Symmetries, conservation laws, and time reversibility for Hamiltonian systems with external forces

Schaft, A.J. van der

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Symmetries, conservation laws, and time reversibility for Hamiltonian systems with external forces

A. J. van der Schaft[a)]
Mathematics Institute, University of Groningen, P.O. Box 800, 9700 AV Groningen, The Netherlands

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A system theoretic framework is given for the description of Hamiltonian systems with external forces and partial observations of the state. It is shown how symmetries and conservation laws can be defined within this framework. A generalization of Noether's theorem is obtained. Finally a precise definition of time reversibility is given and its consequences are explored.

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I. INTRODUCTION

In the last century the mathematical formulation of classical mechanics has culminated in the elegant theory as described for instance in the books of Abraham and Marsden[1] and Arnold.[2] However, the emphasis in this approach has been put on analytical mechanics, i.e., Hamiltonian systems which can be described without external forces. If forces are present, they are assumed to come from a potential field, and therefore can be incorporated in the system by adding a potential function to the Hamiltonian function. Since external forces do come up at various places, for instance experimental devices and technical applications, and mostly cannot be derived from a potential function, this entails indeed quite a loss of generality.

In this paper we elaborate a framework, which can incorporate external forces on a conceptual level. At the same time we also formalize the idea that we may only partially observe the state of a system. Our basic notions stem from system theory, the discipline that explicitly deals with systems with inputs and outputs which in this context are called external forces and observations, respectively.

A simple example will make things more clear. Consider Newton's second law $F = m\ddot{q}$. Notice already that this law cannot be adequately formulated in a framework without external forces. We will look at it as a system with external force $u = F$,

$$\dot{q} = (1/m)p,$$
$$\dot{p} = u,$$

and an observation function y of the state (q,p) given by

$$y = q.$$

We will give, using the language of symplectic geometry, a general framework for the description of such systems. Furthermore we will show how in this framework symmetries, conservation laws and time reversibility can be defined and treated in an appealing way.

The paper is a further elaboration of previous work,[3–7] which was in turn much inspired by work of Brockett,[8] Takens,[9] and Willems.[10] The first definitions in Sec. II were basically given in,[3] while Sec. III is already partially contained in.[5] We have tried to make the paper more or less self-contained, at least with respect to definitions and statements of theorems, but for more background and detail we refer to the references mentioned (see also the survey[11]).

Some notation: For symplectic geometry we refer to Refs. 1 and 2. If (M,ω) is a symplectic manifold (of dimension $2n$) with symplectic form ω, and $H:M\rightarrow\mathbb{R}$ a smooth function, then the Hamiltonian vector field X_H on M is defined by $\omega(X_H, -) = dH$. Let $F,H:M\rightarrow\mathbb{R}$ be two functions, then the Poisson bracket $\{F,H\}$ is again a function on M defined by $\{F,H\} := \omega(X_F,X_H)$. Let $F_1,...,F_k$ be functions on M. Take all functions on M which are functions of the F_i, i.e., all functions $\varphi\circ(F_1,...,F_k)$: $M\rightarrow\mathbb{R}$. This generates a linear subspace $\mathscr{F}$ of the space of functions on M. We call $\mathscr{F}$ a Poisson algebra (or "function group," see Ref. 12) if $\{f,g\}\in\mathscr{F}$ for all $f,g\in\mathscr{F}$. For every $x\in M$ we define $d\mathscr{F}(x)$ as the linear subspace of T^*_xM, given by $d\mathscr{F}(x) = \{df(x)|f\in\mathscr{F}\}$.

A submanifold $N\subset M$ is Lagrangian if $\omega|_N = 0$ and if the dimension of N is n. If $(q_1,...,q_n,p_1,...,p_n)$ are symplectic coordinates (i.e., $\omega = \sum_{i=1}^n dq_i\wedge dp_i$) and N can be parametrized by $q_1,...,q_n$, then there exists (locally) a function $S(q_1,...,q_n)$ such that $N = \{(q,p)|p_i = \partial S/\partial q_i, i = 1,...,n\}$. S is the generating function of N.

If f is a function on a manifold W, we can define the function $\dot{f}$ on TW by $\dot{f}(v) := df(v)$, for $v\in TW$. Therefore if $(x_1,...,x_n)$ are coordinates for W, then $(x_1,...,x_n, \dot{x}_1,...,\dot{x}_n)$ are coordinates for TW. Given an one-form α on W, in local coordinates given by $\alpha = \sum_{i=1}^n f_i\, dx_i$, with $f_i:W\rightarrow\mathbb{R}$, then we can define the one-form $\dot{\alpha}$ on TW by $\dot{\alpha} = \sum_{i=1}^n(\dot{f}_i\, dx_i + f_i\, d\dot{x}_i)$. Let $\omega = \sum_{i=1}^n(dq_i\wedge dp_i$ be a symplectic form on M, then $\dot{\omega} := \sum_{i=1}^n(dq_i\wedge d\dot{p}_i + d\dot{q}_i\wedge dp_i)$is a symplectic form on TM. All these constructions can also be done in a coordinate-free way.[12–14] Let X be a vector field on M with one-parameter group $X_t:M\rightarrow M$, $t\in\mathbb{R}$ and small. Then $(X_t)_*:TM\rightarrow TM$ is the one-parameter group of a vector field on TM, which we denote by $\dot{X}$. If $(y_1,...,y_m)$ are coordinates for a manifold Y then the natural coordinates $(y_1,...,y_m, u_1,...,u_m)$ for T^*Y are defined by letting $(\bar{y}_1,...,\bar{y}_m, \bar{u}_1,...,\bar{u}_m)$ correspond to the one-form $\sum_{i=1}^m \bar{u}_i\, dy_i$. We recall that cotangent bundles have a canonically defined symplectic form.[1,2]

II. HAMILTONIAN SYSTEMS WITH EXTERNAL FORCES AND PARTIAL OBSERVATIONS

The usual description of a Hamiltonian system (see for instance Refs. 1 and 2 is that of a triplet (M,ω,H), where M is a smooth symplectic manifold denoting the phase space and ω is a symplectic form on M which formalizes the typical structure of the phase space, namely the existence of "conju-

[a)] Current address: Department of Applied Mathematics, Twente University of Technology, P.O. Box 217, 7500 AE Enschede, The Netherlands.

