thereby completing the proof. \( \square \)
Remark 4.9 (Time-varying velocities). Instead of a constant forward velocity \( v^*_i \) also time-varying forward velocities \( v^*_i(t) \) can be incorporated in the approach. Then transformation (4.24) needs to be redefined accordingly and the state feedback term \( \beta_i(p_i) \) requires an additional term \( M_i^T v^*_i(t) \), corresponding to the time derivative of the reference velocity \( v^*_i(t) \). The details are left to the reader.

Finally, the next section deals with the formation keeping controller.

Formation control using virtual couplings

The third controller is a distributed formation controller, where robots exchange local information. This controller assigns virtual couplings between the front ends of the wheeled robots (point \((x_{B,i}, y_{B,i})\) in Figure 4.1) [115]. Each virtual coupling consists of a virtual spring and a virtual damper. The virtual coupling dynamics are given in (4.13).

The way in which robots are interconnected by a virtual coupling is modeled by a tree graph \( G(V, E) \). The node-set \( V \) corresponds to \( N \) wheeled robots, while the edge-set \( E \) corresponds to \( E \) virtual couplings. Let \( B \) denote the incidence matrix associated to \( G(V, E) \) then the coupling to the formation controller is given by [3, 100]

\[
\begin{align*}
\begin{cases}
  u_3 = -\bar{G}(\phi) (B \otimes I_2) \tau, \\
  w = (B^T \otimes I_2) \bar{G}^T(\phi) y,
\end{cases}
\end{align*}
\]

(4.27)

with \( u_3 = (u_{31}, \ldots, u_{3N})^T \), \( y = (y_1, \ldots, y_N)^T \), \( w = (w_1, \ldots, w_E)^T \), \( \tau = (\tau_1, \ldots, \tau_E)^T \). The only difference with respect to (4.14) is matrix \( G(\phi) \), which assigns the virtual couplings to the front ends of robots in a similar way as (4.8) transforms the inputs and outputs from (4.5) to (4.9). Recall that \( \bar{G}(\phi) = \text{block.diag}(\bar{G}_1(\phi_1), \ldots, \bar{G}_N(\phi_N)) \), where

\[
\bar{G}_i(\phi_i) = \begin{pmatrix}
\cos \phi_i & \sin \phi_i \\
-d_{AB,i} \sin \phi_i & d_{AB,i} \cos \phi_i
\end{pmatrix}.
\]

The main result of this approach is now stated as follows.

Theorem 4.10. Using \( u = u_1 + u_2 + u_3 \), where coupling (4.27) uses a tree graph topology, the system (4.5) converges to \( \phi = \phi^*, p = M^r v^*, z = z^* \), thereby achieving the control goals (4.20).

Proof. For simplicity of notation let \( \bar{z}_j = z_j - z^*_j \). To analyze the closed-loop dynamics of the network, let \( \bar{q} = (\bar{q}_1, \ldots, \bar{q}_N)^T \), \( \bar{p} = (\bar{p}_1, \ldots, \bar{p}_N)^T \), \( \bar{z} = (\bar{z}_1, \ldots, \bar{z}_E)^T \), and define matrices \( D^r = \text{diag}(D^r_1, \ldots, D^r_N) \), \( D^t = \text{diag}(D^t_1, \ldots, D^t_N) \), \( D^c = \text{diag}(D^c_1, \ldots, D^c_E) \), \( K^c = \text{diag}(K^c_1, \ldots, K^c_E) \), \( \bar{S}(\bar{q}) = \text{block.diag}(\bar{S}_1(\bar{q}_1), \ldots, \bar{S}_N(\bar{q}_N)) \).
From (4.13), (4.26), (4.27) the closed-loop dynamics are obtained as

\[
\begin{pmatrix}
\dot{\bar{\varphi}} \\
\dot{\bar{p}} \\
\dot{\bar{z}}
\end{pmatrix} =
\begin{pmatrix}
0 & \bar{S}(\bar{q}) & 0 \\
-\bar{D}(\bar{q}) & (B^T \otimes I_2) \bar{G}^T(\bar{q}) & 0 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\frac{\partial \bar{H}}{\partial \bar{q}} \\
\frac{\partial \bar{H}}{\partial \bar{p}} \\
\frac{\partial \bar{H}}{\partial \bar{z}}
\end{pmatrix},
\]

(4.28)

with \( \bar{D}(\bar{q}) = D^r + D^t + \bar{G}(\bar{q})(B \otimes I_2)D^c(B^T \otimes I_2)\bar{G}^T(\bar{q}) \). Note that in (4.28) the controller state \( \bar{\phi} \) is contained in \( \bar{q} \). The closed-loop Hamiltonian \( \bar{H} \) is given by

\[
\bar{H}(\bar{\varphi}, \bar{p}, \bar{z}) = \bar{H}^r(\bar{\varphi}, \bar{p}) + H^s(\bar{z})
= -\sum_{i=1}^{N} k_{\phi}^i \ln |\cos \bar{\phi}_i| + \frac{1}{2} \bar{p}^T M^{-1} \bar{p} + \frac{1}{2} \bar{z}^T K^c \bar{z}.
\]

