Essential singularities of rigid analytic functions

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INTRODUCTION

The Picard theorem for a complex analytic function can be formulated as follows:

"Let $f$ be a holomorphic function on $\{z \in \mathbb{C} \mid 0 < |z| < 1\}$ with values in $\mathbb{C} - \{0,1\}$ then $f$ can be extended as meromorphic function on $\{z \in \mathbb{C} \mid |z| < 1\}$".

A short proof of this statement would be the following: The group $\Gamma(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod 2 \right\}$ acts freely as a group of fractional linear transformations on the upper half-space $H$. The group has 3 parabolic points and the genus of the corresponding algebraic curve is 0. This means that $H/\Gamma(2) \cong \mathbb{C} - \{0,1\}$ and as a consequence $\pi : H \to \mathbb{C} - \{0,1\}$ is the universal covering of $\mathbb{C} - \{0,1\}$.

Let

$U_1 = \{z \in \mathbb{C} \mid 0 < |z| < 1, \arg(z) = \pi\}$;

$U_2 = \{z \in \mathbb{C} \mid 0 < |z| < 1, \arg(z) = 0\}$

$U_1 \cap U_2 = U^+ \cup U^-$ where

$U^+ = \{z \in U_1 \cap U_2 \mid \text{im}(z) > 0\}$
and

\[ U^- = \{ z \in U_1 \cap U_2 \mid \text{im}(z) < 0 \}. \]

There are lifts \( f_i : U_i \to H \) of \( f/U_i \) (i.e. \( \pi \circ f_i = f/U_i \) for \( i = 1, 2 \)) such that \( f_i(\xi i) = f_j(\xi j) \). So \( f_1 \) coincides with \( f_2 \) on \( U^+ \). There is a unique \( \gamma \in \Gamma(2) \) with \( f_1 = \gamma \circ f_2 \) on \( U^- \).

We divide \( H \) by the action of \( \langle \gamma \rangle \), the subgroup of \( \Gamma(2) \) generated by \( \gamma \). The result \( H' = H/\langle \gamma \rangle \) is analytically isomorphic to one of the following spaces

(a) \( \{ z \in \mathbb{C} \mid |z| < 1 \} \) if \( \gamma = \text{id} \).

(b) \( \{ z \in \mathbb{C} \mid 0 < |z| < 1 \} \) if \( \gamma \) is parabolic

(c) \( \{ z \in \mathbb{C} \mid r < |z| < 1 \} \) for some \( r > 0 \) if \( \gamma \) is hyperbolic.

Let \( \pi' : H' \to \mathbb{C} - \{0, 1\} \) denote the natural map induced by \( \pi \). From the above it follows that \( f \) lifts to a holomorphic map \( F : \{ z \in \mathbb{C} \mid 0 < |z| < 1 \} \to H' \) such that \( \pi' \circ F = f \). Since \( F \) is bounded, it follows that \( F \) (and so also \( f \)) extends to \( \{ z \in \mathbb{C} \mid |z| < 1 \} \).

We consider a field \( K \), complete with respect to a non-archimedean valuation. In order to simplify the exposition we suppose that \( K \) is algebraically closed. Let \( \mathbb{P} = \mathbb{P}(K) \) denote the projective line over \( K \). In many situations one has to study holomorphic or meromorphic functions on an open set \( \Omega \subset \mathbb{P} \) of the form \( \Omega = \mathbb{P} - L \), where \( L \) is a compact set. We call \( L \) an essential singularity for the meromorphic function \( f \) on \( \Omega \) if \( f \) does not extend to a meromorphic function on any \( \Omega' = \mathbb{P} - L' \) where \( L' \) is a proper closed subset of \( L \).

If \( L \) has at least one isolated point then it turns out that \( f(\Omega) \) omits at most one value of \( \mathbb{P} \). However if \( L \) is perfect then \( f(\Omega) \) may omit a finite number of values in \( \mathbb{P} \) (§ 2, example 1) or \( f(\Omega) \) may even omit a compact infinite subset of \( \mathbb{P} \) (§ 2, example 2).

