Control of port-Hamiltonian systems
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5

Velocity Observers for Mechanical Systems with Kinematic Constraints

"All happy families (linear systems) are alike, but every unhappy family (nonlinear systems) is unhappy in its own way." - Leo Tolstoy.

5.1 Introduction and problem formulation

In chapters 3 and 4, we had constructed velocity observers for special classes of mechanical systems using the Immersion and Invariance principle and the passivity based approach respectively. In this chapter, we consider a general $n$ degrees-of-freedom mechanical system with $k$ kinematic (possibly non-holonomic) constraints ($k < n$) and prove the existence of a $3n - 2k + 1$-dimensional globally exponentially convergent velocity observer for them. An observer for unconstrained mechanical systems is obtained as a particular case of this general result. For the construction of the velocity observer, we employ the Immersion and Invariance technique with dynamic scaling which was proposed in the recent paper [44]. Reference [44] considers the special class of nonlinear systems that are affine in the unmeasured states and thus the observer design methodology adopted in that paper would apply to the $S_{PLvCC}$ mechanical systems we considered in chapter 3 and the special class of port-Hamiltonian systems we had considered in chapter 4. The main advantage of this method is that it relaxes the integrability condition (3.60) and correspondingly (4.10), but the dynamic scaling factor introduces a high gain into the loop which might be undesirable in some applications (reference [44] makes an attempt to alleviate this effect). Later, in the paper [8], the procedure has been generalized for mechanical systems with kinematic constraints and was successfully demonstrated on two well known non-trivial physical examples.
5. Velocity Observers for Mechanical Systems with Kinematic Constraints

The problem is stated as follows. We consider general $n$ degrees–of–freedom mechanical systems with kinematic constraints described in Lagrangian form by [15], [58],

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + \nabla U(q) = G(q)u + Z(q)\lambda_C,$$

$$Z^\top(q)\dot{q} = 0,$$

where $Z(q)\lambda_C$ are the constraint forces with $Z : \mathbb{R}^n \to \mathbb{R}^{n \times k}$, $\text{rank}(Z) = k$, and $\lambda_C(t) \in \mathbb{R}^k$. Compare (5.1) and (5.2) with (2.3). We consider $q(t)$ to be measurable and assume that the input $u(t)$ is such that $q(t)$ and $\dot{q}(t)$ exist for all time, that is, the system is forward complete. Our objective is to design a globally asymptotically convergent observer for $\dot{q}(t)$.

5.2 Main Result

**Proposition 5.1.** Consider the system (5.1), (5.2), and assume $u(t)$ is such that trajectories exist for all $t \geq 0$. There exist smooth mappings $A : \mathbb{R}^{3n-2k+1} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{3n-2k+1}$, $B : \mathbb{R}^n \to \mathbb{R}^{(n-k) \times (3n-2k+1)}$ and a full rank matrix $N : \mathbb{R}^n \to \mathbb{R}^{(n-k) \times n}$ satisfying the condition

$$\text{rank} \begin{bmatrix} N(q) \\ Z^\top(q) \end{bmatrix} = n,$$

such that the dynamical system

$$\dot{\Pi} = A(\Pi, q, u),$$

with state $\Pi(t) \in \mathbb{R}^{3n-2k+1}$, inputs $q(t)$ and $u(t)$, and output

$$\bar{\eta} = B(q)\Pi,$$

has the following property. For some $\sigma > 0$, all trajectories of the interconnected system (5.1), (5.2), (5.3), (5.4) are such that

$$\lim_{t \to \infty} e^{\sigma t} [N(q)\dot{\eta}(t) - \bar{\eta}(t)] = 0,$$

for all initial conditions

$$(q(0), \dot{q}(0), \Pi(0)) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{3n-2k+1}.$$  

That is, (5.3), (5.4) is a globally exponentially convergent velocity observer for the mechanical system (5.1), (5.2). The estimate of $\dot{q}$ is given by

$$\hat{\dot{q}} = \begin{bmatrix} N(q) \\ Z^\top(q) \end{bmatrix}^{-1} \begin{bmatrix} \bar{\eta} \\ 0 \end{bmatrix}.$$
5.2 Observers and alternate passive input-output pairs for PHSD

Remark 5.2. The system (5.1), (5.2)—with $k$ kinetic constraints—evolves in the $2n - k$ dimensional space

$$\mathcal{X}_c = \{(q, \dot{q})|Z^\top(q)\dot{q} = 0\}. \quad (5.7)$$

Therefore, only $n - k$ components of the velocity vector are relevant. For unconstrained systems $k = 0$ and $\mathcal{N} = I$. Please also refer to Remark 5.4.

5.2.1 A preliminary lemma

Before giving the proof of the main result, we propose a state representation of the system (5.1), (5.2) that is fundamental for all subsequent derivations. Towards this end, we recall (also from Chapter 1) that the system (5.1), (5.2) can be expressed in port-Hamiltonian form \[87\] as

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} \nabla_q H(q,p) \\ \nabla_p H(q,p) \end{bmatrix} + \begin{bmatrix} 0 \\ G(q) \end{bmatrix} u + \begin{bmatrix} 0 \\ Z(q) \end{bmatrix} \lambda_C, \quad (5.8)$$

$$Z^\top(q)\nabla_p H(q,p) = 0, \quad (5.9)$$

where $p = M(q)\dot{q}$ are the generalized momenta, and

$$H(q,p) = \frac{1}{2}p^\top M^{-1}(q)p + U(q) \quad (5.10)$$

is the total energy stored in the system. Further, as per \[88\], the system (5.8), (5.9), restricted to the constrained space (5.7), can be represented in the form

$$\begin{bmatrix} \dot{q} \\ \dot{\tilde{p}} \end{bmatrix} = \begin{bmatrix} 0 & \tilde{S}(q) \\ -\tilde{S}^\top(q) & \tilde{J}(q,\tilde{p}) \end{bmatrix} \begin{bmatrix} \nabla_q H_c(q,\tilde{p}) \\ \nabla_p H_c(q,\tilde{p}) \end{bmatrix} + \begin{bmatrix} 0 \\ G_c(q) \end{bmatrix} u, \quad (5.11)$$

where

$$H_c(q,\tilde{p}) = \frac{1}{2}\tilde{p}^\top \tilde{M}^{-1}(q)\tilde{p} + U(q), \quad (5.12)$$

$$\tilde{p} = \tilde{S}^\top(q)p, \quad (5.13)$$

with $\tilde{S} : \mathbb{R}^n \to \mathbb{R}^{n \times (n-k)}$ being a full–rank right annihilator (as discussed in \[30\]), the matrix $\tilde{S}$ is not uniquely defined) of $Z^\top$ and $G_c : \mathbb{R}^n \to \mathbb{R}^{(n-k) \times m}$. The matrix $\tilde{J} : \mathbb{R}^n \times \mathbb{R}^{n-k} \to \mathbb{R}^{(n-k) \times (n-k)}$ is skew-symmetric and its $(ij)$-th element is given by

$$\tilde{J}_{ij}(q,\tilde{p}) = -\tilde{p}^\top \tilde{S}[\tilde{S}_i, \tilde{S}_j], \quad (5.14)$$

with $\tilde{S}_i$ being the $i$-th column of $\tilde{S}$, and $[\tilde{S}_i, \tilde{S}_j]$ is the standard Lie bracket.
5. Velocity Observers for Mechanical Systems with Kinematic Constraints

The matrix \( \bar{M} : \mathbb{R}^n \to \mathbb{R}^{(n-k) \times (n-k)} \) is symmetric positive definite and can be computed as follows. Since \( \bar{S} \) is a full-rank annihilator of \( Z^\top \), the \( n \times n \) matrix \( [\bar{S}(q)Z(q)] \) is full rank, hence invertible. Consider the pseudomomenta:

\[
\begin{bmatrix}
\tilde{p} \\
\tilde{p}_c
\end{bmatrix} =
\begin{bmatrix}
\tilde{S}^\top(q) \\
Z^\top(q)
\end{bmatrix} p,
\]

(5.15)

where \( \tilde{p} \in \mathbb{R}^{n-k}, \tilde{p}_c \in \mathbb{R}^k \). The Hamiltonian function in (5.10), when expressed in the new coordinates \((q, \tilde{p}, \tilde{p}_c)\), takes the form,

\[
H_c(q, \tilde{p}, \tilde{p}_c) = \frac{1}{2} \begin{bmatrix}
\tilde{p} \\
\tilde{p}_c
\end{bmatrix}^\top \bar{M}^{-1} \begin{bmatrix}
\tilde{p} \\
\tilde{p}_c
\end{bmatrix} + U(q),
\]

(5.16)

where the symmetric positive definite matrix \( \bar{M}^{-1} \) is given as

\[
\bar{M}^{-1} = \begin{bmatrix}
\tilde{S}^\top(q) \\
Z^\top(q)
\end{bmatrix} \begin{bmatrix}
\tilde{S}^\top(q) \\
Z^\top(q)
\end{bmatrix}^{-1} = \begin{bmatrix}
m_{11}(q) & m_{12}(q) \\
m_{12}(q)^\top & m_{22}(q)
\end{bmatrix}
\]

(5.17)

with \( m_{11} \in \mathbb{R}^{(n-k) \times (n-k)}, m_{12} \in \mathbb{R}^{(n-k) \times k} \) and \( m_{22} \in \mathbb{R}^{k \times k} \). Further, on the constraint space \( X_c \), the Hamiltonian satisfies the condition [87]

\[
\nabla_{\tilde{p}_c} H_c = 0.
\]

(5.18)

Equations (5.16), (5.17), (5.18) yield

\[
\tilde{p}_c = -m_{22}(q)m_{12}^{-1}(q)\tilde{p}.
\]

(5.19)

By substituting (5.19) in (5.16) we obtain the Hamiltonian function defined in (5.12) where

\[
\hat{M}^{-1}(q) = m_{11}(q) - m_{12}(q)m_{12}^{-1}(q)m_{12}(q).
\]

(5.20)

The matrix in (5.20) is the Schur complement of the positive definite matrix \( \bar{M}^{-1} \), defined in equation (5.17), and hence is also positive definite.

