Essential singularities of rigid analytic functions
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INTRODUCTION

The Picard theorem for a complex analytic function can be formulated as follows:

"Let $f$ be a holomorphic function on $\{z \in \mathbb{C} \mid 0 < |z| < 1\}$ with values in $\mathbb{C} - \{0,1\}$ then $f$ can be extended as meromorphic function on

$\{z \in \mathbb{C} \mid |z| < 1\}".

A short proof of this statement would be the following: The group

$\Gamma(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}(2,\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod 2 \right\}$

acts freely as a group of fractional linear transformations on the upper half-space $H$. The group has 3 parabolic points and the genus of the corresponding algebraic curve is 0. This means that $H/\Gamma(2) \cong \mathbb{C} - \{0,1\}$ and as a consequence $\pi : H \rightarrow \mathbb{C} - \{0,1\}$ is the universal covering of $\mathbb{C} - \{0,1\}$.

Let

$U_1 = \{z \in \mathbb{C} \mid 0 < |z| < 1, \arg(z) = \pi\}$;

$U_2 = \{z \in \mathbb{C} \mid 0 < |z| < 1, \arg(z) = 0\}$

$U_1 \cap U_2 = U^+ \cup U^-$ where

$U^+ = \{z \in U_1 \cap U_2 \mid \text{im}(z) > 0\}$

$U^- = \{z \in U_1 \cap U_2 \mid \text{im}(z) < 0\}$
There are lifts $f_i : U_i \to H$ of $f/\pi_i$ (i.e. $\pi \circ f_i = f/\pi_i$ for $i = 1, 2$) such that $f_i(\{i\}) = f_j(\{i\})$. So $f_1$ coincides with $f_2$ on $U^+$. There is a unique $\gamma \in \Gamma(2)$ with $f_1 = \gamma \circ f_2$ on $U^-$. We divide $H$ by the action of $\langle \gamma \rangle$, the subgroup of $\Gamma(2)$ generated by $\gamma$. The result $H' = H/\langle \gamma \rangle$ is analytically isomorphic to one of the following spaces:

(a) $\{ z \in \mathbb{C} \mid |z| < 1 \}$ if $\gamma = \text{id}$.
(b) $\{ z \in \mathbb{C} \mid 0 < |z| < 1 \}$ if $\gamma$ is parabolic
(c) $\{ z \in \mathbb{C} \mid r < |z| < 1 \}$ for some $r > 0$ if $\gamma$ is hyperbolic.

Let $\pi' : H' \to \mathbb{C} - \{0, 1\}$ denote the natural map induced by $\pi$. From the above it follows that $f$ lifts to a holomorphic map $F : \{ z \in \mathbb{C} \mid 0 < |z| < 1 \} \to H'$ such that $\pi' \circ F = f$. Since $F$ is bounded, it follows that $F$ (and so also $f$) extends to $\{ z \in \mathbb{C} \mid |z| < 1 \}$.

We consider a field $K$, complete with respect to a non-archimedean valuation. In order to simplify the exposition we suppose that $K$ is algebraically closed. Let $\mathbb{P} = \mathbb{P}(K)$ denote the projective line over $K$. In many situations one has to study holomorphic or meromorphic functions on an open set $\Omega \subset \mathbb{P}$ of the form $\Omega = \mathbb{P} - L$, where $L$ is a compact set. We call $L$ an essential singularity for the meromorphic function $f$ on $\Omega$ if $f$ does not extend to a meromorphic function on any $\Omega' = \mathbb{P} - L'$ where $L'$ is a proper closed subset of $L$.

If $L$ has at least one isolated point then it turns out that $f(\Omega)$ omits at most one value of $\mathbb{P}$. However if $L$ is perfect then $f(\Omega)$ may omit a finite number of values in $\mathbb{P}$ (§ 2, example 1) or $f(\Omega)$ may even omit a compact infinite subset of $\mathbb{P}$ (§ 2, example 2).