The examples are derived from the theory of discontinuous groups over a non-archimedean valued field. In this respect the theory seems quite far from its archimedean analogue. We refer to [1] and [2] for non-archimedean function theory of one variable and for discontinuous groups.

§ 1. POSITIVE RESULTS ON THE VALUES OF HOLOMORPHIC MAPS

A connected affinoid subset \( X \) of \( \mathbb{P} \) is a subset of the form \( X = \mathbb{P} - (B_1 \cup \ldots \cup B_n) \) where \( B_1, \ldots, B_n \) are disjoint open disks in \( \mathbb{P} \). The \( B_1, \ldots, B_n \) are usually called the holes of \( X \); their number is \( n \).

(1.1) PROPOSITION. Let \( f \) be a non-constant holomorphic function on a connected affinoid subset \( X \) of \( \mathbb{P} \). Then \( f(X) \) is a connected affinoid subset of \( \mathbb{P} \). Moreover the number of holes of \( f(X) \) is less than or equal to the number of holes of \( X \).

PROOF. The canonical reduction \( \overline{X} \) of \( X \) is the maximal ideal space of \( \overline{\partial(X)} \) ([2] p. 113). According to [2] p. 78, 79 the ring \( \overline{\partial(X)} \) has the form \( \mathbb{K}[z_1, \ldots, z_n]/I \)
where $I$ is an ideal generated by elements $E_{i,j}(i \neq j, 1 \leq i, j \leq n)$ of the form
\[ E_{ij} = z_i z_j + \alpha_{ij} z_i + \beta_{ij} z_j \] with $\alpha_{ij}, \beta_{ij} \in \mathbb{K}$.

It follows that each component $L$ of $X$ is isomorphic to $\mathbb{P}(\mathbb{K}) - V(L)$ where $V(L)$ is a finite non-empty subset of $\mathbb{P}(\mathbb{K})$. We construct $\hat{X}$, the completion of $X$, by completing each component $L$ of $X$ to a $\mathbb{P}(\mathbb{K})$. The total number of "missing" points of $X$ (i.e. the points of $\hat{X} - \hat{X}$) is equal to $\Sigma \neq V(L) = n$ - the number of holes of $X$.

The set $Y = f(X)$ is according to [2] p. 110, lemma (2.7), the union of an affinoid set and a finite set. Since $X$ is connected it follows that $Y$ is actually a connected affinoid subset of $\mathbb{P}$.

The surjective map $f : X \to Y$ induces a morphism $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$ which is an isometry with respect to the spectral norms $||.||_{sp}$ on $X$ and $Y$. We obtain an induced, injective $\overline{f}^* : \mathcal{O}(\overline{Y}) \to \mathcal{O}(\overline{X})$ and a surjective (since [2] p. 114, lemma (2.9.1)) morphism $\overline{f} : \overline{X} \to \overline{Y}$.

The restriction of $\overline{f}$ to any component $L = \mathbb{P}(\mathbb{K}) - V(L)$ of $\overline{X}$ extends uniquely to a morphism of $\mathbb{P}(\mathbb{K}) \to \overline{Y}$. So $\overline{f}$ extends to a morphism $\overline{f} : \overline{X} \to \overline{Y}$. The last map is surjective since $\overline{f}(\overline{X})$ is complete and contains $\overline{Y}$. Hence the number of missing points of $\overline{Y}$ is $\leq n$. This proves the proposition.

We propose now a second proof of the last statement of the proposition. In [1] § 1, (1.8.9) one has established an exact sequence
\[ 0 \to A(X) \to \mathcal{O}(X)^* \to \mathbb{Z}^{n-1} \to 0 \]
in which $\mathcal{O}(X)^*$ is the group of invertible holomorphic functions on $X$; $n$ is the number of holes of $X$; $A(X) = \{ \lambda(1 + h) \mid \lambda \in \mathbb{K}^*, h \in \mathcal{O}(X), ||h||_{sp} < 1 \}$.

Let $m$ be the number of holes of $Y$. The map $f$ induces $f^* : \mathcal{O}(Y)^* \to \mathcal{O}(X)^*$ such that $(f^*)^{-1}(A(X)) = A(Y)$. So we find an injective map $\mathbb{Z}^{m-1} \to \mathbb{Z}^{n-1}$ and we have shown that $m \leq n$.