Next, in order to streamline the presentation, we introduce a factorization of the mass matrix, namely

\[
\hat{M}^{-1}(q) = T(q)T^\top(q),
\]

(5.21)

where \( T : \mathbb{R}^n \to \mathbb{R}^{(n-k) \times (n-k)} \) is a full-rank matrix and define the mappings \( L : \mathbb{R}^n \to \mathbb{R}^{n \times (n-k)} \) and \( F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{n-k} \) as

\[
L(q) = \tilde{S}(q)T(q),
\]

(5.22)

\[
F(q, u) = T^\top(q)[G_c(q)u - \tilde{S}^\top(q)\nabla U(q)].
\]

(5.23)

Notice that, since \( q \) and \( u \) are measurable, the values of these mappings are known.
5.2 Observers and alternate passive input-output pairs for PHSD

Lemma 5.3. The system (5.11), (5.12) admits a state space representation of the form
\[
\begin{align*}
\dot{y} &= L(y)x, \\
\dot{x} &= S(y, x)x + F(y, u),
\end{align*}
\] (5.24)
(5.25)
where \((y, x) = (q, T^T(q)\tilde{p})\), \(L\) is a left-invertible matrix and
\[
S = T^TJT + \sum_{i=1}^{n} [(T^{-1}(\nabla_q T))^T x(L^T e_i)^T - (L^T e_i)((T^{-1}(\nabla_q T))^T x)^T],
\] (5.26)
with \(e_i\) the \(i\)-th basis vector of \(\mathbb{R}^{n-k}\). Furthermore, \(S\) verifies the following properties.

(i) \(S\) is skew-symmetric, that is,
\[
S + S^T = 0.
\]

(ii) \(S\) is linear in the second argument, that is,
\[
S(y, a_1 x + a_2 \bar{x}) = a_1 S(y, x) + a_2 S(y, \bar{x}),
\]
for all \(y \in \mathbb{R}^n, x \in \mathbb{R}^{n-k}, \bar{x} \in \mathbb{R}^{n-k}, a_1 \in \mathbb{R}\) and \(a_2 \in \mathbb{R}\).

(iii) There exists a mapping \(\bar{S} : \mathbb{R}^n \times \mathbb{R}^{n-k} \to \mathbb{R}^{(n-k) \times (n-k)}\) such that
\[
S(y, x)\bar{x} = \bar{S}(y, \bar{x})x,
\] (5.27)
for all \(y \in \mathbb{R}^n, x \in \mathbb{R}^{n-k}\) and \(\bar{x} \in \mathbb{R}^{n-k}\).

Proof. We obtain (5.24) differentiating \(y\) and using (5.11), (5.12) and (5.22). Note now that
\[
\dot{x} = \dot{T}^T \tilde{p} + T^T \dot{\tilde{p}},
\] (5.28)
\[
= \dot{T}^T \tilde{p} - T^T \tilde{S}^T \nabla_q (\frac{1}{2} \tilde{p}^T \tilde{M}^{-1} \tilde{p}) - T^T \tilde{S}^T \nabla U
+ T^T JT x + T^T G_c u,
\] (5.29)
\[
= \dot{T}^T \tilde{p} - L^T \nabla_q (\frac{1}{2} \tilde{p}^T \tilde{M}^{-1} \tilde{p}) + F + T^T JT x,
\] (5.30)
where we have made use of (5.12), (5.21), (5.22) and (5.23). Furthermore
\[
\dot{T}^T \tilde{p} = \sum_{i=1}^{n} (\nabla_q T^T)(e_i^T \dot{q}) \tilde{p},
\]
\[
= \sum_{i=1}^{n} (\nabla_q T^T)\tilde{p}(e_i^T \tilde{S} \tilde{M}^{-1} \tilde{p}),
\]
\[
= \sum_{i=1}^{n} (T^{-1}(\nabla_q T))^T x(e_i^T Lx),
\] (5.31)
and
\[
\nabla_q \left\{ \frac{1}{2} \tilde{p}^T \tilde{M}^{-1} \tilde{p} \right\} = \nabla_q \left\{ \frac{1}{2} \tilde{p}^T TT^T \tilde{p} \right\} = \sum_{i=1}^n e_i \left\{ (\nabla_q T^T) \tilde{p} \right\}^T x = \sum_{i=1}^n e_i \left\{ (T^{-1} (\nabla_q T))^T x \right\}^T x.
\]

Substituting (5.31) and (5.32) in (5.30), and using (5.26), yields (5.25).

Properties (i)–(iii) follow immediately from skew–symmetry and linearity with respect to \( p \) (hence, to \( \tilde{p} \) and \( x \)) of \( \tilde{J} \), and the definition of \( S \) in (5.26). □

Lemma 1 implies that the velocity observer problem for system (5.1), (5.2) can be recast as an observer problem for system (5.24), (5.25) with output \( y \).

**Remark 5.4.** As will become clear in the proof below, the derivation of the observer for the case of unconstrained systems, that is systems for which \( k = 0 \), is obtained as a special case. For these systems, upon choosing the coordinates \( (x, y) = (q, T(y) \dot{q}) \), Lemma 5.3 holds with
\[
L = T^{-1}, \quad F = T^{-\top} (Gu - \nabla U), \quad S = (\dot{T} - T^{-\top} C)T^{-1}.
\]

where \( T \) would now be the factorization of the inertia matrix given as \( M = T^\top T \). The dynamics in the \((x, y)\) coordinates would still be of the form (5.24)-(5.25). Please refer to [8] for additional details. Further, it can be verified that for PLvCC systems studied in Chapter 3, the term \( S(y, x)x \) becomes identically equal to zero.

### 5.3 Proof of the main result

The observer is constructed in four steps.

(S1) Following the I&I procedure [6], we define a manifold (in the extended state-space of the plant and the observer) that should be rendered attractive and invariant and is such that the unmeasurable part of the state can be reconstructed from the function that defines the manifold. As is well–known, to achieve the latter objective a partial differential equation (PDE) should, in principle, be solved.

(S2) To avoid the need to solve the PDE the “approximation” technique proposed in [44] is adopted. Using this approximation induces some errors in the observer error dynamics.
5.3 Observers and alternate passive input-output pairs for PHSD

(S3) Borrowing from [44], we introduce a dynamic scaling that dominates—an in a Lyapunov–like analysis—the effect of the aforementioned disturbance terms, proving that the scaled observer error converges to zero.

(S4) To prove that the dynamic scaling factor is bounded and, consequently, that the actual observer error converges, exponentially, to zero, high-gain terms are introduced in the observer dynamics to, again, dominate sign–indefinite terms in a Lyapunov–like analysis.

Step 1. (Definition of the manifold) For the system (5.24), (5.25) we propose the manifold

\[ \mathcal{M} := \{ (y, x, \xi, \hat{y}, \hat{x}) : \xi - x + \beta(y, \hat{y}, \hat{x}) = 0 \} \subset \mathbb{R}^{5n-3k}, \]

where \( \xi \in \mathbb{R}^{n-k}, \hat{y} \in \mathbb{R}^{n-k}, \hat{x} \in \mathbb{R}^{n} \) are (part of) the observer state, the dynamics of which are to be defined, and the mapping \( \beta : \mathbb{R}^{3n-2k} \rightarrow \mathbb{R}^{n-k} \) is to be defined.

To prove that the manifold \( \mathcal{M} \) is attractive and invariant it is shown that the off–the–manifold coordinate

\[ z = \xi - x + \beta(y, \hat{y}, \hat{x}), \]

the norm of which determines the distance of the state to the manifold \( \mathcal{M} \), is such that:

(C1) \( z(0) = 0 \Rightarrow z(t) = 0 \), for all \( t \geq 0 \) (invariance);

(C2) \( z(t) \) asymptotically (exponentially) converges to zero (attractivity).

Notice that, if \( z(t) \rightarrow 0 \), an asymptotic estimate of \( x \) is given by \( \xi + \beta \).

To obtain the dynamics of \( z \) differentiate (5.34), yielding

\[ \dot{z} = \dot{\xi} - \dot{x} + \dot{\beta} \]

\[ = \dot{\xi} - S(y, x)x - F + \nabla_y \beta \hat{y} + \nabla_{\hat{y}} \beta \hat{y} + \nabla_{\hat{x}} \beta \hat{x}. \]

Let

\[ \dot{\xi} = F - \nabla_{\hat{y}} \beta \hat{y} - \nabla_{\hat{x}} \beta \hat{x} + S(y, \xi + \beta)(\xi + \beta) - \nabla_y \beta L(y)(\xi + \beta), \]

where the dynamics of \( \hat{y} \) and \( \hat{x} \) are yet to be prescribed. Substituting (5.35) in the equation of \( \dot{z} \) above, and invoking properties (ii) and (iii) of Lemma 5.3, yields

\[ \dot{z} = -S(y, \xi + \beta - z)(\xi + \beta - z) + \\
+ S(y, \xi + \beta)(\xi + \beta) - \nabla_y \beta L(y)z, \]

\[ = S(y, x)z + S(y, z)(\xi + \beta) - \nabla_y \beta L(y)z, \]

\[ = S(y, x)z + \bar{S}(y, \xi + \beta)z - \nabla_y \beta L(y)z, \]

(5.36)
5. Velocity Observers for Mechanical Systems with Kinematic Constraints

From (5.36) it is clear that condition (C1) above is satisfied. Furthermore, condition (C2) will be satisfied if we can find a function $\beta$ solving the PDE

$$\nabla_y \beta = [k_1 I + S(y, \xi + \beta)] L^I(y),$$

with $k_1 > 0$, where $L^I : \mathbb{R}^n \to \mathbb{R}^{(n-k)\times n}$ is a full rank left inverse of the matrix $L$. Indeed, in this case, the $z$-dynamics reduce to

$$\dot{z} = (S - k_1)z,$$

which, invoking the skew-symmetry of $S$, ensures the desired exponential convergence property. Unfortunately, solving the PDE (5.37) is a daunting task and therefore, in the next step of the design we proceed to “approximate its solution”. The observer construction steps described in the rest of this section are based on the ideas in [44].