Take the closed-loop Hamiltonian \( \bar{H}(\bar{q}, \bar{p}, \bar{z}) \) as a candidate Lyapunov function. It is easily verified that \( \dot{\bar{H}}(\bar{q}, \bar{p}, \bar{z}) \geq 0 \) and that the time derivative is given by

\[
\dot{\bar{H}}(\bar{q}, \bar{p}, \bar{z}) = -\frac{\partial^T \bar{H}}{\partial \bar{p}} \bar{D}(\bar{q}) \bar{p}^T \frac{\partial \bar{H}^r}{\partial \bar{p}}.
\]

Invoking LaSalle’s invariance principle (Theorem 2.4) gives that (4.26) converges to the largest invariant set where \( \dot{\bar{H}} = 0 \). On this set \( \frac{\partial^T \bar{H}}{\partial \bar{p}} = 0 \), which implies that \( \bar{p} = 0 \) and \( \dot{\bar{p}} = 0 \). Substituting \( \bar{p} = 0, \dot{\bar{p}} = 0 \) into (4.28) yields

\[
-\bar{S}(\bar{q}) \frac{\partial \bar{H}}{\partial \bar{q}} - \bar{G}(\bar{q})(B \otimes I_2) \frac{\partial \bar{H}}{\partial \bar{z}} = 0.
\]

(4.29)

Rearranging the terms, (4.29) can be rewritten as

\[
\cos(\bar{\phi} + \phi^*) BK_x \bar{z}_x + \sin(\bar{\phi} + \phi^*) B K_y \bar{z}_y = 0,
\]

(4.30)

\[K^\phi \tan \bar{\phi} - D_{AB} \sin(\bar{\phi} + \phi^*) B K_x \bar{z}_x + D_{AB} \cos(\bar{\phi} + \phi^*) B K_y \bar{z}_y = 0,
\]

(4.31)

with \( K^\phi = \text{diag}(k_{\phi}^1, \ldots, k_{\phi}^N) \), \( \tan \bar{\phi} = (\tan \bar{\phi}_1, \ldots, \tan \bar{\phi}_N)^T \), and \( D_{AB} = \text{diag}(d_{AB,1}, \ldots, d_{AB,N}) \). Now multiply (4.30) by \( \sin(\bar{\phi} + \phi^*) \), multiply (4.31) by \( \cos(\bar{\phi} + \phi^*) \) and sum the result to obtain

\[
\cos(\bar{\phi} + \phi^*) D_{AB}^{-1} K^\phi \tan \bar{\phi} + B K_y \bar{z}_y = 0.
\]

(4.32)

Since the graph is connected, it follows that \( \ker B^T = \alpha I_N \) with \( \alpha \) some arbitrary
constant. Multiplying (4.32) by $\mathbf{1}^T_N$ from the left results in

$$
\mathbf{1}^T_N \cos(\bar{\phi} + \phi^*) D^{-1}_{AB} K^\phi \tan \bar{\phi} = 0.
$$

(4.33)

Then, since $\cos(\bar{\phi} + \phi^*)$, $D_{AB}$, and $K^\phi$ are diagonal matrices, (4.33) can be rewritten element wise as

$$
\sum_{i=1}^{N} \frac{k_i^\phi}{d_{AB,i}} \cos(\bar{\phi}_i + \phi^*_i) \tan \bar{\phi}_i = 0.
$$

(4.34)

To prove that (4.34) implies that $\bar{\phi}_i = 0$, first show that under Assumption 4.5 it follows that $\bar{\phi}_i(t) \in (-\frac{\pi}{2}, \frac{\pi}{2})$ for all $t$. From Assumption 4.5 and $\phi^*_i = \frac{\pi}{2}$ it follows that $\bar{\phi}_i(0) = \phi_i(0) - \phi^*_i \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Then, since $H(t)$ is bounded from above ($H(t) \leq H(0) \leq M < \infty$ for all $t$) and $H_i \to \infty$ as $\bar{\phi}_i \to \pm \frac{\pi}{2}$ it follows that $\bar{\phi}_i(t) \in (-\frac{\pi}{2}, \frac{\pi}{2})$ for all $t$.

Now it is proven that (4.34) implies that $\bar{\phi}_i = 0$ by contradiction. Assume that $\bar{\phi}_i \neq 0$, then either $\bar{\phi}_i \in (-\frac{\pi}{2}, 0)$ or $\bar{\phi}_i \in (0, \frac{\pi}{2})$ since $\bar{\phi}_i(t) \in (-\frac{\pi}{2}, \frac{\pi}{2})$ for all $t$. For the case that $\bar{\phi}_i \in (-\frac{\pi}{2}, 0)$ it immediately follows that $\cos(\bar{\phi}_i + \phi^*_i) > 0$ while $\tan \bar{\phi}_i < 0$. On the other hand for $\bar{\phi}_i \in (0, \frac{\pi}{2})$ it follows that $\cos(\bar{\phi}_i + \phi^*_i) < 0$ and $\tan \bar{\phi}_i > 0$. Hence, if $\bar{\phi}_i \neq 0$ always $\cos(\bar{\phi}_i + \phi^*_i) \tan \bar{\phi}_i < 0$ for all $i = 1, \ldots, N$.

Substituting into (4.34) gives

$$
\sum_{i=1}^{N} \frac{k_i^\phi}{d_{AB,i}} \cos(\bar{\phi}_i + \phi^*_i) \tan \bar{\phi}_i < 0,
$$

which contradicts (4.34) and hence one may conclude that $\bar{\phi} = 0$. Substituting $\bar{\phi} = 0, \phi^* = \frac{\pi}{2}$ into (4.30)-(4.31) yields

$$
B K_y \bar{z}_y = 0,
$$

$$
D_{AB} B K_x \bar{z}_x = 0.
$$

The graph is assumed to be acyclic, implying that $\ker B = 0$. Since $D_{AB}$ is a positive definite diagonal matrix, it follows that $\bar{z}_x = 0, \bar{z}_y = 0$, thus completing the proof.