(1.2) PROPOSITION. Let $L$ be a compact subset of $\mathbb{P}$ and let $\Omega = \mathbb{P} - L$ denote the analytic subspace of $\mathbb{P}$ defined by the family
\[ \{ F \mid F \text{ affinoid in } \mathbb{P}; F \cap L = \emptyset \} . \]
For any non-constant holomorphic map $f : \Omega \to \mathbb{P}$ the set $\mathbb{P} - f(\Omega)$ is compact.

PROOF. We consider the subspace $\Omega'$ of $\mathbb{P}$ defined by the family of affinoid sets $\{ f(X) \mid X \text{ affinoid}; X \cap L = \emptyset \}$. If $\Omega'$ is not of the form $\mathbb{P} - \{ \text{a compact set} \}$ then, according to [2] p. 145, (2.5), there exists a non-constant bounded holomorphic function $h$ on $\Omega'$. The holomorphic function $h \circ f$ on $\Omega$ is also bounded and must be constant according to the same result. This implies however that $f$ is constant. So the proposition is proved and we have proved slightly more, namely: every affinoid subset, lying in $f(\Omega)$, is the image of an affinoid subset of $\Omega$ under the map $f$. 

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(1.3) PROPOSITION. (A version of Picard’s theorem). Let \( f \) be meromorphic function on \( \{ z \in K \mid R < |z| \} \) which cannot be extended at \( \infty \). Then \( f \) omits at most one value.

PROOF. We note that this result must be known. By lack of reference we include two proofs. Suppose that \( f \) omits at least one value, then we may take \( f \) to be holomorphic on \( \{ z \in K \mid R < |z| \} \).

(1) FIRST PROOF. We may express \( f \) as a convergent Laurent-series

\[
\sum_{n=-\infty}^{\infty} a_n z^n
\]

which has infinitely many \( a_n \neq 0 \) for \( n > 0 \).

For \( \varrho \in |K^*|, R < \varrho < \infty \), we form \( \max |a_n| \varrho^n = \alpha(\varrho) \) and we denote the smallest integer \( n \) with \( |a_n| \varrho^n = \alpha(\varrho) \) by \( n(\varrho) \).

Clearly \( \lim_{\varrho \to \infty} n(\varrho) = \lim_{\varrho \to \infty} \alpha(\varrho) = \infty. \) We will suppose that \( \varrho \gg R \) such that \( n(\varrho) > 0 \). The set \( X_\varrho = \{ z \in K \mid |z| = \alpha(\varrho) \} \) can have the following form:

(a) Suppose that there is only one \( n \) with \( |a_n| \varrho^n = \alpha(\varrho) \), then

\[
X_\varrho = \{ z \in K \mid |z| = \alpha(\varrho) \}
\]

(b) Suppose that there are more positive integers \( n \) with \( |a_n| \varrho^n = \alpha(\varrho) \), then

\[
X_\varrho = \{ z \in K \mid |z| \leq \alpha(\varrho) \}.
\]

The above follows from the well-known statement:

\[
\sum_{n=-\infty}^{\infty} b_n z^n \in \theta(\{ z \in K \mid |z| = 1 \})
\]

has no zeros if and only if there is precisely one \( m \) with \( |b_m| = \max_n |b_n| \).

Situation (b) occurs for an infinite sequence \( \varrho_1, \varrho_2, \ldots \) with \( \lim \varrho_i = \infty. \) Hence

\[
f(\{ z \in K \mid R < |z| \}) = K.
\]

(2) SECOND PROOF. Suppose that the holomorphic map \( f \) omits at least two values in \( \mathbb{P} \). Then we may suppose that \( f \) omits 0 and \( \infty \). In other words \( f \in \theta(\{ z \in K \mid R < |z| \})^* \). Using [1] § 1, (1.8.9) one sees that \( f \) has the form \( \lambda z^n (1 + h) \) where \( \lambda \in K^* \), \( n \in \mathbb{Z} \) and \( h \) is holomorphic on \( \{ z \in K \mid R < |z| \} \) such that \( |h(z)| < 1 \) for all \( z \). But then \( h \) can be extended to \( \infty \) and so also \( f \) extends at \( \infty \).