Step 2. (“Approximate solution” of the PDE) We start by defining the function

$$H(y, \hat{x}) := [k_1 I + S(y, \hat{x})] L^I(y),$$

and denote the columns of this $(n-k) \times n$ matrix by $H_i : \mathbb{R}^n \times \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$ for $i = 1, \ldots, n$, that is,

$$H(y, \hat{x}) = \left[ \begin{array}{c} H_1(y, \hat{x}) \\ \vdots \\ H_n(y, \hat{x}) \end{array} \right].$$

Next, we define$^1$ the function $\beta(y, \hat{y}, \hat{x})$ as

$$\beta(y, \hat{y}, \hat{x}) := \int_0^{y_1} H_1([s, \hat{y}_2, \ldots, \hat{y}_n], \hat{x}) ds + \cdots + \int_0^{y_n} H_n([\hat{y}_1, \ldots, \hat{y}_{n-1}, s], \hat{x}) ds.$$  

(5.40)

From the definition of $\beta$, we obtain

$$\nabla_y \beta(y, \hat{y}, \hat{x}) = \left[ \begin{array}{c} H_1([y_1, \hat{y}_2, \ldots, \hat{y}_n], \hat{x}) \\ \vdots \\ H_n([\hat{y}_1, \ldots, \hat{y}_{n-1}, y_n], \hat{x}) \end{array} \right].$$  

(5.41)

Next, upon adding and subtracting $H(y, \xi + \beta)$ from the right hand side of equation (5.41), we get

$$\nabla_y \beta(y, \hat{y}, \hat{x}) = H(y, \xi + \beta) - \left\{ H(y, \xi + \beta) - \left[ \begin{array}{c} H_1([y_1, \hat{y}_2, \ldots, \hat{y}_n], \hat{x}) \\ \vdots \\ H_n([\hat{y}_1, \ldots, \hat{y}_{n-1}, y_n], \hat{x}) \end{array} \right] \right\}.  

(5.42)

Notice that the functions $H_i : \mathbb{R}^n \times \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$ for $i = 1, \ldots, n$ are smooth, and the term

$$H(y, \xi + \beta) - \left[ \begin{array}{c} H_1([y_1, \hat{y}_2, \ldots, \hat{y}_n], \hat{x}) \\ \vdots \\ H_n([\hat{y}_1, \ldots, \hat{y}_{n-1}, y_n], \hat{x}) \end{array} \right]$$

$^1$We attract the readers attention to the particular selection of the arguments used in the integrands. Namely that, with some abuse of notation, the vector $\hat{y}$ has been spelled out into its components.
5.3 Observers and alternate passive input-output pairs for PHSD

is equal to zero if \( \hat{y} = y \) and \( \hat{x} = \xi + \beta \). Consequently, there exist mappings \( \Delta_y : \mathbb{R}^n \times \mathbb{R}^{n-k} \times \mathbb{R}^n \rightarrow \mathbb{R}^{(n-k) \times n} \) and \( \Delta_x : \mathbb{R}^n \times \mathbb{R}^{n-k} \times \mathbb{R}^{n-k} \rightarrow \mathbb{R}^{(n-k) \times n} \) such that

\[
H(y, \xi + \beta) = \begin{bmatrix} H_1([y_1, \hat{y}_2, \ldots, \hat{y}_n], \hat{x}) & \ldots & H_n([\hat{y}_1, \ldots, \hat{y}_{n-1}, y_n], \hat{x}) \end{bmatrix} = \Delta_y(y, \hat{x}, e_y) + \Delta_x(y, \hat{x}, e_x),
\]

(5.43)

with

\[
e_y := \hat{y} - y, \quad e_x := \hat{x} - (\xi + \beta),
\]

(5.44)

and such that

\[
\Delta_y(y, \hat{x}, 0) = 0, \quad \Delta_x(y, \hat{x}, 0) = 0,
\]

(5.45)

for all \( y, \in \mathbb{R}^n, \hat{y}, \in \mathbb{R}^n, x \in \mathbb{R}^{n-k} \) and \( \hat{x} \in \mathbb{R}^{n-k} \). Now, upon substituting (5.43) in (5.42) we get

\[
\nabla_y \beta(y, \hat{y}, \hat{x}) = H(y, \xi + \beta) - \Delta_y(y, \hat{x}, e_y) - \Delta_x(y, \hat{x}, e_x).
\]

(5.46)

We thus see that, instead of directly trying to find the solution to the original (difficult to solve) PDE in (5.37), we have obtained (in some sense) an approximate solution of it, given by our newly defined \( \beta \) function in (5.41). However, this methodology (as can also be seen from (5.46)) introduces errors in the form of the mappings \( \Delta_x \) and \( \Delta_y \). Indeed, replacing (5.39) and (5.46) in (5.36) yields (compare with (5.38)

\[
\dot{\eta} = (S - k_1)\eta + (\Delta_y + \Delta_x)L(y)\eta.
\]

(5.47)

Recalling that \( S \) is skew–symmetric and \( k_1 > 0 \), it is clear that the mappings \( \Delta_y \) and \( \Delta_x \) play the role of disturbances that we will try to dominate with a dynamic scaling in the next step of the design.

Step 3. (Dynamic scaling) Define the scaled off–the–manifold coordinate

\[
\eta = \frac{1}{r} z,
\]

(5.48)

with \( r \) a scaling dynamic factor to be defined. Differentiating (5.48), and using (5.47), yields

\[
\dot{\eta} = \frac{1}{r} \dot{z} - \frac{1}{r} \dot{\eta}
\]

\[
= (S - k_1)\eta + (\Delta_y + \Delta_x)L(y)\eta - \frac{\dot{r}}{r} \eta.
\]

(5.49)

Consider now the function

\[
V_1(\eta) = \frac{1}{2} |\eta|^2,
\]

(5.50)
and note that its time derivative is such that

$$
\dot{V}_1 = -(k_1 + \frac{\dot{r}}{r})|\eta|^2 + \eta^\top(\Delta_y + \Delta_x)L(y)\eta,
$$

\[\leq -(k_1 + \frac{\dot{r}}{r})|\eta|^2 + ||[\Delta_y + \Delta_x]L|||\eta||^2, \tag{5.51}\]

where $\| \cdot \|$ is the matrix induced 2-norm. Now, upon invoking the standard Young’s inequality argument (refer to footnote 1 of Chapter 3) we obtain,

$$
||[\Delta_y + \Delta_x]L|||\eta||^2 \leq \frac{k_1}{2} |\eta|^2 + \frac{1}{2k_1}||[\Delta_y + \Delta_x]L|||\eta||^2. \tag{5.52}\]

Substituting (5.52) in (5.51) and further simplifying leads to the following inequality

$$
\dot{V}_1 \leq -\left( \frac{k_1}{2} + \frac{\dot{r}}{r} - \frac{1}{2k_1}||[\Delta_y + \Delta_x]L||^2 \right) |\eta|^2
\leq -\left( \frac{k_1}{2} + \frac{\dot{r}}{r} - \frac{1}{k_1} \left(||\Delta_y L||^2 + ||\Delta_x L||^2\right) \right) |\eta|^2, \tag{5.53}\]

Let

$$
\dot{r} = -\frac{k_1}{4}(r - 1) + \frac{r}{k_1} \left(||\Delta_y L||^2 + ||\Delta_x L||^2\right), \quad r(0) \geq 1. \tag{5.54}\]

Notice that the set $\{ r \in \mathbb{R} \mid r \geq 1 \}$ is invariant for the dynamics (5.54). Replacing (5.54) in (5.53) yields

$$
\dot{V}_1 \leq -\left( \frac{k_1}{2} - \frac{k_1}{4} \frac{r - 1}{r} \right) |\eta|^2
\leq -\frac{k_1}{4} |\eta|^2, \tag{5.55}\]

where the property $\frac{r - 1}{r} \leq 1$ has been used to get the second bound. From (5.55) we conclude that $\eta(t)$ converges to zero exponentially.

**Step 4. (High–gain injection)** From (5.48) and the previous analysis it is clear that $z(t) \to 0$ if we can prove that $r \in L_\infty$, which is the property established in this step. To enhance readability the procedure is divided into two parts. First, we make the function

$$
V_2(\eta, e_y, e_x) = V_1(\eta) + \frac{1}{2}(|e_y|^2 + |e_x|^2),
$$

a strict Lyapunov function. Then, the derivative of the function

$$
V_3(\eta, e_y, e_x, r) = V_2(\eta, e_y, e_x) + \frac{1}{2}r^2, \tag{5.56}\]

104
5.3 Observers and alternate passive input-output pairs for PHSD

is shown to be non–positive—establishing boundedness of \( r \). In both steps
the objectives are achieved adding, via a suitable selection of the observer dy-
namics, negative quadratic terms in \( \eta, e_y, e_x \) in the Lyapunov function deriva-
tive. We recall that \( e_y \) and \( e_x \) are measurable quantities, defined in (5.44).