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gate" coordinates q_i and p_i. Finally, H is a smooth function on M which represents the energy of the system. The dynamical behavior of the system is in local coordinates x for M given by $\dot{x} = X_H(x)$, where X_H is the Hamiltonian vector field on M defined by

$$\omega(X_H, -) = dH. \quad (1.1)$$

Using Darboux's theorem we can take local coordinates $(q_1,...,q_n,p_1,...,p_n)$ for M such that $\omega = \Sigma_{i=1}^n dq_i \wedge dp_i$. Then (1.1) comes down to the familiar Hamiltonian equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad i = 1,...,n. \quad (1.2)$$

As mentioned in the Introduction, our starting point will be a generalization of this structure, incorporating the possible external forces on the system. Moreover, we will formalize the idea that we may not be able to observe the whole state of the system but only the values of some functions of the state. Therefore we introduce, apart from the state space manifold M, an observation (or output) manifold Y (dim $Y = m$). Then we define the space of external forces (or inputs) in every point of Y as the fiber of the cotangent bundle T^*Y in that point. This seems natural since an element α of a fiber of T^*Y is a linear function on the tangent vectors $\dot{y}$ of Y in that same point. Therefore $\alpha(\dot{y})$ (force times velocity) is defined and represents the external work performed on the system.

Recall that T^*Y has a canonically defined symplectic form, which we denote by ω^e. Since M has a symplectic form ω, we can also define (see the Introduction) a symplectic form $\dot{\omega}$ on TM. This enables us to define the symplectic form Ω on $TM \times T^*Y$ by setting $\Omega := \pi_1{}^*\dot{\omega} - \pi_2^*\omega^e$ (π_1 and π_2 are the projections of $TM \times T^*Y$ on TM, resp. T^*Y).

Definition 1.1 (Hamiltonian system): Let (M,ω) be a symplectic manifold. Let Y be an observation manifold. A Hamiltonian system $\Sigma(M,T^*Y,L)$ or shortly Σ is given by a submanifold $L \subset TM \times T^*Y$ such that

(i) L can be parametrized by the coordinates of M and the coordinates of the fibers of T^*Y.

(ii) L is a Lagrangian submanifold of $(TM \times T^*Y, \Omega)$.

(iii) The value of the Y-coordinates of a point on L is only a function of the M-coordinates of this point.

Proposition 1.2: Let $\Sigma(M,T^*Y,L)$ be a Hamiltonian system as above. Then in local coordinates the system is given by

$$\dot{x} = X_H(x) + \sum_{i=1}^m u_i X_{C_i}(x),$$
$$y_i = C_i(x), \quad i = 1,...,m, \quad (1.3)$$

with x local coordinates for M, $y = (y_1,...,y_m)$ local coordinates for Y, and $u = (u_1,...,u_m)$ the corresponding natural coordinates for the fibers of T^*Y. We will call $C_i: M \to \mathbb{R}$ the observation (or output) functions.

Proof[3,6]: Because of (ii) L has a generating function. Because of (i) and (iii) this generating function has the form $H(x) + \Sigma_{i=1}^m u_i C_i(x)$, which we will abbreviate as $H + u^T C$. Therefore the $\dot{x}$-coordinates of a point on L are given by $\dot{x} = X_H(x) + \Sigma_{i=1}^m u_i X_{C_i}(x)$ and the y-coordinates by

$$y_i = C_i(x), \; i = 1,...,m.$$

Remark 1: If we drop the assumption that M is a symplectic and the conditions (ii) and (iii) in Definition 1.1, we call $\Sigma(M,T^*Y,L)$ just a system (see Refs. 15 and 4 and references therein).

Remark 2: Without condition (iii) we arrive at the more general class of Hamiltonian systems where the external forces (inputs) enter the equations in a nonlinear way. In this case we can also replace T^*Y by a general symplectic manifold (cf. Ref. 3).

Remark 3: From a mathematical point of view the above definition stresses again the importance of the concept of Lagrangian submanifolds, as already done before by many authors. The use of Lagrangian submanifolds in formulating "reciprocity" and "symmetry" in the description of static systems is successfully advocated in many works (see for some references Ref. 1, Sec. 5.3; a particularly nice account is given in Ref. 15). If we generalize Definition 1.1 in the direction given in Remark 2, such static systems correspond to a Hamiltonian system without state space M; i.e., a Lagrangian submanifold of T^*Y or a more general symplectic manifold (see Ref. 3). The description of a Hamiltonian vector field on M as a Lagrangian submanifold of $(TM,\dot{\omega})$ figures prominently in many works of Tulczyjew and coworkers.[13,16,15] Definition 1.1 (and its generalization indicated in Remark 2, see Ref. 3) combines both aspects and gives more rigor to idea of external forces and observations by using a system theoretic framework.

Another way to look at Eqs. (1.3) is to start from a triplet (M,ω,H), to add an observation map $C: M \to Y$, and to define the input vector fields (the directions in which we can exert external forces) as the Hamiltonian vector fields with Hamiltonian functions C_i, where in coordinates for $Y, C = (C_1,...,C_m)$. So we have formalized the idea that we may influence the system by adding to the internal energy H a Hamiltonian function $\Sigma_{i=1}^m u_i(t) C_i$, depending on the observations made on the system.