§ 2. TWO EXAMPLES

(2.1) The first example imitates the proof of Picard’s theorem that we have given in the introduction.

Let \( k = \mathbb{F}_q ((1/t)) \) be the Laurent-series field in the variable \( 1/t \) and with coefficients in the finite field \( \mathbb{F}_q \). Let \( K \) denote the completion of the algebraic closure of \( k \).

The group \( \Gamma(t) \) is the subgroup of \( \Gamma(1) = \text{PSL}(2, \mathbb{F}_q[t]) \) consisting of the matrices

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ modulo}(t).
\]

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In [2], Chapter 10, it is calculated that: \( f(t) \) has \((q + 1)\) inequivalent parabolic points and that the genus of the corresponding algebraic curve is zero.

So the holomorphic map

\[
f : \mathbb{P}(K) - \mathbb{P}(k) \rightarrow \mathbb{P}(K) - \mathbb{P}(k)/\Gamma(t) = \mathbb{P}(K) - \mathbb{P}(\mathbb{F}_q)
\]

omits exactly \( q + 1 \) values. We still have to verify that \( f \) has an essential singularity at the compact subset \( \mathbb{P}(k) \) of \( \mathbb{P} \).

Let \( L \) be the smallest compact subset of \( \mathbb{P} \) such that \( f \) admits an extension as meromorphic function on \( \mathbb{P} - L \). One easily sees that \( L \) always exists and that \( L \) is invariant under \( \Gamma(t) \). If \( L \neq \phi \) then \( L \) turns out to be \( \mathbb{P}(k) \) since it is invariant. Further \( L = \phi \) would mean that \( f \) is a rational function on \( \mathbb{P} \). But only a constant rational function can be invariant under \( \Gamma(t) \).

In this example one can clearly vary the finite field \( \mathbb{F}_q \) and moreover one can compose \( f \) with a rational function on \( \mathbb{P} \). This shows the following statement:

"Let the field \( K \) have characteristic \( \neq 0 \) and let \( \{a_1, \ldots, a_n\} \) be a subset of \( \mathbb{P}(K) \). There exists a perfect compact subset \( L \) of \( \mathbb{P}(K) \) and a meromorphic function \( f \) with an essential singularity at \( L \) such that \( f(\mathbb{P} - L) = \mathbb{P} - \{a_1, \ldots, a_n\} \)".

(2.2) The second example works for fields \( K \) of any characteristic and residue characteristic. However to simplify matters we assume that the residue field \( \mathfrak{R} \) has a characteristic \( \neq 2 \).

Our construction is a variant of the construction of Whittaker groups done in [2], Chapter 9.

Let the 12 points \( a_1, b_1, \ldots, a_6, b_6 \) in \( \mathbb{P} \) be such that the reduction \( \mathbb{P} \) with respect to this set is:

\[
\begin{array}{cccc}
a_1 & a_2 & \cdots & a_6 \\
b_6 & b_6 & \cdots & b_6 \\
\end{array}
\]

In other terms this means that the position of the 12 points (after an automorphism of \( \mathbb{P} \)) is such that:

1) all \( |a_i| = |b_i| = 1 \)
2) \( |a_i - a_j| = 1 \) for \( i \neq j \)
3) \( |b_i - b_j| = 1 \) for \( i \neq j \)
4) \( |a_i - b_j| = 1 \) for \( i \neq j \)
5) \( |a_i - b_j| < 1 \) for all \( i \).