Towards this end, let

\[
\dot{\hat{y}} = L(y)(\xi + \beta) - \psi_1(y, r) e_y,
\]

with \( \psi_1 : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) a gain function to be defined. The error dynamics, obtained combining (5.24) and (5.57), are

\[
\dot{e}_y = Lz - \psi_1 e_y.
\]

Now, select

\[
\dot{\hat{x}} = F(y, u) + S(y, \xi + \beta)(\xi + \beta) - \psi_2(y, r)e_x,
\]

with \( \psi_2 : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) a gain function to be defined. Recalling (5.35) the error dynamics for \( e_x \) become

\[
\dot{e}_x = \nabla_y \beta Lz - \psi_2 e_x.
\]

Using (5.55), (5.58) and (5.60) and doing some basic bounding, yields

\[
\dot{V}_2 \leq - \frac{k_1}{4} |\eta|^2 + re_y^\top L \eta - \psi_1 |e_y|^2 +
+ re_x^\top \nabla_y \beta L \eta - \psi_2 |e_x|^2
\]

\[
\leq - (\frac{k_1}{4} - 1) |\eta|^2 - \left( \psi_1 - \frac{r^2}{2} \|L\|^2 \right) |e_y|^2 -
- \left( \psi_2 - \frac{r^2}{2} \|\nabla_y \beta\|^2 \|L\|^2 \right) |e_x|^2.
\]

Selecting

\[
\psi_1 = k_2 + \psi_3 + \frac{r^2}{2} \|L\|^2,
\]

\[
\psi_2 = k_3 + \psi_4 + \frac{r^2}{2} \|\nabla_y \beta\|^2 \|L\|^2,
\]

with \( k_2 > 0, k_3 > 0 \) and \( \psi_3, \psi_4 : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) to be defined, we conclude that

\[
\dot{V}_2 \leq - \frac{1}{2} (k_1 - 2) |\eta|^2 - k_2 |e_y|^2 - k_3 |e_x|^2,
\]

which, selecting \( k_1 > 2 \), establishes that \( \eta, e_y, e_x \in \mathcal{L}_2 \cap \mathcal{L}_\infty \) and the origin of the (non-autonomous) subsystem with state \( \eta, e_y, e_x \) is uniformly globally exponentially stable.
5. Velocity Observers for Mechanical Systems with Kinematic Constraints

We are now ready for the selection of $\psi_3$ and $\psi_4$ to guarantee that $r \in \mathcal{L}_\infty$. For, recall (5.45), which ensures the existence of mappings $\bar{\Delta}_y : \mathbb{R}^n \times \mathbb{R}^{n-k} \times \mathbb{R}^n \rightarrow \mathbb{R}^{(n-k) \times n}$, $\bar{\Delta}_x : \mathbb{R}^n \times \mathbb{R}^{n-k} \times \mathbb{R}^{n-k} \rightarrow \mathbb{R}^{(n-k) \times n}$ such that

\[
\| \Delta_y(y, \hat{x}_e) \| \leq \| \bar{\Delta}_y(y, \hat{x}_e) \| | e_y |,
\]  
\[
\| \Delta_x(y, \hat{x}_e) \| \leq \| \bar{\Delta}_x(y, \hat{x}_e) \| | e_x |.
\]  
(5.64)

Evaluate the time derivative of $V_3$, defined in (5.56), replace (5.63) in (5.62), and use the bounds (5.64) to get

\[
\dot{V}_3 \leq -(k_1 - 1) | \eta |^2 - \left( \psi_3 - \frac{r^2}{k_1} \| \bar{\Delta}_y \|^2 \| L \|^2 \right) | e_y |^2 - \left( \psi_4 - \frac{r^2}{k_1} \| \bar{\Delta}_x \|^2 \| L \|^2 \right) | e_x |^2.
\]

Setting $k_1 > 4$,

\[
\psi_3 = \frac{r^2}{k_1} \| \bar{\Delta}_y \|^2 \| L \|^2,
\]  
\[
\psi_4 = \frac{r^2}{k_1} \| \bar{\Delta}_x \|^2 \| L \|^2.
\]

ensures $\dot{V}_3 \leq 0$, which guarantees that $r \in \mathcal{L}_\infty$.

To prove condition (5.5) note that (5.50) and (5.55) imply that

\[
| \eta(t) | \leq | \eta(0) | e^{-\frac{k_1}{2} t},
\]

hence

\[
| z(t) | \leq \frac{r(t)}{r(0)} | z(0) | e^{-\frac{k_1}{2} t} \leq \sup_{t \geq 0} \{ r(t) \} | z(0) | e^{-\frac{k_1}{2} t},
\]

which yields the claim, by boundedness of $r(t)$.

We now define the state vector of the observer as $\Pi = (\hat{x}, \hat{y}, \xi, r)$ whose dynamics can be obtained from (5.59), (5.57), (5.35), and (5.54). We further define

\[
B(y) := \begin{bmatrix} T^{-\top}(y) & 0 & 0 \end{bmatrix},
\]  
(5.65)
\[
\mathcal{N}(y) := \tilde{S}(y)M^{-1}(y).
\]
(5.66)

We next show how to obtain the estimate of $\dot{q}$ and also explain the choice of the matrices $B$ and $\mathcal{N}$.

Firstly, by using (5.65) in (5.4) we get $\tilde{\eta} = T^{-\top} \tilde{x}$. Now, since $e_x = \hat{x} - \{ \zeta + \beta \}$ and $z = \zeta + \beta - \hat{x}$ converge to zero exponentially, we see that $\hat{x}$ exponentially converges to $x$ (also refer to Remark 5.5 below). Hence, $\tilde{\eta}$ is the estimate
5.3 Observers and alternate passive input-output pairs for PHSD

of $\dot{p}$ (since $x = T(q)\dot{p}$). Next, by using the relation $\dot{p} = \tilde{S}^T M^{-1} \dot{q}$ and that $Z^T(q)\dot{q} = 0$, we get

$$\dot{\tilde{q}} = \left[ \begin{array}{c} \tilde{S}^T(q)M^{-1}(q) \\ Z^T(q) \end{array} \right]^{-1} \left[ \begin{array}{c} \tilde{p} \\ 0 \end{array} \right].$$

(5.67)

Hence, the estimate of $\dot{q}$ can be obtained by the following equation

$$\dot{\tilde{q}} = \left[ \begin{array}{c} \tilde{S}^T(q)M^{-1}(q) \\ Z^T(q) \end{array} \right]^{-1} \left[ \begin{array}{c} \tilde{\eta} \\ 0 \end{array} \right],$$

(5.68)

which thus explains the choice of $N$ in (5.66). This concludes the proof.

**Remark 5.5.** The four components $\dot{x}$, $\dot{y}$, $\xi$ and $r$ of the state vector of the observer can be given the following interpretation. The component $\dot{x}$ is the estimate of $x$ and a filtered version of $\xi + \beta$. The component $\dot{y}$ is a filtered version of the measured variable $y$. The $\xi$-dynamics render the set $z = 0$ invariant (refer to (5.36)), regardless of the selection of the other dynamics, and $\xi$ can be regarded as the state of a reduced order observer. To clarify this point note that, ideally, if the PDE (5.37) is directly solvable, then $\beta$ would be a function of $y$ alone which implies that the observer dynamics would then consist only of the component $\xi$. In this regard, please refer to the examples stated in [43] and also the examples from Chapter 3 of this thesis. Thus, in such cases, the variable $\xi$ would then play the role of the state of the (reduced) order observer. The $r$-dynamics are introduced in order to compensate for the “disturbances” $\Delta y$ and $\Delta x$ appearing in the error dynamics (5.47).

**Remark 5.6.** Although the analysis of the performance of the proposed observer in the presence of noise is not within the scope of this chapter, it is worth noting the following. The Lyapunov argument establishing uniform asymptotic stability of the zero equilibrium of the $(\eta, e_y, e_x)$-subsystem yields, via total stability arguments, robustness against small additive perturbations on the measured variables $u$ and $y$. In the presence of such perturbations the variables $e_y$ and $e_x$ do not converge to zero. Nevertheless, as long as they are sufficiently small, equation (5.54) can be regarded as describing a linear (non-autonomous) scalar differential equation in which, by equations (5.45), the coefficient of the linear term is uniformly negative. This ensures boundedness of $r(t)$. Moreover, the high-gain effect due to the scaling factor $r$, which might be undesirable in certain situations, is also alleviated to a certain extent because of the boundedness of $r$.

**Remark 5.7.** Reference [12] considers a general class of (fully actuated) mechanical systems without kinematic constraints and constructs velocity observers for them by using position measurements. The observer is constructed by invoking passivity based concepts and an observer-controller combination
is proposed which solves the output feedback (reference) trajectory tracking control problem for robots. However, the proposed observer is semi-global, that is, the observer error dynamics converges to zero within a region of attraction that depends on the initial state of the system. In contrast, the observers constructed in the Chapters 3, 4, 5 are global. On the other hand, in comparison to [12], we have only looked at (in Chapters 3 and 4) the output feedback set-point stabilization problem and it is yet to be seen how our designed observers would work in conjugation with a suitable trajectory tracking control law.

5.4 Physical examples

In this section, the proposed observer design is illustrated on two physical examples.