Remark 4.11 (Cyclic graphs). This section considered acyclic graphs to model the interaction topology. Since the interaction topology is a design freedom, considering only acyclic graphs is not restrictive. To deal with cyclic graphs an additional a priori condition is required on the desired relative displacements (see Remark 4.3).

Remark 4.12 (Velocity tracking without local feedback control). Experimental and simulation results seem to indicate that it is possible to achieve formation control with velocity tracking using two virtual robots which drag the whole network, without requiring a local velocity tracking controller for each robot [52, 92].
to the lack of friction compensation the geometrical shape of the formation is disturbed to a greater or lesser extent depending on the virtual spring constant. Section 6.2 provides more details and a direction for future research along this avenue.

**Remark 4.13 (Control law formation control with velocity tracking).** The control law $u$ for the wheeled robots is easily derived from Proposition 4.7, Proposition 4.8 and Theorem 4.10 and is given by

$$u = u_1 + u_2 + u_3 = -L K^\phi \tan(\phi - \phi^*) + D^r v^* - D^t(y - \alpha)$$

where $L = \text{block.diag}(L_1, \ldots, L_N)$ with $L_i$ defined below (4.22). From (4.35) it follows that the control input $u_i$ for robot $i$ requires knowledge of its heading, forward and angular velocity (if $D^t \neq 0$), while the formation controller requires the relative displacement and velocity with respect to its neighbors.

Note that (4.35) has a clear physical interpretation. The first term corresponds to a nonlinear spring force, which ensures each robot’s heading $\phi$ is converging to the desired heading $\phi^*$. The second term inserts a damping corresponding to the reference velocity $v^*$, while $D^t(y - \alpha)$ inserts damping with respect to the error velocity. Finally, the fourth and fifth term correspond to the spring and damper forces along $x$ and $y$, which are assigned between the robots. Due to the incidence matrix $B$ each agent only requires local information from its neighbors, thereby making (4.35) a distributed control law.

To illustrate the effectiveness of Theorem 4.10, simulation and experimental results are provided in the next section.

### 4.4.2 Simulation and experimental results

Consider a network of $N = 3$ wheeled robots of the form (4.5), with model parameters $m_i = 0.167 \text{ kg}$, $I_{cm,i} = 9.69 \cdot 10^{-5} \text{ kg m}^2$, $d_{f,i} = 2 \text{ kg/s}$, $d_{\phi,i} = 0.2 \text{ kg m}^2/\text{s}$, $d_{AB,i} = 0.06 \text{ m}$ for $i = 1, 2, 3$. The model parameters are chosen in accordance with the e-puck wheeled robot (Figure C.1) which are given in Table C.1.

The three robots are interconnected using virtual couplings according to the incidence matrix

$$B = \begin{pmatrix}
-1 & 0 \\
+1 & -1 \\
0 & +1
\end{pmatrix},$$
4.4. Formation control with velocity tracking

which is associated to a so-called path graph. Since each robot is strictly passive (i.e., \( d_{f,i} > 0, d_{\phi,i} > 0 \)) no virtual dampers are required for asymptotic stability \( (d_{x,j} = d_{y,j} = 0 \text{ kg/s for } j = 1, 2) \). For the heading control and velocity tracking control the control gains are set at respectively \( \phi_i^* = \frac{\pi}{2} \text{ rad}, k_i^\phi = 0.1 \text{ kg/s}^2 \) and \( v_{f,i}^* = 0.05 \text{ m/s}, D_{i}^t = 0 \) for \( i = 1, 2, 3 \). Finally, for the formation control set \( z_{x,j}^* = 0.4 \text{ m}, z_{y,i}^* = 0 \text{ m}, k_j^x = 2 \text{ kg/s}^2, k_j^y = 2 \text{ kg/s}^2, D_{xj}^c = D_{yj}^c = 0 \text{ kg/s for } j = 1, 2 \). These settings correspond to a line formation.

The simulations are performed using MATLAB and Simulink, while the experiments make use of the experimental setup discussed in Appendix A. The experimental setup consists of \( 2.6 \times 2.0 \text{m} \) table with an overhead camera for localization. Each robot is identified and localized using a data-matrix, attached on top of the e-puck (see Figure C.1 (right)). A vision algorithm runs in parallel to MATLAB and provides the ID, position, and heading of each e-puck. MATLAB calculates the corresponding control inputs (4.35) and sends them to the e-pucks via a Bluetooth connection.

Both the simulation and the experiment are run for \( t = 21 \text{ s} \), starting from the same initial conditions: \( x_B(0) = (0.83, 1.00, 1.15) \text{ m}, y_B(0) = (0.17, 0.19, 0.19) \text{ m}, \phi(0) = (1.68, 1.89, 1.48) \text{ rad}, p_f(0) = (0, 0, 0) \frac{\text{kg m}}{\text{s}}, h(0) = (0, 0, 0) \frac{\text{kg m}^2}{\text{s}} \). Note that the run time is significantly shorter than for the simulation and experiments in

\[ \text{Figure 4.7: Trajectories of the wheeled robots for the simulation (dashed) and the experiment (solid). The crosses (×) and circles (○) denote the initial and final positions respectively.} \]
Section 4.3 due to the limited size of the table in the experimental setup (i.e., the e-pucks would run into the boundaries of the setup for a longer run time). The results are shown in Figure 4.3-4.5.