Let \( s_i \) \((i = 1, \ldots, 6)\) denote the elliptic element of order 2 with fixed points \( a_i, b_i \). In [2] p. 281 it is shown that the group \( \Gamma_0 = \langle s_1, \ldots, s_6 \rangle \) generated by the six reflexions is discontinuous and it is shown that the only relations among the generators are \( s_1^2 = s_2^2 \cdots = s_6^2 = 1 \). Let \( \Omega \) denote the set of ordinary points of \( \Gamma_0 \).
We introduce now four subgroups $\Gamma_i$ ($i = 1, 2, 3, 4$) of $\Gamma_0$ of finite index. Consider the surjective group homomorphism $\phi : \Gamma_0 \to \mathbb{Z}/2 \oplus \mathbb{Z}/2$ given by $\phi(s_i) = (1, 0)$ for $i = 1, 2, 3$ and $\phi(s_i) = (0, 1)$ for $i = 4, 5, 6$. The kernel $\Gamma_4$ of $\phi$ is a Schottky group on 9 free generators. The generators are

$$s_1s_2, s_1s_3, s_4s_5, s_4s_6, s_1s_4s_1s_2s_4s_1, s_1s_4s_1s_3s_4s_1, s_1s_4s_4s_1, s_1s_6s_4s_1, s_1s_4s_4s_1$$

as one easily verifies.

The group $\Gamma_1$ is generated by $\Gamma_4$ and $s_1$; the group $\Gamma_2$ is generated by $\Gamma_4$ and $s_4$, the group $\Gamma_3$ is generated by $\Gamma_4$ and $s_1s_4$. Hence $\Gamma_4 \subset \Gamma_i \subset \Gamma_0$ for $i = 1, 2, 3$ and $[\Gamma_0 : \Gamma_i] = 2$ for $i = 1, 2, 3$.

The group $\Gamma_3$ turns out to be a free group on 5 generators, namely on $\{s_1s_2, s_1s_3, s_4s_5, s_4s_6, s_1s_4\}$.

The groups $\Gamma_i$ ($i = 0, 1, 2$) are not free. One easily calculates that the rank of the abelianized groups $\Gamma_i/[\Gamma_i, \Gamma_i]$ is 2 for $i = 1, 2$.

We write $X_i$ for the algebraic curve $\Omega/\Gamma_i$ ($i = 0, \ldots, 4$). Although the curve is not always parametrized by a Schottky group (cases $i = 0, 1, 2$) the curve is certainly "locally isomorphic to $\mathbb{P}$" and hence a Mumford curve. (See [2] p. 177). Let $g_i$ denote the genus of $X_i$, then we have $g_0 = 0$, $g_1 = g_2 = 2$, $g_3 = 5$, $g_4 = 9$ by using [2] p. 250, 251. Moreover we have a diagram of holomorphic maps of degree two between the various curves:

$$\xymatrix{ X_1 & \ar[l] \ar[r] & X_0 = \mathbb{P} \\ X_4 \ar[r] & X_2 \ar[l] & X_3 \ar[l] }$$

We are especially interested in the morphism $X_4 \to X_1$. The curve $X_1$ is a Mumford curve of genus 2 and can also be parametrized by a Schottky group $\Delta$ with $\Omega'$ as set of ordinary points.

The map $f : X_4 \to X_1$ lifts to a holomorphic map $F : \Omega \to \Omega'$ since $\pi : \Omega \to X_4$ and $\pi_1 : \Omega' \to X_1$ are the universal coverings. (Compare [2] p. 149-153). The holomorphic map $F$ omits an infinite compact set since $\mathbb{P} - \Omega'$ is infinite.

Our example is completed with the following lemma.

**Lemma.** $F$ has an essential singularity at the compact perfect set $\mathbb{P} - \Omega$.

**Proof.** Using the Riemann-Hurwitz formula one finds that $f : X_4 \to X_1$ is ramified in 12 points. Let $p \in \Omega$ be a point such that its image in $X_4$ is one of those 12 points. Since $\pi_4 : \Omega \to X_4$ and $\pi_1 : \Omega' \to X_1$ are locally isomorphisms it follows that also $F$ is ramified (of index 2) at $p$. The whole orbit $\Gamma_4(p)$ consists clearly of ramification points of $F$. Since $p$ is an ordinary point for $\Gamma_4$ the limit points for this orbit are precisely $\mathbb{P} - \Omega$. This implies that $F$ cannot be extended since in any neighbourhood of any $\lambda \in \mathbb{P} - \Omega$ there are infinitely many ramification points of $F$. So $F$ has an essential singularity at $\mathbb{P} - \Omega$.
REFERENCES