5.4.1 The Chaplygin Sleigh [15]

We briefly considered the example of the Chaplygin Sleigh in Chapter 1. We now illustrate our observer design theory on this system which is otherwise, a benchmark example of a system with nonholonomic kinetic constraints. Its inertia matrix $M$ and the constraint matrix $Z$ are given by

$$M(q) = \begin{bmatrix} m & 0 & -ma \sin(q_3) \\ 0 & m & ma \cos(q_3) \\ -ma \sin(q_3) & ma \cos(q_3) & I_0 + ma^2 \end{bmatrix}, \quad Z(q) = \begin{bmatrix} -\sin(q_3) \\ \cos(q_3) \\ 0 \end{bmatrix},$$

where $m$ is the mass of the rigid body, $I_0$ is the moment of inertia of the rigid body about its center of mass and $a$ denotes the fixed distance between the knife edge and the center of mass.

We assume the body to be moving on a horizontal plane, that is, $U(q) = 0$ and that the body is free from any external force, that is, $u = 0$, hence $F = 0$. A full rank matrix $\tilde{S}(q)$ that satisfies $Z^T(q)\tilde{S}(q) = 0$ is

$$\tilde{S} = \begin{bmatrix} \cos(q_3) & 0 \\ \sin(q_3) & 0 \\ 0 & 1 \end{bmatrix},$$

and correspondingly the new momentum coordinates on the constrained space are given by

$$\begin{bmatrix} \tilde{p}_1 \\ \tilde{p}_2 \end{bmatrix} = \begin{bmatrix} p_1 \cos(q_3) + p_2 \sin(q_3) \\ p_3 \end{bmatrix}.$$
5.4 Observers and alternate passive input-output pairs for PHSD

The Cholesky factorization of \( \tilde{M} \) is obtained as

\[
T = \begin{bmatrix}
\frac{\sqrt{m}}{\sqrt{I_0 + ma^2}} & 0 \\
0 & \sqrt{I_0 + ma^2}
\end{bmatrix}
\]

and thus the new coordinates are given by \( y = [q_1, q_2, q_3] \) and \( x = \left[ \frac{\tilde{p}_1}{\sqrt{m}}, \frac{\tilde{p}_2}{\sqrt{I_0 + ma^2}} \right] \).

The matrix \( L \) in (5.22) and the skew-symmetric matrix \( S \) in (5.26) are

\[
L = \begin{bmatrix}
\frac{\cos(y_3)}{\sqrt{m}} & 0 & 0 \\
\frac{\sin(y_3)}{\sqrt{m}} & 0 & 0 \\
0 & -\frac{\sqrt{I_0 + ma^2}}{\sqrt{I_0 + ma^2}} & 1
\end{bmatrix},
S = \begin{bmatrix}
0 & -\frac{a\sqrt{m}}{I_0 + ma^2}x_2 \\
-\frac{a\sqrt{m}}{I_0 + ma^2}x_2 & 0
\end{bmatrix}.
\]

The observer is constructed following the steps in Section 4, yielding

\[
\bar{S}(x, y) = \begin{bmatrix}
0 & -\frac{a\sqrt{m}}{I_0 + ma^2}x_2 \\
-\frac{a\sqrt{m}}{I_0 + ma^2}x_2 & 0
\end{bmatrix},
\]

\[
H(y, x) = \begin{bmatrix}
k_1\sqrt{m}\cos(y_3) & k_1\sqrt{m}\cos(y_3) & \tilde{k}x_2 \\
0 & 0 & k_1\sqrt{I_0 + ma^2} \\
0 & 0 & -\tilde{k}x_1
\end{bmatrix},
\]

\[
\beta(y, \hat{y}, \hat{x}) = \begin{bmatrix}
k_1\sqrt{m}\{y_1\cos(\hat{y}_3) + y_2\sin(\hat{y}_3)} + \tilde{k}x_2 \\
0 & k_1\sqrt{I_0 + ma^2} - y_3\tilde{k}x_1
\end{bmatrix},
\]

\[
\Delta_y = k_1\sqrt{m}\begin{bmatrix}
\{\cos(y_3)\} & \{\sin(y_3)\} & 0 \\
-\cos(\hat{y}_3) & -\sin(\hat{y}_3) & 0 \\
0 & 0 & 0
\end{bmatrix},
\]

\[
\Delta_x = \begin{bmatrix}
0 & 0 & -\tilde{k}e_{x_2} \\
0 & 0 & -\tilde{k}e_{x_1}
\end{bmatrix},
\]

where \( \tilde{k} = \frac{a\sqrt{m}}{\sqrt{I_0 + ma^2}} \). Finally, we obtain

\[
\|\bar{\Delta}_y(y, \hat{x}, e_y)\| = 2k_1\sqrt{m},
\]

\[
\|\bar{\Delta}_x(y, \hat{x}, e_x)\| = 2\tilde{k},
\]

\[
\|L\| \leq \text{Max}\{\frac{1}{\sqrt{m}}, \frac{1}{\sqrt{I_0 + ma^2}}\},
\]

\[
\|\nabla_y\beta\| \leq \sqrt{k_1^2m + (k_1\sqrt{I_0 + ma^2} - \tilde{k}x_1)^2 + \tilde{k}^2x_2^2}.
\]
The overall observer dynamics are

\[
\begin{align*}
\dot{y}_1 &= \frac{\cos(y_3)\{\xi_1 + \beta_1\}}{\sqrt{m}} - \psi_1\{\dot{y}_1 - y_1\}, \\
\dot{y}_2 &= \frac{\sin(y_3)\{\xi_1 + \beta_1\}}{\sqrt{m}} - \psi_1\{\dot{y}_2 - y_2\}, \\
\dot{y}_3 &= \frac{\{\xi_2 + \beta_2\}}{\sqrt{I_0 + ma^2}} - \psi_1\{\dot{y}_3 - y_3\}, \\
\dot{x}_1 &= \frac{a\sqrt{m}\{\xi_2 + \beta_2\}^2}{I_0 + ma^2} - \psi_2\{\dot{x}_1 - \xi_1 - \beta_1\}, \\
\dot{x}_2 &= -\frac{a\sqrt{m}\{\xi_1 + \beta_1\}\{\xi_2 + \beta_2\}}{I_0 + ma^2} - \psi_2\{\dot{x}_2 - \xi_2 - \beta_2\}, \\
\dot{r} &= -\frac{k_1}{4}\{r - 1\} + \frac{r\|L\|^2}{k_1}\left\{\frac{4a^2m\|e_x\|^2}{I_0 + ma^2} + 4k_1^2m\right\}, \\
\dot{\xi}_1 &= -k_1\{\xi_1 + \beta_1\}\cos(y_3 - \hat{y}_3) - \frac{a\sqrt{m}\{\xi_2 + \beta_2\}e_{xx}}{I_0 + ma^2} \\
&\quad + k_1\sqrt{m}\{y_1\sin(\hat{y}_3) - y_2\cos(\hat{y}_3)\}\left\{\frac{\xi_2 + \beta_2}{\sqrt{I_0 + ma^2}} - \psi_1 e_{y_3}\right\} \\
&\quad + \frac{a\sqrt{m}y_3}{\sqrt{I_0 + ma^2}}\left\{\frac{a\sqrt{m}\{\xi_2 + \beta_2\}^2}{I_0 + ma^2} + \psi_2 e_{x_1}\right\}, \\
\dot{\xi}_2 &= \frac{a\sqrt{m}}{\sqrt{I_0 + ma^2}}\left\{\frac{a\sqrt{m}\{\xi_2 + \beta_2\}^2}{I_0 + ma^2} - \psi_2 e_{x_1}\right\} \\
&\quad + \left\{\frac{a\sqrt{me_{xx}}}{I_0 + ma^2} - k_1\right\}\{\xi_2 + \beta_2\},
\end{align*}
\]

where

\[
\begin{align*}
\psi_1 &= k_2 + \frac{1}{2}r^2\|L\|^2\{8k_1m + 1\}, \\
\psi_2 &= k_3 + \frac{1}{2}r^2\|L\|^2\\{\|\nabla y\|^2 + \frac{8\hat{k}^2}{k_1}\}.
\end{align*}
\]

Simulations have been run to demonstrate the properties of the observer. The initial robustness and the parameter values are given in Table 1. To check the robustness of the observer, the measurements of \(q\), are perturbed by additive disturbances of maximum amplitude equal to 1% of the maximum value of the measured signals during the simulation time.

Simulation results are shown in Figs. 5.1, 5.2 and 5.3. The graphs in Fig. 5.1 show the time histories of the positions \(q(t)\) and of the velocities \(\dot{q}(t)\) (solid lines) along with the filtered variables \(\hat{q}(t)\), \(\hat{q}(t)\) (dashed lines for \(k_1 = 4.1\) and dotted lines for \(k_1 = 10\)). From the left column of Fig. 5.1, it can be seen that \(\dot{q}\) converges to \(q\) in a very short time interval and hence their respective plots
5.4 Observers and alternate passive input-output pairs for PHSD

| \( \begin{array}{l}
m = 1 \\
a = 2 \\
l = 5 \\
k_1 = 4.1, 10 \\
k_2 = 4 \\
k_3 = 5
\end{array} \) | \( \begin{array}{l}
y(0) = (1, 3, 1.5) \\
x(0) = (5, 10) \\
\dot{y}(0) = (2, 5, 5) \\
\dot{x}(0) = (3, 7) \\
r(0) = 3
\end{array} \) | \( \begin{array}{l}
\xi_1(0) = (4.0745, 18.4); (6.4083, 14.35)
\end{array} \) |

Table 5.1: Simulation parameters for the Chaplygin sleigh example

appear as one single plot. From the right column of Fig. 5.1, it can be seen that \( \dot{\hat{q}}(t) \) converges to \( \dot{q}(t) \) faster for \( k_1 = 10 \) than for \( k_1 = 4.1 \). This becomes more clear in Fig. 5.2 which shows the time histories of the actual estimation errors \( \dot{q}(t) - \dot{\hat{q}}(t) \) for \( k_1 = 4.1 \) (dashed line) and \( k_1 = 10 \) (solid line). The graphs in Fig. 5.3 show the time histories of the dynamic scaling factor \( r(t) \) and the Lyapunov function \( V_1(t) \). As before, the simulations are shown for \( k_1 = 4.1 \) and \( k_1 = 10 \). As predicted by the theory, \( r(t) \) remains bounded, the Lyapunov function \( V_1(t) \) converges to zero quickly and both are only minimally affected by the presence of the random noise in the output measurements. Further, as expected, the larger the value of \( k_1 \), the faster the convergence to zero of \( V_1 \).