The examples are derived from the theory of discontinuous groups over a non-archimedean valued field. In this respect the theory seems quite far from its archimedean analogue. We refer to [1] and [2] for non-archimedean function theory of one variable and for discontinuous groups.

§ 1. POSITIVE RESULTS ON THE VALUES OF HOLOMORPHIC MAPS

A connected affinoid subset $X$ of $\mathbb{P}$ is a subset of the form $X = \mathbb{P} - (B_1 \cup \ldots \cup B_n)$ where $B_1, \ldots, B_n$ are disjoint open disks in $\mathbb{P}$. The $B_1, \ldots, B_n$ are usually called the holes of $X$; their number is $n$.

(1.1) PROPOSITION. Let $f$ be a non-constant holomorphic function on a connected affinoid subset $X$ of $\mathbb{P}$. Then $f(X)$ is a connected affinoid subset of $\mathbb{P}$. Moreover the number of holes of $f(X)$ is less than or equal to the number of holes of $X$.

PROOF. The canonical reduction $\mathcal{X}$ of $X$ is the maximal ideal space of $\mathcal{O}(X)$ ([2] p. 113). According to [2] p. 78, 79 the ring $\mathcal{O}(X)$ has the form $K[z_1, \ldots, z_n]/I$
where \( I \) is an ideal generated by elements \( E_{i,j}(i \neq j, 1 \leq i, j \leq n) \) of the form
\[
E_{i,j} = z_i z_j + \alpha_{ij} z_i + \beta_{ij} z_j \text{ with } \alpha_{ij}, \beta_{ij} \in K.
\]

It follows that each component \( L \) of \( \hat{X} \) is isomorphic to \( \mathbb{P}(K) - V(L) \) where \( V(L) \) is a finite non-empty subset of \( \mathbb{P}(K) \). We construct \( \hat{X} \), the completion of \( X \), by completing each component \( L \) of \( X \) to a \( \mathbb{P}(K) \). The total number of "missing" points of \( \hat{X} \) (i.e. the points of \( \hat{X} - X \)) is equal to \( \sum \neq V(L) = n \) the number of holes of \( X \).

The set \( Y = f(X) \) is according to [2] p. 110, lemma (2.7), the union of an affinoid set and a finite set. Since \( X \) is connected it follows that \( Y \) is actually a connected affinoid subset of \( \mathbb{P} \).

The surjective map \( f : X \to Y \) induces a morphism \( f^* : \mathcal{O}(Y) \to \mathcal{O}(X) \) which is an isometry with respect to the spectral norms \( \| \cdot \|_{sp} \) on \( X \) and \( Y \). We obtain an induced, injective \( \bar{f}^* : \overline{\mathcal{O}(Y)} \to \overline{\mathcal{O}(X)} \) and a surjective (since [2] p. 114, lemma (2.9.1)) morphism \( \bar{f} : \hat{X} \to \hat{Y} \).

The restriction of \( \bar{f} \) to any component \( L = \mathbb{P}(K) - V(L) \) of \( \hat{X} \) extends uniquely to a morphism of \( \mathbb{P}(K) \to \hat{Y} \). So \( \bar{f} \) extends to a morphism \( \hat{f} : \hat{X} \to \hat{Y} \). The last map is surjective since \( \hat{f}(\hat{X}) \) is complete and contains \( \hat{Y} \). Hence the number of missing points of \( \hat{Y} \) is \( \leq n \). This proves the proposition.

We propose now a second proof of the last statement of the proposition. In [1] § 1, (1.8.9) one has established an exact sequence
\[
0 \to A(X) \to \mathcal{O}(X)^* \to \mathbb{Z}^{n-1} \to 0
\]
in which \( \mathcal{O}(X)^* \) is the group of invertible holomorphic functions on \( X \); \( n \) is the number of holes of \( X \); \( A(X) = \{ \lambda(1 + h) | \lambda \in K^*, h \in \mathcal{O}(X), \| h \|_{sp} < 1 \} \).