The trajectories of the e-pucks are shown in Figure 4.7. The (small) differences are due to localization errors of the vision algorithm, model parameter uncertainty and a difference in the actuators of the model and the e-puck (the robot dynamics (4.5) do not consider the differential drive of the e-puck).

To illustrate that the control objectives (4.20) are achieved, Figure 4.8 and 4.9 show the time evolution of the heading $\phi$, forward velocity $v_f$, and relative displacement $z_x$. The dotted lines correspond the reference values $\phi^*, v_f^*, z_x^*$ respectively. The angular velocity $v_\phi$ and relative displacement $z_y$ show a similar trend and are given in Appendix D. Figure 4.8 and 4.9 show that for both the simulation and the experiment all variables converge to their reference value in accordance with Theorem 4.10.

Preliminary results on tuning of the controller gains in an extensive simulation and experimental setting may be found in respectively [52] and [20, 92]. For the velocity tracking controller considered in [52, 92] only some robots in the network require knowledge on the reference velocity. While the simulation and experimental results are promising, no formal proofs are available (see page 121 for a recommendation for future research).

In addition to formation control and velocity tracking, [20, 52] consider obstacle avoidance using artificial potential fields. The performance indicators considered in tuning the controller gains include energy consumption [20, 52, 92], formation error [92], number of collisions [20, 52], settling time [20] and coverage [52].
Figure 4.8: Time evolution of the heading $\phi$, forward velocity $v_f$ and relative displacement $z_x$ (simulation). The dotted lines show the reference values.
Figure 4.9: Time evolution of the heading $\phi$, forward velocity $v_f$ and relative displacement $z_x$ (experiment). The dotted lines show the reference values.
4.5 Formation control in the presence of matched input disturbances

In practice there are often disturbances influencing the robots’ behavior. In Sections 4.3 and 4.4 it was assumed that no disturbances were present. In this section matched input disturbances are considered, which act on the same channel as the control input. The disturbances are assumed to be generated by an external exosystem with internal state \( w \). An internal model controller with state \( \theta \) is designed, which asymptotically compensates the unknown disturbance.

The objectives of formation control in the presence of matched input disturbances can now be formulated in terms of (4.15) as

\[
\begin{cases}
    p \to 0, \\
    z \to z^*, \\
    \theta \to w,
\end{cases}
\quad \text{as } t \to \infty.
\tag{4.36}
\]

Note that the first two objectives are the same as the formation control objectives (4.11). Different is the addition of an objective for the internal model controller state, which guarantees that the disturbances considered here are rejected. In contrast with (4.20) velocity tracking and heading control are not considered here.

The results presented here to achieve (4.36) are published in [60, 116] and are based on a collaboration with Matin Jafarian and Claudio De Persis.

4.5.1 Matched input disturbance rejection using an internal-model-based approach

Before continuing with the controller design, first the matched input disturbance is introduced. A matched input disturbance acts on the same channel as the control input \( u_i \) in (4.9) such that the dynamics of robot \( i \) subject to a matched input disturbance \( d_i \) is given by

\[
\begin{pmatrix}
    \dot{q}_i \\
    \dot{p}_i
\end{pmatrix} = \begin{pmatrix}
    0 & S_i(q_i) \\
    -S_i^T(q_i) & -D_i^r
\end{pmatrix} \left( \frac{\partial H_i^r}{\partial q_i} \right) + \begin{pmatrix}
    0 \\
    \bar{G}_i(q_i)
\end{pmatrix} (\bar{u}_i + d_i),
\]

\[
\bar{y}_i = \bar{G}_i^T(q_i) \frac{\partial H_i^r}{\partial p_i}.
\tag{4.37}
\]
with $q_i \in \mathbb{R}^3$, $p_i, \bar{u}_i, d_i, \bar{y}_i \in \mathbb{R}^2$. In a similar fashion as for (4.10) the compact dynamics for $N$ robots of the form (4.37) are given by

\[
\begin{pmatrix}
\dot{q} \\
\dot{p}
\end{pmatrix} = \begin{pmatrix}
0 & S(q) \\
-S^T(q) & -D^r
\end{pmatrix} \begin{pmatrix}
\frac{\partial H^r}{\partial q} \\
\frac{\partial H^r}{\partial p}
\end{pmatrix} + \begin{pmatrix}
0 \\
\bar{G}(q)
\end{pmatrix} (\bar{u} + d),
\]

(4.38)

with $d = (d_1, \ldots, d_N)^T$.

The controller to achieve the control objectives (4.36) for (4.37) consists of two parts. The first part is a formation controller of the form (4.12), which provides the control input $\bar{u}_i$ given in (4.14). The second part is an internal-model-based disturbance rejection controller. The corresponding control input $\mathbf{\tilde{d}}_i$ is designed below.