Examples:

1. Newton's second law as treated in the Introduction.
2. Consider the Euler–Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = F_i.$$

Assume that the Legendre transformation $\dot{q} \to p := \partial L / \partial \dot{q}$ is nondegenerate (otherwise see Ref. 3), giving the Hamiltonian function

$$H\left(q, \frac{\partial L}{\partial \dot{q}}\right) = \dot{q}\frac{\partial L}{\partial \dot{q}} - L(q,\dot{q}).$$

Then the Euler–Lagrange equations can be written as

$$\dot{q}_i = \frac{\partial H}{\partial p_i},$$
$$\dot{p}_i = -\frac{\partial H}{\partial q_i} + F_i,$$

together with the observation functions $y_i = q_i$.

3. Consider a capacitor C and inductance L, coupled in series, with an external voltage V_e. With q_c the charge on the capacitor and φ_L the magnetic flux on the inductance we obtain

$$\begin{pmatrix} \dot q_c \\ \dot\varphi_L \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{L} \\ \frac{1}{C} & 0 \end{pmatrix} \begin{pmatrix} q_c \\ \varphi_L \end{pmatrix} + \begin{pmatrix} 0 \\ V_e \end{pmatrix}$$

and the observation function $y = q_c$ (note that $\dot q_c$ is the extenral current).

As already announced, external work can be naturally defined in our framework. We need one more definition.

Definition 1.3 (External behavior): Let $\Sigma(M,T^*Y,L)$ be a Hamiltonian system locally given by

$$\dot x = X_H(x) + \sum_{i=1}^m u_i X_{C_i}(x), \tag{1.4a}$$

$$y_i = C_i(x), \quad i = 1,\dots,m. \tag{1.4b}$$

Let $\bar w$: $[0,T] \to T^*Y$ be a curve in T^*Y which in natural coordinates (y,u) for T^*Y can be written as

$$\bar w(t) = (\bar y(t), \bar u(t)), \quad t\in[0,T] \quad \text{for a } T > 0.$$

Then $\bar w$ belongs to the external behavior of the system, if there exists an $x_0 \in M$ such that when we apply the force function $\bar u(\cdot)$ to the system (1.4a) with $x(0) = x_0$, Eq. (1.4b) yields the same observation function $\bar y(\cdot)$.

Let now $\bar w$: $[0,T] \to T^*Y$ belong to the external behavior. Then the performed external work is equal to

$$\int_0^T \sum_{i=1}^m \bar u_i \dot{\bar y}_i \, dt = \int_0^T \sum_{i=1}^m \bar u_i \left\{ H + \sum_{j=1}^m \bar u_j C_j, C_i \right\} dt$$
$$= \int_0^T \sum_{i=1}^m \bar u_i \{H, C_i\} \, dt + \int_0^T \sum_{i,j=1}^m \bar u_i \bar u_j \{C_i, C_j\} \, dt, \tag{1.5}$$

where for the first identity we use

$$\dot{\bar y}_i = \left(X_H + \sum_{j=1}^m \bar u_j X_{C_j} \right)(\bar y_i) = \{H, C_i\} + \sum_{j=1}^m \bar u_j \{C_j, C_i\},$$

since $y_i = C_i(x)$. Because $\{C_i, C_j\} = -\{C_j, C_i\}$ the last term of (1.5) vanishes and we obtain

$$\text{External work} = \sum_{i=1}^m \int_0^T \bar u_i \{H, C_i\} dt.$$

Definition (1.1) formalizes that we can exert external forces in the direction of the Hamiltonian vector field of every observation function. This might be too strong. For instance, we should also like to cover the situation

$$\dot x = X_H(x), \quad y_i = C_i(x), \quad i = 1,\dots,m,$$

i.e., a Hamiltonian system with partial observations of the state but without external forces. For this we give

Definition 1.4 (Degenerate Hamiltonian system): Let $\Sigma(M,T^*Y,L)$ be a Hamiltonian system. Let $P \subset T^*Y$ be a codistribution on Y (i.e., in every point y of Y a linear subspace of T_y^*Y, denoting the possible forces), which is involutive. Assume that $L' := L \cap (TM \times P)$ is a submanifold of $TM \times T^*Y$. Then we call $\Sigma(M,T^*Y,L')$ a degenerate Hamiltonian system.

In local coordinates we obtain the easily proved analog of Proposition 1.2.

Proposition 1.5: Let $\Sigma(M,T^*Y,L' = L \cap (TM \times P))$ be a degenerate Hamiltonian system. Since P is involutive, there exist local coordinates $\{y_1,\dots,y_m\}$ for Y such that $P = \text{span}\{dy_1,\dots,dy_k\}$, $k \leqslant m$. In these coordinates for Y the system is locally given by

$$\dot x = X_H(x) + \sum_{i=1}^k u_i X_{C_i}(x),$$
$$y_i = C_i(x), \quad i = 1,\dots,m. \tag{1.6}$$

Degenerative Hamiltonian systems are frequently encountered as the result of an interconnection of Hamiltonian systems.[3] Apart from interconnections also notions like "coupling Hamiltonians" and "interaction potentials" can be naturally described in our scheme. A time-varying function H_t on M of the form $H_t(x) = \Sigma_{i=1}^m \bar u_i(t) C_i(x)$, with $\bar u(\cdot) = (\bar u_1(\cdot),\dots,\bar u_m(\cdot))$ a certain force function, is sometimes called a coupling Hamiltonian. Consider two Hamiltonian systems $\Sigma(M_i,T^*Y_i,L_i)$, $i = 1,2$. The product is again a Hamiltonian system $\Sigma(M_1 \times M_2, T^*(Y_1 \times Y_2), L_1 \times L_2)$. If the generating function of L_1 is $H_1 + u^{1\,T} C_1$, and of L_2, $H_2 + u^{2\,T} C_2$, then $L_1 \times L_2$ has generating function $H_1 + H_2 + u^{1\,T} C_1 + u^{2\,T} C_2$. A coupling Hamiltonian or interaction potential is a function $V: Y_1 \times Y_2 \to \mathbb{R}$, which changes the generating function into

$$H_1(x) + H_2(x) + V(C_1(x), C_2(x)) + u^{1\,T} C_1(x) + u^{2\,T} C_2(x).$$

We will now bring in some system theoretic concepts, the most important of which is minimality. Intuitively, one says that a system is minimal if the system cannot be reduced to a system living on a lower dimensional state space and with the same external behavior as the original system.

