2

Mathematical Prerequisites

“There is no royal road to geometry” - Euclid of Alexandria

In the previous chapter we gave a brief introduction to port-Hamiltonian systems with dissipation (PHSD) and had seen some benchmark physical examples modeled in the form of a PHSD. Our main aim in this chapter is to provide in more detail the background theory which will be used in the forthcoming chapters.

In the first section, we start by giving a short introduction to the classical Euler-Lagrange equations used for modeling mechanical systems, study some of their properties and then show how the PHSD can be obtained as a generalization of the well known Hamiltonian equations. We then study the underlying geometric structure of a PHSD given as the power conserving interconnection of a Dirac and a resistive structure. The Dirac structure captures the interconnection structure and the physical laws of the system while the resistive structure captures the system’s damping. We subsequently present the various representations of Dirac structures.

In the second section, we explain the basic theory of the Immersion and Invariance principle proposed in [6, 43] and discuss how it can be used in the context of observer design. As we had mentioned before, the contents of chapters 3 and 5 are based on this theory.

2.1 Port-Hamiltonian systems and Dirac structures

We introduce in this section the port-Hamiltonian system and show how to obtain the PHSD from it. We can arrive at the port-Hamiltonian system model by following two different approaches. In the first method, we start from the classical Euler-Lagrange equations and the well-known Hamiltonian equations, used for modeling mechanical systems, and show that the port-Hamiltonian system equations can be obtained as a generalization of these Hamiltonian equations. In the second method, we start with the port-based network modeling approach and arrive at the port-Hamiltonian system
model. The second approach allows us to investigate the underlying geometric structure of port-Hamiltonian systems known as the Dirac structure. We now describe both the approaches in this section.

2.1.1 From Euler-Lagrange to port-Hamiltonian systems with dissipation

The equations of motion for a mechanical system (refer to [87, Chapter 4]) can be derived using the classical Euler-Lagrange equations given as

\[
\frac{d}{dt} (\nabla \dot{q} L(q(t), \dot{q}(t))) - \nabla q L(q(t), \dot{q}(t)) = \tau,
\]

where \( q = (q_1, ..., q_n) \in \mathbb{R}^n \) are the generalized position coordinates, \( \dot{q} = (\dot{q}_1, ..., \dot{q}_n) \in \mathbb{R}^n \) are the generalized velocity coordinates, \( d/dt \) represents the total time derivative and \( \tau = (\tau_1, ..., \tau_n)^T \) is the vector of generalized forces acting on the system. The Lagrangian \( L(q, \dot{q}) \) equals \( K(q, \dot{q}) - U(q) \), where

\[
K(q, \dot{q}) = \frac{1}{2} \dot{q}^T M(q) \dot{q},
\]

represents the kinetic (co)-energy of the system, with the \( n \times n \) inertia (generalized mass) matrix \( M(q) \) being symmetric and positive definite for all \( q \), and \( U(q) \) is the potential energy of the system. Now, using (2.1) and (2.2), we can obtain the equations of motion for a \( n \)-degree of freedom mechanical system in general form as

\[
M(q) \ddot{q} + C(q, \dot{q}) \dot{q} + \nabla U(q) = \tau,
\]

where \( C(q, \dot{q}) \dot{q} \) is the vector of Coriolis and centrifugal forces, with the \((ik)\)-th element of the matrix \( C : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n \times n} \) defined by

\[
C_{ik}(q, \dot{q}) = \sum_{j=1}^{n} C_{ijk}(q) \dot{q}_j,
\]

with \( C_{ijk} : \mathbb{R}^n \to \mathbb{R} \) being the Christoffel symbols of the first kind defined as

\[
C_{ijk}(q) := \frac{1}{2} \left\{ \nabla_q M_{ik} + \nabla_{q_i} M_{jk} - \nabla_{q_k} M_{ij} \right\}. \tag{2.4}
\]

We next recall two well-known properties involving the inertia matrix \( M \) and the matrix \( C \) given as,

\[
\dot{M} = C + C^T, \tag{2.5}
\]

\[
\nabla_q \left( \frac{1}{2} \dot{q}^T M \dot{q} \right) = (\dot{M} - C) \dot{q}. \tag{2.6}
\]
Further, for all vectors \( x, y \in \mathbb{R}^n \), we have

\[
C(q, x) y = \begin{bmatrix}
x^\top C^{(1)}(q) y \\
x^\top C^{(2)}(q) y \\
\vdots \\
x^\top C^{(n)}(q) y
\end{bmatrix},
\tag{2.7}
\]

where the \( ij \) elements of the symmetric matrix \( C^{(k)} : \mathbb{R}^n \to \mathbb{R}^{n \times n} \) are precisely the Christoffel symbols \( C_{ijk} \). The above properties (2.5)-(2.7) can be found in the reference [68].