The disturbance signal $d_i$ is assumed to be generated by an autonomous exosystem. Given two matrices $\Phi^d_i \in \mathbb{R}^{4 \times 4}$ and $\Gamma^d_i \in \mathbb{R}^{2 \times 4}$, whose properties will be made precise later on. Let $w^d_i \in \mathbb{R}^4$ denote the exosystem state, then the exosystem of robot $i$ obeys the following dynamics

\[
\dot{w}^d_i = \Phi^d_i w^d_i,
\]

\[
d^i = \Gamma^d_i w^d_i,
\]

(4.39)

for $i = 1, 2, \ldots, N$. Here it is assumed that $\Phi^d_i$ is a skew-symmetric matrix, implying that (4.39) is able to generate harmonic and constant disturbance signals. Inspired by the theory of output regulation (see e.g. [54]), an internal-model-based controller is adopted to counteract the effect of the disturbance $d_i$ generated by (4.39). Let $\theta^i \in \mathbb{R}^4$, $\tilde{u}^i, \mathbf{\tilde{d}}_i \in \mathbb{R}^2$ denote respectively the state, input and output of the internal-model-based controller. Then the internal model dynamics are given by

\[
\dot{\theta}^i = \Phi^d_i \theta^i + \Gamma^d_i T \tilde{u}^i,
\]

\[
\mathbf{\tilde{d}}^i = \Gamma^d_i \theta^i,
\]

(4.40)

for $i = 1, \ldots, N$. When $\tilde{u}^i = 0$ and the system is appropriately initialized, the latter system is able to generate any $w^d_i$ solution to (4.39). Now, define the error variables $\bar{d}^i = \bar{d}_i - d_i$ and $\bar{\theta}^i = \theta^i - \theta^d_i$, then the error dynamics of the internal model controller including the exosystem are given by

\[
\dot{\bar{\theta}}^i = \Phi^d_i \bar{\theta}^i + \Gamma^d_i T \bar{u}^i,
\]

\[
\bar{d}^i = \Gamma^d_i \bar{\theta}^i.
\]

(4.41)

To compactly denote the dynamics of $N$ internal model controllers of the form
4.5. Formation control in the presence of matched input disturbances

![Diagram](image)

**Figure 4.10:** The exosystem (4.39) together with the internal-model-based disturbance rejecting controller (4.40) form the lossless subsystem (4.41) with port variables $\tilde{u}_i, \tilde{d}_i$.

(4.41) denote vectors $\tilde{\theta} = (\tilde{\theta}_1, \ldots, \tilde{\theta}_N)^T$, $\tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_N)^T$, $\tilde{d} = (\tilde{d}_1, \ldots, \tilde{d}_N)^T$, and matrices $\Phi^d = \text{block.diag}(\Phi^d_1, \ldots, \Phi^d_N)$, $\Gamma^d = \text{block.diag}(\Gamma^d_1, \ldots, \Gamma^d_N)$. Then, the total internal model controller dynamics are given by

$$
\begin{align*}
\dot{\tilde{\theta}} &= \Phi^d \tilde{\theta} + \Gamma^d \tilde{u}, \\
\tilde{d} &= \Gamma^d \tilde{\theta}.
\end{align*}
$$

(4.42)

Note that the exosystem (4.39) itself is not a passive system, since it has no input. However, interconnecting exosystem (4.39) with the internal model controller (4.40) the resulting system (4.41) is lossless with respect to the port variables $\tilde{u}_i, \tilde{d}_i$ (see Figure 4.10). Furthermore (4.42) can easily be represented in the port-Hamiltonian framework by defining the Hamiltonian $H^d(\tilde{\theta}) = \frac{1}{2} \tilde{\theta}^T \tilde{\theta}$, such that (4.42) can be rewritten in the form (2.5) as

$$
\begin{align*}
\dot{\tilde{\theta}} &= \Phi^d \frac{\partial H^d}{\partial \tilde{\theta}} + \Gamma^d \tilde{u}, \\
\tilde{d} &= \Gamma^d \frac{\partial H^d}{\partial \tilde{\theta}}.
\end{align*}
$$

(4.43)

Note that in (4.43) the port-Hamiltonian structure is preserved, since $\Phi^d$ is skew-symmetric.

**Closed-loop analysis**

Now, continue with the closed-loop analysis of the two control systems: one for reaching the desired formation ($\bar{u}_i$ from (4.14)) and one for counteracting matched input disturbances ($\bar{d}_i$ from (4.40)). By interconnecting the systems appropriately, the closed-loop system preserves passivity properties and the port-Hamiltonian structure. The coupling of the two controllers with the robots is power-preserving.
and given by [3, 100]

\[
\begin{aligned}
u &= -(B \otimes I_2)\tau - \ddot{d}, \\
v &= (B^T \otimes I_2)y, \\
\dot{u} &= y,
\end{aligned}
\]  
(4.44)

with \( \ddot{d} = (\ddot{d}_1, \ldots, \ddot{d}_N)^T \). Note that \((u, y), (v, \tau)\) and \((\dot{u}, \dot{d})\) are the port-variables for respectively the robots (4.38), the virtual couplings (4.13) and the internal-model-based controller (4.40). Coupling (4.44) is a combination of formation control coupling (4.14) and standard negative feedback.