Let \( m \) be the number of holes of \( Y \). The map \( f \) induces \( f^* : \mathcal{O}(Y)^* \to \mathcal{O}(X)^* \) such that \( (f^*)^{-1}(A(X)) = A(Y) \). So we find an injective map \( \mathbb{Z}^{m-1} \to \mathbb{Z}^{n-1} \) and we have shown that \( m \leq n \).

(1.2) PROPOSITION. Let \( L \) be a compact subset of \( \mathbb{P} \) and let \( \Omega = \mathbb{P} - L \) denote the analytic subspace of \( \mathbb{P} \) defined by the family
\[
\{ F | F \text{ affinoid in } \mathbb{P}; F \cap L = \phi \}.
\]
For any non-constant holomorphic map \( f : \Omega \to \mathbb{P} \) the set \( \mathbb{P} - f(\Omega) \) is compact.

PROOF. We consider the subspace \( \Omega' \) of \( \mathbb{P} \) defined by the family of affinoid sets \( \{ f(X) | X \text{ affinoid; } X \cap L = \phi \} \). If \( \Omega' \) is not of the form \( \mathbb{P} - \{ \text{a compact set} \} \) then, according to [2] p. 145, (2.5), there exists a non-constant bounded holomorphic function \( h \) on \( \Omega' \). The holomorphic function \( h \circ f \) on \( \Omega \) is also bounded and must be constant according to the same result. This implies however that \( f \) is constant. So the proposition is proved and we have proved slightly more, namely: every affinoid subset, lying in \( f(\Omega) \), is the image of an affinoid subset of \( \Omega \) under the map \( f \).
(1.3) PROPOSITION. (A version of Picard's theorem). Let $f$ be meromorphic function on $\{z \in K \mid R < |z|\}$ which cannot be extended at $\infty$. Then $f$ omits at most one value.

PROOF. We note that this result must be known. By lack of reference we include two proofs. Suppose that $f$ omits at least one value, then we may take $f$ to be holomorphic on $\{z \in K \mid R < |z|\}$.

(1) FIRST PROOF. We may express $f$ as a convergent Laurent-series

$$\sum_{n=-\infty}^{\infty} a_n z^n$$

which has infinitely many $a_n \neq 0$ for $n > 0$.

For $\varrho \in |K^*|$, $R < \varrho < \infty$, we form $\max |a_n| \varrho^n = \alpha(\varrho)$ and we denote the smallest integer $n$ with $|a_n| \varrho^n = \alpha(\varrho)$ by $n(\varrho)$.

Clearly $\lim_{\varrho \to \infty} n(\varrho) = \lim_{\varrho \to \infty} \alpha(\varrho) = \infty$. We will suppose that $\varrho > R$ such that $n(\varrho) > 0$. The set $X_\varrho = \{z \in K \mid |z| = \varrho\}$ can have the following form:

(a) Suppose that there is only one $n$ with $|a_n| \varrho^n = \alpha(\varrho)$, then

$$X_\varrho = \{z \in K \mid |z| = \alpha(\varrho)\}$$

(b) Suppose that there are more positive integers $n$ with $|a_n| \varrho^n = \alpha(\varrho)$, then

$$X_\varrho = \{z \in K \mid |z| = \alpha(\varrho)\}.$$  

The above follows from the well-known statement:

has no zeros if and only if there is precisely one $m$ with $|b_m| = \max_n |b_n|$.  

Situation (b) occurs for an infinite sequence $\varrho_1, \varrho_2, \ldots$ with $\lim \varrho_i = \infty$. Hence $f(\{z \in K \mid R < |z|\}) = K$.

(2) SECOND PROOF. Suppose that the holomorphic map $f$ omits at least two values in $\mathbb{P}$. Then we may suppose that $f$ omits 0 and $\infty$. In other words $f \in \mathcal{O}(\{z \in K \mid R < |z|\})^*$. Using [1] § 1, (1.8.9) one sees that $f$ has the form $h(1 + h)$ where $\lambda \in K^*$, $n \in \mathbb{Z}$ and $h$ is holomorphic on $\{z \in K \mid R < |z|\}$ such that $|h(z)| < 1$ for all $z$. But then $h$ can be extended to $\infty$ and so also $f$ extends at $\infty$.