Definition 1.6 (Minimality): Let $\Sigma(M,T^*Y,L)$ be a (possibly degenerate) Hamiltonian system. Σ is called minimal, if, when there exists another system $\Sigma'(M',T^*Y,L')$ (not necessarily Hamiltonian) and a surjective submersion φ: $M \to M'$ such that $(\varphi_*, \text{id})(L) = L'$ (with id the identity mapping from T^*Y to T^*Y), then necessarily φ is a diffeomorphism.

A local version of minimality, called local minimality[4] has the following neat characterization, which we will frequently use in the sequel.

Proposition 1.7[4,6]: Let $\Sigma(M,T^*Y,L)$ be a Hamiltonian system with generating function $H + \Sigma_{i=1}^m u_i C_i$. Define the Poisson algebra (see the Introduction) $\mathscr{F}$ as the smallest Poisson algebra containing the functions $C_1,\dots, C_m$ and closed under taking Poisson brackets with H and C_i, $i = 1,\dots, m$. Then Σ is locally minimal if and only if $\dim d\mathscr{F}(x) = 2n$, for every $x \in M$.

The proposition above has an interesting system theoretic interpretation, since it implies that a locally minimal Hamiltonian system is "observable" as well as "controllable." Observability means grosso modo that from the knowledge of the external behavior on an interval $[0,T]$ we can deduce the value of the state on time 0. Controllability implies that if we look at Eq. (1.4a) for a certain $x(0) = x_0$ and consider the set of points in M which are reachable from x_0 by applying different force functions, this set has a nonempty interior in M.

In Ref. 6 it is proven that a Hamiltonian system which is not locally minimal can be reduced to a locally minimal system which is again Hamiltonian. Therefore we could also have formulated Definition 1.6 with a Hamiltonian system $\Sigma(M',T^*Y,L')$. Finally we note that we can give a characterization similar to Proposition 1.7 for local minimality of degenerate Hamiltonian systems. However, we remark that if $\Sigma(M,T^*Y,L' = L\cap(TM\times P))$ is degenerate Hamiltonian, then the local minimality of $\Sigma(M,T^*Y,L)$ need not imply the local minimality of $\Sigma(M,T^*Y,L')$.

III. SYMMETRIES AND CONSERVATION LAWS

We recall the usual definition of a symmetry for the triplet (M,ω,H) (cf. Refs. 1 and 2): a diffeomorphism φ: $M\to M$ is Hamiltonian symmetry if (i) $\varphi^*\omega = \omega$, (ii) $\varphi^*H = H$.

This generalizes in our framework to

Definition 2.1 (Hamiltonian symmetry): Let $\Sigma(M,T^*Y,L)$ be a Hamiltonian system. A symmetry for Σ is a pair of diffeomorphisms (φ,ψ), with $\varphi: M\to M$, $\psi: T^*Y\to T^*Y$ such that $(\varphi_*,\psi)(L) = L$. A symmetry is called Hamiltonian if $\varphi^*\omega = \omega$ and $\psi^*\omega^e = \omega^e$.

Remark: Note that if (φ,ψ) is a symmetry, then the external behavior of the system is invariant under ψ.[5]

Example 2.2:

Consider a particle in $\mathbb{R}^3$ with mass m in a potential field V, and subject to an external force F. Then the system is given by

$$\left.\begin{aligned} m\ddot{q}_i &= \frac{\partial V}{\partial q_i} + F_i \\ y_i &= q_i \quad (y \text{ is the observation}) \end{aligned}\right\} i = 1,2,3.$$

Suppose the equations $m\ddot{q}_i = \partial V/\partial q_i$ are invariant under the rotation R around the e_1-axis (this is equivalent to $H = \frac{1}{2}m\dot{q}^2 + V$ invariant). Then we know that $R^*: T^*\mathbb{R}^3\to T^*\mathbb{R}^3$ ($T^*\mathbb{R}^3$ is the phase space) is a Hamiltonian symmetry for the system without external force. In this case also $Y = \mathbb{R}^3$, and so $T^*Y = T^*\mathbb{R}^3$. The pair (R^*,R^*) is a Hamiltonian symmetry in the sense of Definition 2.1. It expresses the fact that for this system the observation corresponding to an external force which is rotated around the e_1-axis is obtained by rotating the observation in the same way. We can immediately prove the following

Theorem 2.3: Let $\Sigma(M,T^*Y,L)$ be a locally minimal Hamiltonian system. Let (φ,ψ) be a symmetry for Σ. Then (i) ψ is a fiber respecting bundle morphism and assuming $\psi^*\omega^e = \omega^e$, then, (ii) $\varphi^*\omega = \omega$, and hence (φ,ψ) is a Hamiltonian symmetry.

Proof (see also Refs. 17 and 5):

(i) follows from the structure of L.