The overall closed-loop dynamics follow from (4.38), (4.13), (4.43), (4.44) and are given by

\[
\begin{pmatrix}
\dot{q} \\
\dot{p} \\
\dot{z} \\
\dot{\theta}
\end{pmatrix} =
\begin{pmatrix}
0 & S(q) & 0 & 0 \\
-S^T(q) & -D(q) & -\dot{G}(q) & -\dot{G}(q)\Gamma^d \\
0 & \dot{G}^T(q) & 0 & 0 \\
0 & \Gamma^dT \dot{G}^T(q) & 0 & \Phi^d
\end{pmatrix}
\begin{pmatrix}
\frac{\partial H}{\partial q} \\
\frac{\partial H}{\partial p} \\
\frac{\partial H}{\partial z} \\
\frac{\partial H}{\partial \theta}
\end{pmatrix},
\]  
(4.45)

with \( \dot{G}(q) = \dot{G}(q)(B \otimes I_2) \) and \( \dot{D}(q) = D^r + \dot{G}(q)D^c\dot{G}^T(q) \). Finally, the closed-loop Hamiltonian is given by

\[H(p, z, \tilde{\theta}) = H^r(p) + H^c(z) + H^d(\tilde{\theta}).\]

Now the results of the stability and convergence analysis of the closed loop system are presented, starting with the following proposition.

**Proposition 4.14.** Interconnect wheeled robots (4.38) with virtual couplings (4.13) and internal model controller (4.43) via coupling (4.44) using a tree graph topology. Then, the closed-loop system (4.45) asymptotically converges to \( p = 0 \) provided that \( D^c > 0 \) and there is at least one robot \( i \) satisfying \( D^r_i > 0 \).

**Proof.** Take the closed-loop Hamiltonian \( H \) as the candidate Lyapunov function. Calculating the time derivative of \( H \), one obtains

\[
\dot{H} = -\frac{\partial^T H}{\partial p} \dot{D}(q) \frac{\partial H}{\partial p} - \frac{\partial^T H}{\partial p} \dot{G}(q) \frac{\partial H}{\partial z} + \frac{\partial^T H}{\partial z} \dot{G}^T(q) \frac{\partial H}{\partial p}
\]  
\[+ \frac{\partial^T H}{\partial \theta} \Phi^d \frac{\partial H}{\partial \theta} - \frac{\partial^T H}{\partial \theta} \dot{G}(q) \Gamma^d \frac{\partial H}{\partial p} + \frac{\partial^T H}{\partial \theta} \Gamma^dT \dot{G}^T(q) \frac{\partial H}{\partial p}.
\]  
(4.46)

Due to skew-symmetry of \( \Phi^d \) it immediately follows that

\[
\dot{H}(x) = -\frac{\partial^T H}{\partial p} \left(D^r + \dot{G}(q)D^c\dot{G}^T(q)\right) \frac{\partial H}{\partial p}.
\]  
(4.47)
Note that since $D^c > 0$ and $D^r_i \geq 0$ it follows that $\dot{H}(x) \leq 0$ and therefore the system is stable. Invoking LaSalle’s invariance principle (Theorem 2.4) provides that the system (4.45) converges to the largest invariant set where

$$
\dot{G}^T(q) \frac{\partial H}{\partial p} = (B^T \otimes I_2)\dot{G}^T(q)(M^r)^{-1}p = 0,
$$

$$
\frac{\partial H}{\partial p_i} = (M^r_i)^{-1}p_i = 0,
$$

where $p_i$ is the momentum of strictly passive robot $i$ (i.e., $D^r_i > 0$).

Since the graph topology is assumed to be undirected and connected, it follows that $\ker B^T = \alpha_1 I_N$, for some constant $\alpha_1 \in \mathbb{R}$. Noting that $G^T(q)$ has no singularity for any $q$ it follows from (4.48) that $p = \alpha_2 \otimes I_N$ for some constant vector $\alpha_2 \in \mathbb{R}^2$. Moreover, from (4.49) it follows that $p_i = 0$ (since $m_i > 0$ and $I_{CM,i} > 0$). This implies that $\alpha_2 = 0$ and therefore $p = 0$, thereby completing the proof.

Remark 4.15 (Strictly passive robots). The results of the Proposition 4.14 still hold if $D^c = 0$ provided that for all robots $D^r_i > 0$ (i.e., a network of strictly passive robots).

Proposition 4.14 guarantees stability of the system. However, it guarantees neither reaching the desired formation (i.e., $z = z^*$) nor rejecting the matched input disturbances (i.e., $\bar{\theta} = 0$). To verify more, two special types of the disturbances generated by the exosystem (4.39) are considered next, namely harmonic and constant disturbances.

**Corollary 4.16 (Harmonic disturbances).** Assume that the exosystems’ matrices $(\Gamma^d_i, \Phi^d_i)$ are of the form

$$
\Phi^d_i = \text{block} \text{-} \text{diag} \left( \begin{pmatrix} 0 & \omega_{i1} \\ -\omega_{i1} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \omega_{i2} \\ -\omega_{i2} & 0 \end{pmatrix} \right),
$$

with $\omega_{i\ell} \neq 0$ for $\ell = 1, 2$, and $\Gamma^d_i = \text{block} \text{-} \text{diag}(\Gamma^d_{i1}, \Gamma^d_{i2})$, with $\Gamma^d_{i\ell} \neq 0$, for all $\ell = 1, 2$, and the pair $(\Gamma^d_i, \Phi^d_i)$ is observable. Then the closed loop system (4.45) converges to $p = 0$, $\bar{\theta} = 0$, and $z = z^*$.