§ 2. TWO EXAMPLES

(2.1) The first example imitates the proof of Picard's theorem that we have given in the introduction.

Let $k = \mathbb{F}_q ((1/t))$ be the Laurent-series field in the variable $1/t$ and with coefficients in the finite field $\mathbb{F}_q$. Let $K$ denote the completion of the algebraic closure of $k$.

The group $I(1)$ is the subgroup of $I(1) = PSL(2, \mathbb{F}_q[t])$ consisting of the matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ modulo}(t).$$

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In [2], Chapter 10, it is calculated that: \( I(t) \) has \((q + 1)\) inequivalent parabolic points and that the genus of the corresponding algebraic curve is zero.

So the holomorphic map

\[ f : \mathbb{P}(K) - \mathbb{P}(k) \to \mathbb{P}(K) - \mathbb{P}(k)/I(t) = \mathbb{P}(K) - \mathbb{P}(\mathbb{F}_q) \]

omits exactly \( q + 1 \) values. We still have to verify that \( f \) has an essential singularity at the compact subset \( \mathbb{P}(k) \) of \( \mathbb{P} \).

Let \( L \) be the smallest compact subset of \( \mathbb{P} \) such that \( f \) admits an extension as meromorphic function on \( \mathbb{P} - L \). One easily sees that \( L \) always exists and that \( L \) is invariant under \( I(t) \). If \( L \neq \phi \) then \( L \) turns out to be \( \mathbb{P}(k) \) since it is invariant. Further \( L = \phi \) would mean that \( f \) is a rational function on \( \mathbb{P} \). But only a constant rational function can be invariant under \( I(t) \).

In this example one can clearly vary the finite field \( \mathbb{F}_q \) and moreover one can compose \( f \) with a rational function on \( \mathbb{P} \). This shows the following statement:

"Let the field \( K \) have characteristic \( \neq 0 \) and let \( \{a_1, \ldots, a_n\} \) be a subset of \( \mathbb{P}(K) \). There exists a perfect compact subset \( L \) of \( \mathbb{P}(K) \) and a meromorphic function \( f \) with an essential singularity at \( L \) such that \( f(\mathbb{P} - L) = \mathbb{P} - \{a_1, \ldots, a_n\} \)."

(2.2) The second example works for fields \( K \) of any characteristic and residue characteristic. However to simplify matters we assume that the residue field \( K \) has a characteristic \( \neq 2 \).

Our construction is a variant of the construction of Whittaker groups done in [2], Chapter 9.

Let the 12 points \( a_1, b_1, \ldots, a_6, b_6 \) in \( \mathbb{P} \) be such that the reduction \( \mathbb{P} \) with respect to this set is:

In other terms this means that the position of the 12 points (after an automorphism of \( \mathbb{P} \)) is such that:

1) \( |a_i| = |b_i| = 1 \)
2) \( |a_i - a_j| = 1 \) for \( i \neq j \)
3) \( |b_i - b_j| = 1 \) for \( i \neq j \)
4) \( |a_i - b_j| = 1 \) for \( i \neq j \)
5) \( |a_i - b_i| < 1 \) for all \( i \).

Let \( s_i \) (\( i = 1, \ldots, 6 \)) denote the elliptic element of order 2 with fixed points \( a_i, b_i \).