(ii) Since $\dot{\omega} - \omega^e|_L = 0$ and $(\varphi_*,\psi)(L) = L$, also $\dot{\omega} - \omega^e|_{(\varphi_*,\psi)(L)} = 0$, and hence $(\varphi_*)^*\dot{\omega} - \psi^*\omega^e|_L = 0$. Because $\psi^*\omega^e = \omega^e$ and $\dot{\omega} - \omega^e|_L = 0$, it follows that $(\varphi_*)^*\dot{\omega} - \dot{\omega}|_L = 0$.

Define $\alpha := \varphi^*\omega - \omega$. Then $\dot{\alpha}|_L = 0$. Let L have as a generating function $H + u^TC$. If follows that $\dot{\alpha}(\dot{X}_{C_i}, -) = 0, i = 1,...,m$, or equivalently, $\alpha(X_{C_i}, -) = 0, i = 1,...,m$.

Since $L_{X_H}\omega = 0$ and $L_{\varphi_*X_H}\omega = 0$, and therefore $\varphi^*(L_{\varphi_*X_H}\omega) = L_{X_H}\varphi^*\omega = 0$, it follows that $L_{X_H}\alpha = 0$. Hence $0 = L_{X_H}(\alpha(X_{C_i}, -)) = \alpha(L_{X_H}(X_{C_i})) = \alpha(X_{\{H,C_i\}}, -)$ for $i = 1,...,m$. In the same way we can prove by induction that for every function $f\in\mathscr{F}$ (see Proposition 1.7): $\alpha(X_f, -) = 0$. Because Σ is locally minimal, $\dim d\mathscr{F}(x) = 2n$ for every $x\in M$ and so $\alpha = 0$, or $\varphi^*\omega = \omega$.

As is common[1,2] we will concentrate in the sequel on infinitesimal symmetries, in which case the analog of Definitio 2.1 becomes

Definition 2.4 (Infinitesimal Hamiltonian symmetry): Let $\Sigma(M,T^*Y,L)$ be a Hamiltonian system. An infinitesimal symmetry is a pair (S,T), with S vector field on M and T vector field on T^*Y, such that for every $z\in L$, $(\dot{S}(z), T(z))\in T_zL$. An infinitesimal symmetry (S,T) is an infinitesimal Hamiltonian symmetry if $L_S\omega = 0$ and $L_T\omega^e = 0$.

Analogous to Theorem 2.3 we can prove that the vector field T is necessary fiber respecting, and that if $\Sigma(M,T^*Y,L)$ is locally mininmal and (S,T) is an infinitesimal symmetry with $L_T\omega^e = 0$, then also $L_S\omega = 0$. We also note that because $L_T\omega^e = 0$ and T is fiber respecting the Hamiltonian function G^e corresponding to T $[\omega^e(T, -) = dG^e]$ has the form $G^e(y,u) = \Sigma_{i=1}^m u_iK_i(y) + V(y)$ with K_i and V smooth functions on Y.[5]

Now we will give a generalization of Noether's theorem in our framework. Recall the setting of Noether's theorem for a triplet (M,ω,H). A vector field S on M is an infinitesimal Hamiltonian symmetry if $L_S\omega = 0$ and $S(H) = 0$. A function $G: M\to\mathbb{R}$ is a conservation law for (M,ω,H) if $X_H(G) = 0$. Now let S be an infinitesimal symmetry, then since $L_S\omega = 0$, there exists (locally) a $G: M\to\mathbb{R}$ such that $S = X_G$. Since $X_G(H) = S(H) = 0$ it follows that $X_H(G) = 0$. So G is a conservation law. Conversely, if G is a conservation law then $L_{X_G}\omega = 0$ and $X_G(H) = -X_H(G) = 0$, so X_G is an infinitesimal symmetry. We first derive

Theorem 2.5[5]: Let $\Sigma(M,T^*Y,L)$ be a Hamiltonian system with generating function $H + \Sigma_{i=1}^m u_iC_i$. Let (S,T) be an infinitesimal Hamiltonian symmetry. Let $G: M\to\mathbb{R}$ and G^e: $T^*Y\to\mathbb{R}$ with $G^e(y,u) := \Sigma_{i=1}^m u_iK_i(y) + V(y)$ the (locally defined) functions such that $\omega(S, -) = dG$ and $\omega^e(T, -) = dG^e$. Then $\{H(x) + u^TC(x), G(x)\} = G^e(C(x),u)$, for every u or equivalently

$$\{H,G\} = V\circ C,$$

$$\{C_i,G\} = K_i\circ C, \quad i = 1,...,m$$

with $\{\ ,\ \}$ the Poisson bracket on M.

The pair (G,G^e) as above can be called a conservation law in our framework. The derivative of the function G (the conserved quantity) along trajectories of the system is a function G^e of the behavior on the boundary of the system. Therefore we have proved in Theorem 2.5 that if (S,T) is an infinitesimal Hamiltonian symmetry, then (G,G^e), with $S = X_G$ and $T = X_{G^e}$, is a conservation law. Conversely, it can be easily seen that if (G,G^e) is a conservation law, i.e., $G^e = \Sigma_{i=1}^m u_iK_i + V$ and Eqs. (2.1) are satisfied, then (X_G, X_{G^e}) is an infinitesimal Hamiltonian symmetry.

Example 2.2 (continued): The group of rotations around the e_1-axis generates an infinitesimal Hamiltonian symmetry S on $M = T^*\mathbb{R}^3$ for the system without external forces. The corresponding conservation law is the angular momentum G around the e_1-axis. For zero external force we obtain $dG/dt = 0$, with $G: = \langle q\times m\dot{q},e_1\rangle$. However, for a nonzero external force F we obtain

$$\frac{dG}{dt} = \langle q\times F,e_1\rangle = \langle y\times u,e_1\rangle, \quad u = F.$$

Now $G^e(y,u) := \langle y \times u, e_1 \rangle$ is a function on $T^*Y = T^*\mathbb{R}^3$, and (G,G^e) is our conservation law.