Next, the vector of generalized momenta \( p = (p_1, ..., p_n)^T \) is defined as

\[
p := \nabla_q \dot{\mathcal{L}}(q, \dot{q}) = M(q) \dot{q}.
\]

Now, by defining the state vector \((q_1, ..., q_n, p_1, ..., p_n)^T\), the \( n \) second order equations (2.3) transform into \( 2n \) first order equations given as

\[
\begin{align*}
\dot{q} &= \nabla_p H(q(t), p(t)) \quad (= M^{-1}(q(t)) p(t)) \\
\dot{p} &= -\nabla_q H(q(t), p(t)) + \tau, \\
y &= \nabla_p H(q(t), p(t)),
\end{align*}
\tag{2.8}
\]

where \( y \) denotes the output of the system and is defined such that the product of the input \( \tau \) and the output \( y \) has dimension of power. The function \( H(q,p) \) is called the Hamiltonian being equal to the sum of the kinetic and the potential energies given as

\[
H(q, p) = \frac{1}{2} p^T M^{-1} p + U(q).
\]

The following energy balance immediately follows from (2.8)

\[
\begin{align*}
\dot{H} &= \nabla_q^T H(q, p) \dot{q} + \nabla_p^T H(q, p) \dot{p} \\
&= \nabla_p H(q, p) \tau = \dot{q}^T \tau = y^T \tau,
\end{align*}
\]

expressing that the increase in energy of the system is equal to the supplied work which shows conservation of energy. System (2.8) is an example of a Hamiltonian system with collocated inputs and outputs which more generally is given in the following form

\[
\begin{align*}
\dot{q} &= \nabla_p H(q,p), \quad (q,p) = (q_1, ..., q_n, p_1, ..., p_n) \\
\dot{p} &= \nabla_q H(q,p) + G(q) u, \quad u \in \mathbb{R}^m, \\
y &= G^T(q) \nabla_p H(q,p) \quad (= G^T(q) \dot{q}), \quad y \in \mathbb{R}^m,
\end{align*}
\tag{2.9}
\]
where $G(q)$ is the input force matrix, with $G(q)u$ denoting the generalized forces resulting from the control inputs $u \in \mathbb{R}^m$. The state space of (2.9) with local coordinates $(q,p)$ is usually called the phase space. In case $m < n$ we speak of an underactuated system. If $m = k$ and the matrix $G(q)$ is everywhere invertible, then the Hamiltonian system is called fully actuated. Because of the form of the output equations $y = G^T(q)\dot{q}$ we again obtain the energy balance

$$\dot{H}(q,p) = u^T y.$$ 

Hence if $H$ is non-negative (or, bounded from below), any Hamiltonian system (2.9) is a lossless state space system.

Later, in [54] (see also [87]) the class of Hamiltonian systems in (2.9) was generalized into the so-called port-Hamiltonian systems which are described in local coordinates as

$$\dot{x} = J(x)\nabla H(x) + g(x)u, \quad x \in \mathcal{X}, u \in \mathbb{R}^m$$

and

$$y = g^T(x)\nabla H(x), \quad y \in \mathbb{R}^m.$$ 

(2.10)

Here $J(x)$ is an $n \times n$ matrix with entries depending smoothly on $x$, which is assumed to be skew-symmetric

$$J(x) = -J^T(x), \quad (2.11)$$

and $x \in (x_1, ..., x_n)$ are local coordinates for an $n$–dimensional state space manifold $\mathcal{X}$. Because of (2.11) we get the energy balance

$$\dot{H}(x(t)) = u^T(t)y(t).$$

This shows that (2.10) is lossless [94] if $H \geq 0$. The system (2.10) is called a port-Hamiltonian system with interconnection matrix $J(x)$ and Hamiltonian $H(x)$. Note that (2.9) (and hence (2.8)) is a particular case of (2.10) with $x = (q,p)$, and $J(x)$ being given by a constant skew-symmetric matrix

$$J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \quad \text{and} \quad g(q,p) = \begin{bmatrix} 0 \\ G(q) \end{bmatrix}.$$

We next show how a port-Hamiltonian system with dissipation (PHSD) (given by (1.1)-(1.2)) can be obtained from (2.10), that is, how the dissipation matrix $R(x)$ enters into (1.1). Energy-dissipation is included in the framework of port-Hamiltonian systems (2.10) by terminating some of the ports by resistive elements. To show this, we consider, instead of the term $g(x)u$ in (2.10), the following term,

$$\begin{bmatrix} g(x) \\ g_R(x) \end{bmatrix} \begin{bmatrix} u \\ u_R \end{bmatrix} = g(x)u + g_R(x)u_R,$$