**Proof.** Corollary 4.16 is a special case of Proposition 4.14. Therefore, one can start by substituting $p = 0$ into (4.45) to obtain

$$
\dot{q} = 0,
0 = -G(q)(B \otimes I_2)K^c(z - z^*) - \Gamma^d_i \bar{\theta},
\dot{z} = 0,
\dot{\bar{\theta}} = \Phi^d \bar{\theta},
$$

(4.51)
First calculate the time derivative of the second equality. Since, \( \dot{q} = 0 \) and \( \dot{z} = 0 \), it follows that \( \dot{\Gamma}d\dot{\theta} = 0 \). Now, replacing \( \dot{\theta} \) from the last equality in (4.51) gives \( \Gamma^d\Phi^d\dot{\theta} = 0 \). Calculate the time derivative of the latter gives \( \dot{\Gamma}d\Phi^d\Phi^d\dot{\theta} = 0 \). Since \( \Phi^d \) is a skew-symmetric matrix with the structure given in (4.50), \( \Phi^d\Phi^d \) is equal to \( \gamma I_4 \), where \( \gamma \) is a constant. Hence, \( \dot{\Gamma}d\Phi^d\Phi^d\dot{\theta} = 0 \) implies that \( \dot{\Gamma}d\dot{\theta} = 0 \). Hence,

\[
\begin{align*}
\dot{\theta} &= \Phi^d\dot{\theta}, \\
\dot{\Gamma}d\dot{\theta} &= 0.
\end{align*}
\]

Since the pair \((\Gamma^d, \Phi^d)\) is observable, it immediately follows that \( \dot{\theta} = 0 \). Now, consider the second equality in (4.51). Substituting \( \dot{\theta} = 0 \) results in

\[
\bar{G}(q)(B \otimes I_2)K^c(z - z^*) = 0.
\]

For an undirected acyclic graph the kernel of the incidence matrix is given by \( \ker(B \otimes I_2) = 0 \). Since \( \bar{G}(q) \) has no singularities for any \( q \) and \( K^c \) is a diagonal matrix with positive constants on the diagonal it immediately follows that \( z = z^* \), thereby completing the proof. \(\square\)

Corollary 4.16 shows that the internal-model-based controller (4.42) can counteract harmonic disturbances. The following corollary presents the result for constant disturbances. Note that constant disturbances are generated by (4.43) by setting \( \Phi^d = 0 \).

**Corollary 4.17 (Constant disturbances).** Assume that \( \Phi^d = 0 \), and \( \Gamma^d \) is nonsingular. Then the closed loop system (4.45) converges to \( p = 0, \dot{\theta} = c_{\dot{\theta}}, \) and \( \dot{z} = c_z \), with \( c_{\dot{\theta}} \in \mathbb{R}^N, c_z \in \mathbb{R}^E \) arbitrary constants.

**Proof.** Corollary 4.17 is again a special case of Proposition 4.14. Substituting \( p = 0 \) into (4.45) gives (4.51). For constant disturbances \( \Phi^d = 0 \), substituting into (4.51) gives

\[
\begin{align*}
\dot{q} &= 0, \\
0 &= -\bar{G}(q)(B \otimes I_2)K^c(z - z^*) - \Gamma^d\dot{\theta}, \\
\dot{z} &= 0, \\
\dot{\theta} &= 0.
\end{align*}
\]

(4.52)

Since \( \dot{z} = 0 \) and \( \dot{\theta} = 0 \), \( \bar{z} \) and \( \dot{\theta} \) converge to arbitrary constants, thereby completing the proof. \(\square\)

**Remark 4.18 (Constant disturbances).** For the constant disturbance case, one can only conclude that the error position vector \( z - z^* \) and the error disturbance vector \( \dot{\theta} \) are constant. Therefore the internal-model-based controller (4.43) guarantees
neither rejecting a constant disturbance nor achieving the desired formation. However, the stability of the network is maintained and the robots’ velocities converge to zero.

The difference between Corollary 4.16 and 4.17 is that for constant disturbances observability of the pair \((\Gamma^d, \Phi^d)\) is lost. This observability property plays a crucial role in the proof of Corollary 4.16 to prove that \(\tilde{\theta} = 0\).

Remark 4.19 (Cyclic graphs). This section considered acyclic graphs to model the interaction topology. Since the interaction topology is a design freedom, considering only acyclic graphs is not restrictive. To deal with cyclic graphs an additional a priori condition is required on the desired relative displacements (see Remark 4.3).

Remark 4.20 (Control input formation control and disturbance rejection). The input to each robot follows directly from (4.13), (4.40) and (4.44) and is given by

\[
u = -(B \otimes I_2)\tau - \tilde{d} - (B \otimes I_2)K^c(z - z^*) - (B \otimes I_2)D^c(B^T \otimes I_2)M^{-1}p - \Gamma^d \theta .
\]

The first two parts in (4.53) correspond to the virtual springs and virtual dampers assigned between the robots. The presence of the incidence matrix \(B\) shows that (4.53) is in fact a distributed control law. The last term in (4.53) corresponds to the local internal model controller, which requires only the controller state \(\theta\).