Using minimality we can sharpen Theorem 2.5 in the following way.

Proposition 2.6: Let $\Sigma(M,T^*Y,L)$ be a locally minimal Hamiltonian system with generating function $H + \Sigma_{i=1}^m u_i C_i$. Let (S_i,T_i), $i = 1,...,k$ be infinitesimal Hamiltonian symmetries with corresponding conservation laws (G_i,G_i^e), $i = 1,...,k$. Let $\mathscr{G}$ be the Poisson algebra on M generated by G_i, $i = 1,...,k$, and let $\mathscr{G}^e$ be the Poisson algebra on T^*Y generated by G_i^e, $i = 1,...,k$. Then the map α: $\mathscr{G}$(modulo constant functions)$\rightarrow \mathscr{G}^e$, defined by α: $G(x) \rightarrow \{H(x) + u^T C(x), G(x)\} =: G^e(C(x),u)$ is a Poisson algebra isomorphism.

Proof: Let $G_1,G_2 \in \mathscr{G}$ and G_1^e, $G_2^e \in \mathscr{G}^e$, with $G_1^e(y,u) = \Sigma_{j=1}^m u_j K_j^i(y) + V^i(y)$ such that $\{H + u^T C, G_i\} = G_i^e \circ (C,\mathrm{id})$, $i = 1,2$ with $(G_1^e \circ (C,\mathrm{id}))(x,u) = G_i^e(C(x),u)$. The Jacobi identity implies

$$\begin{aligned}
\{H + u^T C, \{G_1,G_2\}\} &= \{\{H + u^T C,G_1\},G_2\} - \{\{H + u^T C,G_2\},G_1\} \\
&= \{G_1^e \circ (C,\mathrm{id}),G_2\} - \{G_2^e \circ (C,\mathrm{id}),G_1\} \\
&= \sum_{i,j=1}^m u_i \frac{\partial K_i^1}{\partial y_j}\{C_j,G_2\} + \sum_{j=1}^m \frac{\partial V^1}{\partial y_j}\{C_j,G_2\} \\
&\quad - \sum_{i,j=1}^m u_i \frac{\partial K_i^2}{\partial y_j}\{C_j,G_1\} - \sum_{j=1}^m \frac{\partial V^2}{\partial y_j}\{C_j,G_1\} \\
&= \sum_{i=1}^m u_i \left(\sum_{j=1}^m \frac{\partial K_i^1}{\partial y_j} K_j^2 - \frac{\partial K_i^2}{\partial y_j} K_j^1 \right) \\
&\quad + \sum_{j=1}^m \left(\frac{\partial V^1}{\partial y_j} K_j^2 - \frac{\partial V^2}{\partial y_j} K_j^1 \right) \\
&= \left\{ \sum_{i=1}^m u_i K_i^1 + V^1, \sum_{i=1}^m u_i K_i^2 + V^2 \right\}_{T^*Y} \\
&= \{G_1^e, G_2^e\}_{T^*Y},
\end{aligned}$$

where $\{\ ,\ \}_{T^*Y}$ means Poisson bracket on T^*Y.

Therefore the map α is a Poisson algebra morphism. It is immediate that constant functions are mapped to zero. Suppose that a function $G \in \mathscr{G}$ satisfies $\{H + u^T C, G\} = 0$. Then $\{H,G\} = 0$ and $\{C_i,G\} = 0$, $i = 1,...,m$. Therefore for every $f \in \mathscr{F}$ (see Proposition 1.7, the algebra generated by C_i under taking Poisson brackets with H and C_i), $\{f,G\} = 0$. Since Σ is locally minimal, this implies $G = $ const. So α is an isomorphism. □

Proposition 2.6 implies that for locally minimal systems our definition of a Hamiltonian symmetry really covers the ususal one for a triplet (M,ω,H). Indeed, suppose that $S = X_G$ is an infinitesimal Hamiltonian symmetry, so $\{H,G\} = 0$, which cannot be observed, i.e., $\{C_i,G\} = 0$, $1,...,m$. Then G is constant and therefore $S = 0$.

A maybe unsatisfying feature of Theorem 2.5 is that we obtain $\{H,G\} = V \circ C$, instead of $\{H,G\} = 0$ as in the case without external forces. We will now show how by adding a potential P, only depending on the observations, to the Hamiltonian H we can change $\{H,G\} = V \circ C$ into $\{H + P \circ C, G\} = 0$.

Theorem 2.7: Let $\Sigma(M,T^*Y,L)$ be a Hamiltonian system. Let (S_i,T_i), $i = 1,...,k$, be infinitesimal Hamiltonian symmetries such that $\pi_* T_i$, $i = 1,...,k$, are independent vector fields on Y (π is the projection of T^*Y on Y), which are therefore nowhere zero. Let (G_i,G_i^e) be the corresponding conservation laws. Suppose that $\{G_i^e,G_j^e\}_{T^*Y} = 0$, $i,j = 1,...,k$. Then we can (locally) construct a function P: $Y \rightarrow \mathbb{R}$ such that $\{H + P \circ C, G_i\} = 0$, $i = 1,...,k$.