![Figure 5.1: Time histories of \( q(t) \), \( \dot{y}(t) \) (left column) and of \( \dot{q}(t) \), \( \dot{\hat{q}}(t) \) (right column), for \( k_1 = 4.1 \) (dashed lines) and \( k_1 = 10 \) (dotted lines).](image)
5. Velocity Observers for Mechanical Systems with Kinematic Constraints

Figure 5.2: Time histories of $\dot{q}(t) - \hat{\dot{q}}(t)$, for $k_1 = 4.1$ (dashed line) and $k_1 = 10$ (solid line).

Figure 5.3: Time histories of $r(t)$ and $V_1(t)$, for $k_1 = 4.1$ (dashed line) and $k_1 = 10$ (solid line).
5.4 Observers and alternate passive input-output pairs for PHSD

5.4.2 2-Link Robotic Manipulator

Consider a two degree of freedom robotic manipulator [6] and the problem of constructing an observer for the unmeasured link velocities. Let \( q_1 \) and \( q_2 \) denote the absolute angles of the two links with respect to a reference \( x \)-axis. The unforced dynamics of the manipulator are described by the Euler-Lagrange equations (5.1), with \( u = 0 \), and

\[
M = \begin{bmatrix}
m_1 d_1^2 + m_2 l_1^2 + I_1 & m_2 l_1 d_2 c_{12}(q_1, q_2) \\
m_2 l_1 d_2 c_{12}(q_1, q_2) & m_2 d_2^2 + I_2
\end{bmatrix},
\]

\[
C = \begin{bmatrix}
0 & -m_2 l_1 d_2 s_{12}(q_1, q_2) \dot{q}_1 \\
m_2 l_1 d_2 s_{12}(q_1, q_2) \dot{q}_2 & 0
\end{bmatrix},
\]

\[
U = m_1 d_1 g \sin(q_1) + m_2 g (l_1 \sin q_1 + d_2 \sin q_2),
\]

where \( m_1, m_2 \) denote the masses of the links, \( l_1, l_2 \) denote the length of each link, \( l_{w1}, l_{w2} \) denote the width of each link, \( d_1, d_2 \) denote the location of the center of mass of each link from its end. The functions \( c_{12}(a, b), s_{12}(a, b) \) are defined as

\[
c_{12}(a, b) := \cos(a - b), \quad s_{12}(a, b) := \sin(a - b),
\]

and the moment of inertia of the links are given by

\[
I_1 = \frac{1}{12} m_1 (l_1^2 + l_{w1}^2), \quad I_2 = \frac{1}{12} m_2 (l_2^2 + l_{w2}^2).
\]

Finally, we define the function

\[
D(a, b) := c_1 c_2 - (c_3 c_{12}(a, b))^2
\]

and let

\[
c_1 = m_1 d_1^2 + m_2 l_1^2 + I_1, \quad c_2 = m_2 d_2^2 + I_2, \quad c_3 = m_2 l_1 d_2, \quad c_4 = m_1 d_1 + m_2 l_1, \quad c_5 = m_2 d_2.
\]

This is a system with no constraints (\( k = 0 \)) and hence we choose (as stated in Remark 5.4) our coordinates as \((x, y) = (q, T(y) \dot{q})\) and further obtain

\[
L = T^{-1}, \quad F = T^{-\top} (Gu - \nabla U), \quad S = (\dot{T} - T^{-\top} C) T^{-1}
\]

with \( M = T^\top T \). The lower triangular Cholesky factorization of the mass matrix (5.21) can be computed as

\[
T = \begin{bmatrix}
\sqrt{c_1} & \sqrt{c_3} c_{12}(q_1, q_2) \\
0 & \sqrt{D(q_1, q_2) / \sqrt{c_1}}
\end{bmatrix}.
\]
5. Velocity Observers for Mechanical Systems with Kinematic Constraints

We further compute

\[ S(y, x) = \begin{bmatrix} 0 & \frac{c_3 \sqrt{c_1 s_{12}(y_1, y_2)x_2}}{D(y_1, y_2)} \\ -\frac{c_3 \sqrt{c_1 s_{12}(y_1, y_2)x_2}}{D(y_1, y_2)} & 0 \end{bmatrix}, \quad F = \begin{bmatrix} c_4 g \cos y_1 \\ c_5 g \cos y_2 \end{bmatrix}, \]

and subsequently

\[ \tilde{S}(y, x) = \begin{bmatrix} 0 & \frac{c_3 \sqrt{c_1 s_{12}(y_1, y_2)x_2}}{D(y_1, y_2)} \\ 0 & \frac{c_3 \sqrt{c_1 s_{12}(y_1, y_2)x_1}}{D(y_1, y_2)} \end{bmatrix} \]

with \( \tilde{S}(y, x) \) satisfying \( S(y, x)\bar{x} = \tilde{S}(y, \bar{x})x \).

The observer is constructed following the steps in the proof of Proposition 3.3, yielding

\[ H(y, \zeta + \beta) = \begin{bmatrix} k_1 \sqrt{c_1} \frac{k_1 c_3 s_{12}(y_1, y_2)}{\sqrt{c_1}} + \frac{k_1 c_3 s_{12}(y_1, y_2)\{c_2 + \beta_2\}}{\sqrt{D(y_1, y_2)}} \\ 0 \end{bmatrix}, \]

\[ \beta(x, y, \hat{y}) = \begin{bmatrix} k_1 \sqrt{c_1} y_1 \frac{k_1 c_3 s_{12}(\hat{y}_1, y_2)}{c_1} + \hat{x}_2 \tan^{-1} \frac{c_3 s_{12}(\hat{y}_1, y_2)\beta_2}{\sqrt{D(\hat{y}_1, y_2)}} \\ y_2 \frac{k_1 \sqrt{D(\hat{y}_1, s)}}{c_1} ds \\ -\hat{x}_1 \tan^{-1} \frac{c_3 s_{12}(\hat{y}_1, y_2)\beta_1}{\sqrt{D(\hat{y}_1, y_2)}} \end{bmatrix}, \]

\[ \Delta_y(y, \hat{x}, e_y) = \begin{bmatrix} 0 & \frac{k_1 c_3 \{c_12(y_1, y_2)\} - c_12(\hat{y}_1, y_2)\}}{\sqrt{D(y_1, y_2)}} + \frac{c_3 s_{212}(\hat{y}_1, y_2)\{c_2 + \beta_2\}}{\sqrt{D(y_1, y_2)}} \\ \frac{k_1 c_3 \{c_12(y_1, y_2)\} - c_12(\hat{y}_1, y_2)\}}{\sqrt{D(y_1, y_2)}} + \frac{c_3 s_{212}(\hat{y}_1, y_2)\{c_2 + \beta_2\}}{\sqrt{D(y_1, y_2)}} \end{bmatrix}, \]

\[ \Delta_x(y, \hat{x}, e_x) = \begin{bmatrix} 0 \frac{c_3 s_{12}(\hat{y}_1, y_2)\beta_2}{\sqrt{D(\hat{y}_1, y_2)}} \\ 0 \frac{c_3 s_{12}(\hat{y}_1, y_2)\beta_1}{\sqrt{D(\hat{y}_1, y_2)}} \end{bmatrix}. \]
5.4 Observers and alternate passive input-output pairs for PHSD

As a result

\[ ||\Delta_y(y, \hat{x}, e_y)|| = \frac{k_1 c_3}{\sqrt{c_1}} \left( 1 + \frac{c_3}{\sqrt{c_1 c_2 - c_3}} \right) \left( |\hat{x}_1| + |\hat{x}_2| \right), \]

\[ ||\Delta_x(y, \hat{x}, e_x)|| = \frac{\sqrt{2} c_3}{\sqrt{c_1 c_2 - c_3}}, \]

Finally

\[ \nabla_y \beta = \begin{bmatrix} k_1 \sqrt{c_1} \frac{k_1 c_3 c_{12}(\hat{y}_1, y_2)}{\sqrt{c_2}} + \frac{c_3 s_{12}(\hat{y}_1, y_2) \hat{x}_2}{\sqrt{D}(y_1, y_2)} \\ 0 \end{bmatrix}, \]

\[ L = \begin{bmatrix} 1 \frac{k_1 \sqrt{c_1} D(y_1, y_2)}{\sqrt{c_2}} - c_3 s_{12}(\hat{y}_1, y_2) \hat{x}_1 \sqrt{D}(y_1, y_2) \\ 0 \end{bmatrix}, \]

and, subsequently,

\[ ||\nabla_y \beta|| \leq \sqrt{\bar{c}_1 + \bar{c}_2 [\bar{x}_1^2 + \hat{x}_2^2] + \frac{2k_1 c_3}{\sqrt{c_1}} \left\{ \frac{c_3 |\hat{x}_2|}{\sqrt{c_1 c_2 - c_3}} + |\hat{x}_1| \right\}}, \]

\[ ||L|| \leq \sqrt{\frac{c_1 + c_2}{c_1 c_2 - c_3}}, \]

where \( \bar{c}_1 = k_1^2 (c_1 + c_2 + \frac{c_3^2}{c_1^2}) \), \( \bar{c}_2 = \frac{c_3^2}{c_1 c_2 - c_3^2} \). The dynamics of \( \dot{x} \) and \( \dot{y} \) can be computed using the formulas (5.57), (5.59).