In [2] p. 281 it is shown that the group \( \Gamma_0 = \langle s_1, \ldots, s_6 \rangle \) generated by the six reflexions is discontinuous and it is shown that the only relations among the generators are \( s_1^2 = s_2^2 = \ldots = s_6^2 = 1 \). Let \( \Omega \) denote the set of ordinary points of \( \Gamma_0 \).
We introduce now four subgroups \( \Gamma_i \) (\( i = 1, 2, 3, 4 \)) of \( \Gamma_0 \) of finite index. Consider the surjective group homomorphism \( \phi : \Gamma_0 \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \) given by \( \phi(s_i) = (1, 0) \) for \( i = 1, 2, 3 \) and \( \phi(s_i) = (0, 1) \) for \( i = 4, 5, 6 \). The kernel \( \Gamma_4 \) of \( \phi \) is a Schottky group on 9 free generators. The generators are

\[
s_1s_2, s_1s_3, s_4s_5, s_4s_6, s_1s_4s_1s_2s_4s_1, s_1s_4s_1s_3s_4s_1, s_1s_5s_4s_1, s_1s_6s_4s_1, s_1s_4s_1s_4
\]
as one easily verifies.

The group \( \Gamma_1 \) is generated by \( \Gamma_4 \) and \( s_1 \); the group \( \Gamma_2 \) is generated by \( \Gamma_4 \) and \( s_4 \), the group \( \Gamma_3 \) is generated by \( \Gamma_4 \) and \( s_1s_4 \). Hence \( \Gamma_4 \subset \Gamma_i \subset \Gamma_0 \) for \( i = 1, 2, 3 \) and \([\Gamma_0 : \Gamma_i] = 2\) for \( i = 1, 2, 3 \).

The group \( \Gamma_3 \) turns out to be a free group on 5 generators, namely on \( \{s_1s_2, s_1s_3, s_4s_5, s_4s_6, s_1s_4\} \).

The groups \( \Gamma_i \) (\( i = 0, 1, 2 \)) are not free. One easily calculates that the rank of the abelianized groups \( \Gamma_i/[\Gamma_i, \Gamma_i] \) is 2 for \( i = 1, 2 \).

We write \( X_i \) for the algebraic curve \( \Omega/\Gamma_i \) (\( i = 0, \ldots, 4 \)). Although the curve is not always parametrized by a Schottky group (cases \( i = 0, 1, 2 \)) the curve is certainly "locally isomorphic to \( \mathbb{P} \)" and hence a Mumford curve. (See [2] p. 177). Let \( g_i \) denote the genus of \( X_i \), then we have \( g_0 = 0 \), \( g_1 = g_2 = 2 \), \( g_3 = 5 \), \( g_4 = 9 \) by using [2] p. 250, 251. Moreover we have a diagram of holomorphic maps of degree two between the various curves:

![Diagram](image)

We are especially interested in the morphism \( X_4 \to X_1 \). The curve \( X_1 \) is a Mumford curve of genus 2 and can also be parametrized by a Schottky group \( \Delta \) with \( \Omega' \) as set of ordinary points.

The map \( f : X_4 \to X_1 \) lifts to a holomorphic map \( F : \Omega \to \Omega' \) since \( \pi : \Omega \to X_4 \) and \( \pi_1 : \Omega' \to X_1 \) are the universal coverings. (Compare [2] p. 149–153). The holomorphic map \( F \) omits an infinite compact set since \( \mathbb{P} - \Omega' \) is infinite.

Our example is completed with the following lemma.

**Lemma.** \( F \) has an essential singularity at the compact perfect set \( \mathbb{P} - \Omega \).

**Proof.** Using the Riemann-Hurwitz formula one finds that \( f : X_4 \to X_1 \) is ramified in 12 points. Let \( p \in \Omega \) be a point such that its image in \( X_4 \) is one of those 12 points. Since \( \pi_4 : \Omega \to X_4 \) and \( \pi_1 : \Omega' \to X_1 \) are locally isomorphisms it follows that also \( F \) is ramified (of index 2) at \( p \). The whole orbit \( \Gamma_4 (p) \) consists clearly of ramification points of \( F \). Since \( p \) is an ordinary point for \( \Gamma_4 \) the limit points for this orbit are precisely \( \mathbb{P} - \Omega \). This implies that \( F \) cannot be extended since in any neighbourhood of any \( \lambda \in \mathbb{P} - \Omega \) there are infinitely many ramification points of \( F \). So \( F \) has an essential singularity at \( \mathbb{P} - \Omega \).
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