Proof: Since $\{G_i^e,G_j^e\} = 0$, also $[T_i,T_j] = 0$. This implies $[\pi_* T_i, \pi_* T_j] = 0$, $i,j = 1,...,k$. Therefore we can take local coordinates $\{y_1,...,y_m\}$ for Y such that $\pi_* T_i = \partial/\partial y_i$, $i = 1,...,k$. Denote $v_i := G_i^e$, $i = 1,...,k$. Then we have independent functions $y_1,...,y_m$ and $v_1,...,v_k$, $k \leqslant m$, such that

$$\begin{aligned}
&\{y_i,y_j\} = 0, \quad i,j = 1,...,m, \\
&\{v_i,v_j\} = 0, \quad i,j = 1,...,k, \\
&\{y_i,v_j\} = \delta_{ij}, \quad i = 1,...,m, \quad j = 1,...,k.
\end{aligned}$$

Therefore (cf. Ref. 2 Darboux's theorem) we can construct a complementary set of independent functions $v_{k+1},...,v_m$ such that

$$\begin{aligned}
&\{v_i,v_j\} = 0, \quad i = 1,...,m, \quad j = k+1,...,m, \\
&\{y_i,v_j\} = \delta_{ij}, \quad i = 1,...,m, \quad j = k+1,...,m,
\end{aligned}$$

or equivalently, $\{y_1,...,y_m, v_1,...,v_m\}$ are symplectic coordinates. The submanifold of T^*Y given by $v_1 = \cdots = v_m = 0$ is Lagrangian and has therefore (locally) a generating function P: $Y \rightarrow \mathbb{R}$. Since $\Sigma(M,T^*Y,L)$ has generating function $H + \Sigma_{i=1}^m u_i C_i$ in the old coordinates (y,u), it has generating function $H + P \circ C + \Sigma_{i=1}^m v_i C_i$ in the new coordinates (y,v). Because $G_i^e = v_i$ $i = 1,...,k$ it follows that $\{H + P \circ C, G_i\} = 0$, $i = 1,...,k$.

Remark: Notice that when we write $C = (C_1,...,C_m)$ corresponding to the y-coordinates constructed above we obtain

$$\{C_j,G_j\} = \delta_{ij}, \quad i = 1,...,k, \quad j = 1,...,m.$$

If dim $T^*Y = $ dim $M = 2n$, and if we have n symmetries (S_i,T_i) satisfying the conditions of Theorem 2.7, we can construct (locally) a function P: $Y \rightarrow \mathbb{R}$ such that $\widetilde{H} = H + P \circ C$ satisfies $\{\widetilde{H}, G_i\} = 0$, $i = 1,...,n$. Moreover, Proposition 2.6 implies that since $\{G_i^e,G_j^e\}_{T^*Y} = 0$, also $\{G_i,G_j\}_M = c_{ij}$, with c_{ij} constants. Hence we are very near to the case of complete integrability (cf Refs. 1 and 2), for which we need n symmetries G_i satisfying $\{G_i,G_j\} = 0$. Therefore we cannot construct action-angle coordinates,[1,2] but using the remark above and assuming that ker dC is a Lagrangian submanifold of M (think for instance of observation of the positions), it follows from a result in[18] that in this case we can take symplectic coordinates $\{q_1,...,q_n, p_1,...,p_n\}$ for M such that

$$\begin{aligned}
&C_i = q_i, \quad i = 1,...,n, \\
&G_i = p_i - \frac{1}{2}\sum_{j=1}^n c_{ij} q_j.
\end{aligned}$$

Since $0 = \{\widetilde{H}, G_i\} = \{\widetilde{H},p_i\} - \frac{1}{2}\Sigma_{j=1}^n c_{ij} \{\widetilde{H},q_j\}$ the system in these coordinates, after adding the potential P, is given by

$$\dot{q}_i = \frac{\partial \widetilde{H}}{\partial p_i}, \quad i = 1,...,n,$$

$$\dot p_i = -\frac{\partial \tilde H}{\partial q_i} + F_i = -\frac{1}{2}\sum_{j=1}^{n} c_{ij}\dot q_j + F_i,$$

$y_i = q_i$ (F_i the external force).

For degenerate Hamiltonian systems we obtain the following definition of a Hamiltonian symmetry.

Definition 2.8: Let $\Sigma(M,T^*Y,L = L'\cap(TM\times P))$ be a degenerate Hamiltonian system. A pair of vector fields (S,T) is an infinitesimal Hamiltonian symmetry for $\Sigma(M,T^*Y,L)$ if (S,T) is an infinitesimal Hamiltonian symmetry for $\Sigma(M,T^*Y,L')$ and the vector field T is tangent to $P\subset T^*Y$, i.e., $T(z)\in T_zP$, for every $z\in P$.

Hence we have formalized the idea that the symmetry should only work on the possible external behavior. In local coordinates we obtain

Proposition 2.9: Let (S,T) be an infinitesimal Hamiltonian symmetry for the degenerate Hamiltonian system $\Sigma(M,T^*Y,L = L'\cap(TM\times P))$. Then we can find coordinates $y_1,...,y_m$ for Y such that $P = \text{span}\{dy_1,...,dy_k\}$, $k\leqslant m$, and such that G^e [with $\omega^e(T,-) = dG^e$] has the form

$$G^e(y,u) = \sum_{i=1}^{k} u_iK_i(y_1,...,y_k) + V(y_1,...,y_k) + \sum_{i=k+1}^{m} u_iK_i(y_1,...,y_m).$$

Proof: We know that $G^e = \Sigma_{i=1}^m u_iK_i(y_1,...,y_m) + V(y_1,...,y_m)$. Since T is tangent to P we must have

$$\Sigma_{i=1}^m u_i\frac{\partial K_i}{\partial y_j} + \frac{\partial V}{\partial y_j} = 0 \quad \text{for } u_{k+1} = \cdots = u_m = 0, \quad j = k+1,...,m,$$

or equivalently,

$$\frac{\partial K_i}{\partial y_j} = 0 \quad i = 1,...,k, \quad j = k+1,...,m$$

and

$$\frac{\partial V}{\partial y_j} = 0, \quad j = k+1,...,m. \qquad \square$$

We see that the vector field T in this case projects to a vector field on $T^*\tilde Y$, where $\tilde Y$ has coordinates $y_1,...,y_k$. It can also be seen that the additional potential P in Theorem 2.7 in this case only has to depend on $y_1,...,y_k$.