Simulations have been run to demonstrate the properties of the observer. The initial conditions and the parameter values are given in Table 1. In order to check the robustness of the observer, random disturbances \( k_4, k_5 \) are introduced in the measurements of \( q_1 \) and \( q_2 \) respectively. The disturbances are modelled as

\[ k_4 = -0.03 + 0.06 \text{rand}(1, 1), \]  
\[ k_5 = -0.15 + 0.3 \text{rand}(1, 1), \]  

where \text{rand} is a predefined function in MATLAB and an expression of the form ”\( a + b * \text{rand}(1, 1) \)” with \( a, b \in \mathbb{R} \) generates a random number in the interval \([a, b]\). The simulation results for the 2-dof manipulator system are shown in Figures 1 and 2. The graphs in Figure 1 show the time histories of
5. Velocity Observers for Mechanical Systems with Kinematic Constraints

\begin{table}[h]
\centering
\begin{tabular}{|c|c|}
\hline
\(l_1 = 0.6, l_2 = 0.4\) & \(y_1(0) = y_2(0) = 0\) \\
\hline
\(l_{w1} = l_{w2} = 0.07\) & \(x_1(0) = 1.5\) \\
\hline
\(m_1 = 2\) & \(y_1(0) = 1\) \\
\hline
\(m_2 = 1.4\) & \(\dot{y}_2(0) = 2\) \\
\hline
\(g = 9.81\) & \(\dot{x}_2(0) = 4\) \\
\hline
\(d_1 = 0.45, d_2 = 0.3143\) & \\
\hline
\(k_1 = 5, 10\) & \(\beta_1(0) + \zeta_1(0) = 0.1\) \\
\hline
\(k_2 = 8\) & \(\beta_2(0) + \zeta_2(0) = 2\) \\
\hline
\(k_3 = 25\) & \(r(0) = 2\) \\
\hline
\end{tabular}
\caption{Simulation parameters for the 2-dof manipulator example}
\end{table}

the link angles \(q_1(t)\), \(q_2(t)\) and of the link velocities \(\dot{q}_1(t)\), \(\dot{q}_2(t)\) respectively (solid lines) along with the filtered variables \(\hat{y}(t), T^{-1}(y(t))\hat{x}(t)\) (dashed lines for \(k_1 = 5\) and dotted lines for \(k_1 = 10\)). As can be seen, the filtered variables converge to the plant state. The graphs in Figure 2 show the time histories of the dynamic scaling factor \(r(t)\), Lyapunov function \(V_1(t)\) and of the errors \(\eta_1(t), \eta_2(t)\) respectively. As before, the simulations are shown for \(k_1 = 5, 10\). It can be seen that \(r(t)\) remains bounded while \(\eta_1(t), \eta_2(t)\) and \(V_1(t)\) converge to zero and all signals are only minimally affected in the presence of random noise in the output measurements. Further, as expected, the larger the value of \(k_1\), the faster is the convergence to zero of the estimation errors.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig54a.png}
\includegraphics[width=0.5\textwidth]{fig54b.png}
\caption{Time histories of \(q(t), \hat{y}(t)\) (top graphs) and of \(\dot{q}(t), T^{-1}(y(t))\dot{x}(t)\) (bottom graphs). The dashed lines are for \(k_1 = 5\) and the dotted lines are for \(k_1 = 10\).}
\end{figure}
5.4 Observers and alternate passive input-output pairs for PHSD

Figure 5.5: Time histories of $r(t)$ (top left graph), $V_1(t)$ (top right graph), $\eta_1(t)$ (bottom left half), $\eta_2(t)$ (bottom right half). The dashed lines are for $k_1 = 5$ and the dotted lines are for $k_1 = 10$.

5.4.3 A Walking Robot [34]

Consider a bipedal robot that consists of a torso, hips and two legs of equal length, with no ankles and knees as shown in Fig. 5.6. Usually, the motion of a bi-pedal robot involves two phases: the single support where one leg is in contact with the ground and the double support, where both legs are in contact with the ground. In this section, the single support phase of the robot is considered. In this phase, the robot can be modeled as a three degrees of freedom mechanical system with position coordinates $[q_1, q_2, q_3] = [\theta_1, \theta_2, \theta_3]$, where $\theta_1, \theta_2$ and $\theta_3$ parameterize the position of the stance leg, of the swing leg and of the torso, respectively. The state space of the system is

$$\{(q, \dot{q}) | q_i \in (-\pi, \pi), \dot{q} \in \mathbb{R}^3\}.$$ 

The unforced dynamics are described by the Euler-Lagrange equations (5.1) with $u = 0, k = 0$ and

$$M = \begin{bmatrix} \Delta_1 d^2 & -\frac{1}{2} md^2 c_{12}(q, q) & M_T dl c_{13}(q, q) \\ * & \frac{1}{2} md^2 & 0 \\ * & * & M_T l^2 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & -C_1(q) \dot{q}_2 & C_2(q) \dot{q}_3 \\ C_1(q) \dot{q}_1 & 0 & 0 \\ -C_2(q) \dot{q}_3 & 0 & 0 \end{bmatrix},$$

$$U = \frac{1}{2} g \{2M_H + 3m + 2M_T\} d \cos(q_1) - \frac{1}{2} gmd \cos(q_2) + gM_T l \cos(q_3),$$
where \( m \) denotes the mass of each individual leg which is assumed to be lumped at the center of the leg, \( M_H \) denotes the hip mass and \( M_T \) denotes the mass of the torso. The length of each leg is \( d \), and \( l \) denotes the distance from the center of gravity of the hip to the center of gravity of the torso (distance from \( O_H \) to \( O_T \)). The functions

\[
C_1(q) = \frac{1}{2} \sigma(S) d^2 s_{12}(q), \quad C_2(q) = M_T d l s_{13}(q)
\]

where for ease of notation, we have defined the functions

\[
c_{ij}(v, w) := \cos(v_i - w_j), \quad s_{ij}(v, w) := \sin(v_i - w_j),
\]

with \( i, j \in \{1, 2, 3\} \). We have further defined

\[
\Delta_1 = \frac{5}{4} m + M_H + M_T,
\]

\[
\Delta_2(u, v, w) = \sqrt{\Delta_1 - m c_{12}^2(u, v) - M_T c_{13}^2(u, w)},
\]

This is a system with no constraints \((k = 0)\) and hence we choose (as stated in Remark 5.4) our coordinates as \((x, y) = (q, T(y)\dot{q})\) and further obtain

\[
L = T^{-1}, \quad F = T^{-\top}(G u - \nabla U), \quad S = (\dot{T} - T^{-\top} C)T^{-1}
\]

with \( M = T^\top T \). Note that the lower–triangular Cholesky factorization of \( M \)
5.4 Observers and alternate passive input-output pairs for PHSD

\[
T = \begin{bmatrix}
    d\Delta_2(q, q, q) & 0 & 0 \\
    -\sqrt{mdc_{12}(q, q)} & \frac{1}{\tau}\sqrt{md} & 0 \\
    \sqrt{M_Tdc_{13}(q, q)} & 0 & \sqrt{M_T}\tau
\end{bmatrix},
\]

yielding

\[
S = \begin{bmatrix}
    0 & \frac{\sqrt{m}s_{12}(q, q)\dot{q}_2}{\Delta_2(q, q, q)} & -\frac{\sqrt{M_T}s_{13}(q, q)\dot{q}_3}{\Delta_2(q, q, q)} \\
    -\frac{\sqrt{m}s_{12}(q, q)\dot{q}_2}{\Delta_2(q, q, q)} & 0 & 0 \\
    \frac{\sqrt{M_T}s_{13}(q, q)\dot{q}_3}{\Delta_2(q, q, q)} & 0 & 0
\end{bmatrix},
\]

where we have used the expression of \( S \) in Remark 5.4. Let \( y = q, x = T(y)\dot{q} \) and note that the observer is constructed following the steps in Section 4, yielding \( H(y, \dot{x}) \) as,

\[
\begin{bmatrix}
    k_1d\Delta_2(y, y, y) & \frac{\sqrt{m}s_{12}(y, y)\dot{y}_2}{\Delta_2(y, y, y)} & -\frac{\sqrt{M_T}s_{13}(y, y)\dot{y}_3}{\Delta_2(y, y, y)} \\
    -k_1\sqrt{mdc_{12}(y, y)} & \frac{1}{2}k_1\sqrt{md} & 0 \\
    k_1\sqrt{M_Tdc_{13}(y, y)} & 0 & \frac{\sqrt{M_T}s_{13}(y, y)\dot{y}_1}{\Delta_2(y, y, y)} + \sqrt{M_T}\tau k_1
\end{bmatrix}
\]