### 4.5.2 Simulation results

Consider a network of \(N = 5\) wheeled robots of the form (4.5), with model parameters \(m_i = 0.167 \text{ kg}, I_{cm,i} = 9.69 \cdot 10^{-5} \text{ kg m}^2\), \(d_{AB,i} = 0.06 \text{ m}\) for \(i = 1, \ldots, 5\). In accordance with Proposition 4.14 set \(d_{f,1} = 2 \text{ kg/s}, d_{\phi,1} = 0.2 \text{ kg m}^2/\text{s}\) and \(d_{f,i} = 0 \text{ kg/s}, d_{\phi,i} = 0 \text{ kg m}^2/\text{s}\) for \(i = 2, \ldots, 5\) such that only robot 1 is strictly passive, while the other robots are lossless. The five robots are interconnected using the incidence matrix

\[
B = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
+1 & -1 & 0 & 0 \\
0 & +1 & -1 & 0 \\
0 & 0 & +1 & -1 \\
0 & 0 & 0 & +1
\end{pmatrix} ,
\]

which is associated to a path graph. The virtual coupling parameters are set at \(z^*_{x,k} = 1 \text{ m}, z^*_{y,k} = 0 \text{ m}, \kappa_{x,k} = \kappa_{y,k} = 2 \text{ kg/s}^2, d_{x,k} = d_{y,k} = 1\) for \(k = 1, \ldots, 4\). Note that this choice for \(z^*_{x,k}, z^*_{y,k}\) corresponds to a line formation.
To illustrate the effectiveness of Corollary 4.16 the exosystem parameters generating the harmonic disturbance signal are set at
\[
\Phi^d_i = I_2 \otimes \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \Gamma^d_i = I_2 \otimes \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix},
\]
\[
w^d_i(0) = (1, 1, 1, 1)^T, \quad \theta^d_i(0) = (0, 0, 0, 0)^T,
\]
for \(i = 1, \ldots, 5\). For the illustration of constant disturbance case (Corollary 4.17) set \(\Phi^d_i = 0\) and keep \(\Gamma^d_i, w^d_i(0), \theta^d_i(0)\) the same.

The simulations are performed using MATLAB and Simulink and were run for \(t = 100\) s, starting from the initial conditions \(x_B(0) = (0.0035, 0.0981, 0.2354, 0.3467, 0.4797) m, y_B(0) = (0.0599, -0.0537, 0.0582, -0.0529, -0.0565) m, \phi(0) = (1.5124, 4.2482, 1.8162, 4.2211, 4.3677) \text{ rad}, p_f(0) = 0 \text{ kg m/s}, h(0) = 0 \text{ kg m}^2/\text{s}.
\]
The results are shown in Figures 4.11, 4.12, and 4.13. Only the results along the \(x\) direction are shown here, the plots along the \(y\) direction show a similar trend and are given in Appendix D.

Figure 4.11 shows the time evolution of the velocity \(v_x\), relative displacement \(z_x\), and internal model controller state \(\tilde{\theta}_x\) in the presence of harmonic disturbances, while Figure 4.12 shows the same variables for constant disturbances. For the harmonic disturbances, all three variables converge to zero in accordance with Corollary 4.16 (see Figure 4.11). For the constant disturbance case \(v_x\) does converge to zero, while \(z_x\) and \(\tilde{\theta}_x\) converge to a constant different from the desired value (see Figure 4.12).

Finally, to illustrate Remark 4.15 a network where all five robots are strictly passive is simulated (i.e., \(d_f,i = 2 \text{ kg/s}, d_{\phi,i} = 0.2 \text{ kg m}^2/\text{s} \text{ for } i = 1, \ldots, 5\)). The virtual damping coefficients are set at \(d_{x,k} = d_{y,k} = 0\) for \(k = 1, \ldots, 4\). Here, the robots are subject to the same harmonic disturbances as in Figure 4.11. For this setting all variables converge to zero as pointed out by Remark 4.15 (see Figure 4.13).
4.5. Formation control in the presence of matched input disturbances

Figure 4.11: Time evolution of the velocity $v_x$, relative displacement $z_x$ and internal model controller state $\tilde{\theta}_x$ in the presence of harmonic disturbances. The dotted lines show the reference values.
Figure 4.12: Time evolution of the velocity \( v_x \), relative displacement \( z_x \) and internal model controller state \( \hat{\theta}_x \) in the presence of constant disturbances. The dotted lines show the reference values.
4.5. Formation control in the presence of matched input disturbances

Figure 4.13: Time evolution of the velocity $v_x$, relative displacement $z_x$ and internal model controller state $\tilde{\theta}_x$ in the presence of harmonic disturbances when all robots are strictly passive and $D^e = 0$. The dotted lines show the reference values.
4. Formation control of nonholonomic wheeled robots

4.6 Concluding remarks

This chapter considers three formation control problems for a network of nonholonomic wheeled robots. The wheeled robots are modeled as rigid bodies with a nonholonomic constraint on the wheel axle. Solving for the constraint provides the wheeled robot dynamics on the constrained state space.

The formation control problem is tackled by assigning virtual couplings between the front ends of the robots, where the interconnection topology is modeled as a tree graph. Assigning the couplings to the front ends differs from the coupling assignment in Chapter 3, where the springs are assigned to the center of mass.

The same approach is then used to achieve formation control while tracking a reference velocity along a desired heading. The velocity controller builds upon generalized canonical transformations to derive and stabilize the error dynamics with respect to the reference velocity. The nonlinear heading controller guarantees that under a natural assumption on the initial heading, all robots move along the desired heading.

Finally an internal-model-based controller is designed to reject matched input disturbances. The internal model controller is able to compensate for harmonic disturbances if at least one robot is strictly passive and the other robots are connected using (a chain of) virtual dampers in parallel with the virtual springs. For constant disturbances the controller does prevent the robots from drifting away (i.e., stability is guaranteed), but it does not completely reject the disturbance. Ongoing research investigates alternative controller designs to deal with constant disturbances.

The results in this chapter are illustrated by simulation and experimental results using the e-puck wheeled robot.