IV. TIME-REVERSIBLE HAMILTONIAN SYSTEMS

Time reversibility of a system is a widely used but many times rather vaguely defined notion. In our framework it can be defined in the following way.[19] We say that the external behavior of a system $\Sigma(M,T^*Y,L)$ is time reversible if, when $(\bar y(t),\bar u(t))$, $t\in\mathbb{R}$ belongs to the external behavior of the system; also the time-reversed signal $(\bar y(-t),\bar u(-t))$, $t\in\mathbb{R}$ is a feasible external behavior.

Let now $\Sigma(M,T^*Y,L)$ be a Hamiltonian system, locally given by

$$\begin{cases} \dot x = X_H(x) + \sum_{i=1}^{m} u_iX_{C_i}(x), & x\in M, \\ y_i = C_i(x), & i = 1,...,m. \end{cases} \tag{3.1}$$

If the external behavior of this system is time reversible then the Hamiltonian system

$$\dot x = -X_H(x) - \sum_{i=1}^{m} u_iX_{C_i}(x), \quad x\in M,$$
$$y_i = C_i(x), \quad i = 1,...,m \tag{3.2}$$

has the same external behavior.

If the system Σ is (locally) minimal it seems then reasonable to ask that there exists a diffeomorphism $\varphi:M\to M$ which carries Eqs. (3.1) over in (3.2). In the case of a linear Hamiltonian system this can actually be proven.[7]

Definition 3.1 (Time reversibility): Let $\Sigma(M,T^*Y,L)$ be a locally minimal Hamiltonian system, locally given by (3.1). Then Σ is time reversible if there exists a diffeomorphism $\varphi:M\to M$ such that

$$\varphi_*X_H = -X_H, \quad \varphi_*X_{C_i} = -X_{C_i},$$
$$\text{and } \varphi^*C_i = C_i, \quad i = 1,...,m.$$

In this case we will call φ a time-reversing symmetry.

Using local minimality we can prove two important properties.

Theorem 3.2: Let φ be a time-reversing symmetry for a locally minimal Hamiltonian system $\Sigma(M,T^*Y,L)$. Then

(i) φ is an involution, i.e., $\varphi^2 = \text{id}$.

(ii) φ is an anti-symplectomorphism, i.e., $\varphi^*\omega = -\omega$.

Proof: Denote $A := X_H$ and $B_i := X_{C_i}$.

(i) We will prove that every function f on M, generated by taking (repeated) Poisson brackets of C_i, $i = 1,...,m$, and H and C_i satisfies $\varphi^*f = \pm f$. By assumption $\varphi^*C_i = C_i$, and for instance

$$\varphi^*\{H,C_i\} = \varphi^*(L_AC_i) = L_{\varphi.} - 1_A\varphi^*C_i = L_{-A}C_i = -\{H,C_i\}.$$

By induction it follows that $\varphi^*f = \pm f$, for every f constructed as above. Hence $(\varphi^2)^*f = f$ for every $f\in\mathcal{F}$ (see Proposition 1.7). Since Σ is locally minimal, this implies $\varphi^2 = \text{id}$, if we assume that φ^2 has at least one fixed point.

(ii) Define $\alpha = \varphi^*\omega + \omega$. Since $\varphi^*\omega(B_i,-) = \omega(\varphi_*B_i,\varphi_*-) = -\omega(B_i,\varphi_*-) = -dC_i(\varphi_*-) = -\varphi_*dC_i = -d\varphi^*C_i = -dC_i = -\omega(B_i,-)$, we obtain that $\alpha(B_i,-) = 0$. $i = 1,...,m$.

Furthermore

$$L_A\alpha = L_A\omega + L_A\varphi^*\omega = \varphi^*(L_{\varphi.A}\omega) = \varphi^*(L_{-A}\omega) = 0$$

and

$$0 = L_A(\alpha(B_i,-)) = \alpha(L_AB_i,-) = \alpha(X_{\{H,C_i\}},-),$$

and therefore by induction

$$\alpha(X_f,-) = 0$$

for every f constructed as above. Local minimality implies $\alpha = 0$, or $\varphi^*\omega = -\omega$.

Remark: Note that $\varphi_*X_H = X_H$ together with $\varphi^*\omega = -\omega$ implies $\varphi^*H = H + \text{const}$.

Maps $\varphi:M\to M$ with exactly these properties (i) and (ii) have been studied in Ref. 20 (see also the references cited there). It can be proved[20] that the points $p\in M$ such that $\varphi(p) = p$ form a Lagrangian submanifold Q of M. Furthermore we can find a neighborhood U of Q and local coordinates $(q_1,...,q_n,p_1,...,p_n)$ for U such that Q is the submanifold given by the equations $p_1 = \cdots = p_n = 0$, $\omega = \Sigma_{i=1}^n dq_i\wedge dp_i$ and φ is given by

$$\varphi:(q_1,\ldots,q_n,p_1,\ldots,p_n)\mapsto(q_1,\ldots,q_n,-p_1,\ldots,-p_n).$$

Consequently if H and C_k, $k=1,\ldots,m$ are at most quadratic in the p_j-coordinates, it follows from $\varphi^*H = H +$ const and $\varphi^*C_k = C_k$ that H and C_k have the form

$$H(q,p) = \sum_{i,j=1}^{n} g_{ij}(q)p_ip_j + V(q),$$

$$C_k(q,p) = \sum_{i,j=1}^{n} h^k_{ij}(q)p_ip_j + W^k(q), \quad k=1,\ldots,m,$$

with $g_{ji}=g_{ij}$ and $h^k_{ij}=h^k_{ji}$. Especially the form of H is very appealing; it denotes a Hamiltonian consisting of the sum of a potential $V(q)$ and a kinetic energy given by a "Riemannian metric."

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