and

\[
\beta(y, \dot{y}, \dot{x}) = \begin{bmatrix}
    k_1 \int_{y_1}^{y_1} \Delta_2(s, \dot{y}, \dot{y}) ds \\
    + \dot{x}_2 \tan^{-1} \left( \frac{\sqrt{m}c_{12}(\dot{y}, \dot{y})}{\Delta_2(y, y, y)} \right) \\
    -\dot{x}_3 \tan^{-1} \left( \frac{\sqrt{M_T}c_{13}(y, y)}{\Delta_2(y, y, y)} \right)
\end{bmatrix}
\]

\[
(5.71)
\]

\[
\Delta_x = \begin{bmatrix}
    0 & -\frac{\sqrt{m}s_{12}(y, y)e_{s_2}}{\Delta_2(y, y, y)} & \frac{\sqrt{M_T}s_{13}(y, y)e_{s_3}}{\Delta_2(y, y, y)} \\
    0 & \frac{\sqrt{m}s_{12}(y, y)e_{s_2}}{\Delta_2(y, y, y)} & 0 \\
    0 & 0 & -\frac{\sqrt{M_T}s_{13}(y, y)e_{s_3}}{\Delta_2(y, y, y)}
\end{bmatrix}
\]
5. Velocity Observers for Mechanical Systems with Kinematic Constraints

\[
\Delta_y = \begin{bmatrix}
  k_1 r \delta_{\Delta_2} & \sqrt{m} \dot{x}_2 \delta_{s_{12}} & -\sqrt{M_T} \dot{x}_3 \delta_{s_{13}} \\
  -k_1 \sqrt{md} \delta_{c_{12}} & -\sqrt{m} \dot{x}_1 \delta_{s_{12}} & 0 \\
  k_1 \sqrt{M_T} \dot{d} \delta_{c_{13}} & 0 & \sqrt{M_T} \dot{x}_1 \delta_{s_{13}}
\end{bmatrix}
\]

where

\[
f(y_1, y_2, y_3) = \frac{\cos(y_1 - y_2)}{\sqrt{\Delta_2(y, y, y)}},
\]

\[
\delta_{\Delta_2} = \Delta_2(y, y, y) - \Delta_2(y_1, \dot{y}_2, \dot{y}_3),
\]

\[
\delta_{s_{12}} = \frac{s_{12}(y, y)}{s_{12}(y_1, y_2)},
\]

\[
\delta_{s_{13}} = \frac{s_{13}(y, y)}{s_{13}(y_1, y_3)},
\]

\[
\delta_{c_{12}} = c_{12}(y, y) - c_{12}(y_1, \dot{y}_2),
\]

\[
\delta_{c_{13}} = c_{13}(y, y) - c_{13}(y_1, \dot{y}_3).
\]

As a result,

\[
\|\tilde{\Delta}_x(y, \dot{x}, e_x)\| \leq 2\left(\frac{\sqrt{m} + \sqrt{M_T}}{\sqrt{M_H + \frac{m}{4}}}\right),
\]

\[
\|\tilde{\Delta}_y(y, \dot{x}, e_y)\| \leq \tilde{\Delta}_{y1} + \tilde{\Delta}_{y2} + \tilde{\Delta}_{y3},
\]

\[
\tilde{\Delta}_{y1} \leq \sqrt{\frac{\{2\|\dot{x}\|^2\}{\{m\{\Delta_1 - m\}^2 + M_T\{\Delta_1 - M_T\}^2\}}}{(M_H + \frac{m}{4})^3}},
\]

\[
\tilde{\Delta}_{y2} \leq \sqrt{k_1^2 d^2 m\{\frac{2m + 4M_H}{m + 4M_H}\} + \frac{M_T m^2\|\dot{x}\|^2}{2\{\frac{m}{4} + M_H\}^3}},
\]

\[
\tilde{\Delta}_{y3} \leq \sqrt{k_1^2 d^2 m\{\frac{m + M_T + 4M_H}{m + 4M_H}\} + \frac{M_T^2 m\|\dot{x}\|^2}{2\{\frac{m}{4} + M_H\}^3}}.
\]

Finally,

\[
L(y) = \begin{bmatrix}
  \frac{1}{d \Delta_2(y,y,y)} & 0 & 0 \\
  \frac{2c_{12}(y,y)}{d \Delta_2(y,y,y)} & \frac{2}{\sqrt{m} d} & 0 \\
  \frac{-c_{13}(y,y)}{d \Delta_2(y,y,y)} & 0 & \frac{1}{\sqrt{M_T} l}
\end{bmatrix},
\]

and we can compute \(\nabla_y \beta\) from (5.71) which yields

\[
\|\nabla_y \beta\| \leq \sqrt{K_1 + K_2\|\dot{x}\|^2 + K_3\|\dot{x}\|},
\]

\[
\|L\| \leq \sqrt{\frac{1}{\sqrt{\frac{m}{4} + M_H}\{\frac{5}{d^2} + \frac{1}{l^2}\} + \frac{4}{md^2} + \frac{1}{M_T l^2}}},
\]

120
5.5 Observers and alternate passive input-output pairs for PHSD

<table>
<thead>
<tr>
<th>$m = 1, M_H = 2, M_T = 3$</th>
<th>$y(0) = (1, 3, 2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$l = 2$</td>
<td>$x(0) = (1, 2, 3)$</td>
</tr>
<tr>
<td>$d = 1.5$</td>
<td>$\hat{y}(0) = (4, 5, 3)$</td>
</tr>
<tr>
<td>$k_1 = 4.5, 30$</td>
<td>$\xi(0) = (-3.7676, 2.3557, 0.0337); (-8.4065, 0.6835, -6.8149)$</td>
</tr>
<tr>
<td>$k_2 = 5$</td>
<td>$\hat{x}(0) = (2, 5, 5)$</td>
</tr>
<tr>
<td>$k_3 = 10$</td>
<td>$r(0) = 2$</td>
</tr>
</tbody>
</table>

Table 5.3: Simulation parameters for the Walking Robot example

where the constants $K_1 = k_1^2 \left\{d^2 \left(\frac{5m}{2} + M_H + 2M_T \right) + M_T l^2 \right\}$, $K_2 = \frac{2(m+M_T)}{\frac{d^2}{4} + M_H}$, $K_3 = \frac{k_1(M_Tl+md)}{\left(\frac{d^2}{4} + M_H\right)^2}$. From these equations it is possible to obtain the explicit expression of the observer.

Simulations have been run to demonstrate the properties of the observer. The initial conditions and the parameter values are given in Table 2. To check the robustness of the observer, the measurements of $q_i$ are perturbed by additive disturbances of maximum amplitude equal to 1% of the maximum value of the measured signals during the simulation time. Simulation results are shown in Figs. 5.7, 5.8 and 5.9. The graphs in Fig. 5.7 show the time histories of the positions $q(t)$ and of the velocities $\dot{q}(t)$ respectively (solid lines) along with the filtered variables $\hat{y}(t), \hat{\dot{q}}(t)$ (dashed lines for $k_1 = 4.5$ and dotted lines for $k_1 = 30$). As can be seen from the left column, $\hat{y}(t)$ converges to $q(t)$ very quickly and hence their respective plots appear as one single plot. From the right column, it can be seen that for $k_1 = 30$, the filtered variable $\hat{\dot{q}}(t)$ converges very quickly to $\dot{q}(t)$ and hence there is no much distinction in this case between the plots of $\dot{q}(t), \hat{\dot{q}}(t)$, whereas for $k_1 = 4.5$, the convergence is relatively slow. This becomes clear in Fig. 5.8 which shows the time histories of the actual estimation errors $\dot{q}(t) - \hat{\dot{q}}(t)$. The graphs in Fig. 5.9 show the time histories of the dynamic scaling factor $r(t)$ and of the Lyapunov function $V_1(t)$. As before, the simulations are performed for $k_1 = 4.5$ and $k_1 = 30$. As can be seen, $r(t)$ remains bounded and is minimally affected by the noise. On the other hand, $V_1(t)$ converges to zero and stays close to it for $k_1 = 30$, but the effect of noise is more significant in the case $k_1 = 4.5$. As expected, the larger the value of $k_1$, the faster the convergence to zero of $V_1(t)$.

5.5 Conclusions

We have given an affirmative answer to the question of existence of a globally convergent velocity observer for general mechanical systems with kinematic constraints of the form (5.2) and have outlined a constructive procedure for
Figure 5.7: Time histories of $q(t)$, $\dot{q}(t)$ (left column) and of $\ddot{q}(t)$, $\dot{\ddot{q}}(t)$ (right column). The dotted lines are for $k_1 = 30$ and the dashed lines are for $k_1 = 4.5$. 
5.5 Observers and alternate passive input-output pairs for PHSD

Figure 5.8: Time histories of $\dot{q}(t) - \hat{\dot{q}}(t)$. The solid line is for $k_1 = 30$ and the dashed line is for $k_1 = 4.5$

Figure 5.9: Time histories of $r(t)$, $V_1(t)$ for $k_1 = 4.5$ (dashed line) and $k_1 = 30$ (solid line).
the observer design. No assumption is made on the existence of an upper-bound for the inertia matrix, hence the result is applicable for robots with prismatic joints. The only requirement is that the system is forward complete, i.e., that trajectories of the system exist for all times $t \geq 0$.

There are however some issues when it comes to the practical implementation of our observer design. Leaving aside the high complexity of the observer dynamics that can be easily retraced from the proof of Section 5.3, the difficulty stems from the fact that the key function $\beta$ is defined via the integrals (4.10), whose explicit analytic solution cannot be guaranteed a priori. Of course, the (scalar) integrations can always be numerically performed leading to a numerical implementation of the observer.

Finally, the established observer has been shown to be implementable for three practically interesting, non-trivial, examples.