(2.12)

and correspondingly, the extended output equations,

$$\begin{bmatrix} y \\ y_R \end{bmatrix} = \begin{bmatrix} g^T(x)\nabla H(x) \\ g_R^T(x)\nabla H(x) \end{bmatrix}.$$ 

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Thus, we have the more general form of a port-Hamiltonian system\(^1\)
\[
\dot{x} = J(x)\nabla H(x) + g(x)u + g_R(x)u_R, \quad (2.14) \\
y = g_\top(x)\nabla H(x), \quad (2.15) \\
y_R = g_R(x)\nabla H(x), \quad (2.16)
\]
where \(u_R, y_R \in \mathbb{R}^p\) denote the resistive ports. These resistive ports are terminated by static resistive elements
\[
u_R = -\bar{F}(y_R)
\]
where the resistive relationship \(\bar{F} : \mathbb{R}^p \to \mathbb{R}^p\) satisfies \(y_\top\bar{F}(y_R) \geq 0\). If we assume that \(\bar{F}\) is linear, that is
\[
u_R = -S_R y_R \quad (2.17)
\]
with \(S_R = S_R^\top \geq 0\) then, upon substituting this in (2.14), we obtain the equations of PHSD given in (1.1)-(1.2), with \(R(x) = g_R(x)S_Rg_\top R(x)\).

In the next section, we look at the underlying geometric structure of the port-Hamiltonian system (2.14)-(2.16).

2.1.2 Dirac structures

We first recall the definition of a state modulated Dirac structure given in [27], [28]. For this, we consider an \(n\)-dimensional manifold \(\mathcal{X}\) with tangent bundle \(T\mathcal{X}\), cotangent bundle \(T^*\mathcal{X}\) and define \(T\mathcal{X} \oplus T^*\mathcal{X}\) as the smooth vector bundle over \(\mathcal{X}\) with fibers at each \(x \in \mathcal{X}\) being given by \(T_x\mathcal{X} \times T^*_x\mathcal{X}\). Further, if \(X\) be a smooth vector field and \(\alpha\) a smooth one-form on \(\mathcal{X}\) respectively, then the pair \((X, \alpha)\) belongs to a smooth vector subbundle \(\mathcal{D} \subset T\mathcal{X} \oplus T^*\mathcal{X}\) if \((X(x), \alpha(x)) \in \mathcal{D}(x)\) for every \(x \in \mathcal{X}\). We next state the following definitions.

**Definition 2.1.** For every smooth vector subbundle \(\mathcal{D} \subset T\mathcal{X} \oplus T^*\mathcal{X}\), there exists another smooth vector subbundle \(\mathcal{D}^\perp \subset T\mathcal{X} \oplus T^*\mathcal{X}\) defined as
\[
\mathcal{D}^\perp := \{(X, \alpha) \in T\mathcal{X} \oplus T^*\mathcal{X} | \langle \alpha|\dot{X} > + \langle \dot{\alpha}|X >= 0\}, \quad (2.18)
\]
for all \((\dot{X}, \dot{\alpha} \in \mathcal{D})\), where \(\langle |\rangle\) denotes the natural pairing between a one-form and a vector field.

**Definition 2.2.** A generalized Dirac structure on \(\mathcal{X}\) is defined to be a smooth vector subbundle \(\mathcal{D} \subset T\mathcal{X} \oplus T^*\mathcal{X}\) such that \(\mathcal{D} = \mathcal{D}^\perp\).

\(^1\)In the sequel, we would refer to the equations (2.14)-(2.16) for describing a port-Hamiltonian system and continue to refer (as before) to the equations (1.1)-(1.2) for describing a port-Hamiltonian system with dissipation
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Using Definitions 2.1 and 2.2, if we substitute \( \hat{\alpha} = \alpha \) and \( \hat{X} = X \) in (2.18), we obtain that
\[
<\alpha | X> = 0, \forall (X, \alpha) \in D.
\] (2.19)

Conversely, if (2.19) holds, then for every \((X, \alpha), (\hat{X}, \hat{\alpha}) \in D\), we have
\[
0 = <\alpha + \hat{\alpha} | X + \hat{X}> = <\alpha | X> + <\alpha | \hat{X}> + <\hat{\alpha} | X> + <\hat{\alpha} | \hat{X}>
= <\alpha | \hat{X}> + <\hat{\alpha} | X>,
\] (2.20)

which implies that \( D \subset D^\perp \) and hence a Dirac structure \( D \) is maximal with respect to the property (2.19) or (2.20). We now define on the vector bundle \( T\mathcal{X} \oplus T^*\mathcal{X} \) the notion of power as
\[
P := <\alpha | X>, (X, \alpha) \in T\mathcal{X} \oplus T^*\mathcal{X}. \] (2.21)

From (2.19), (2.21), we see that any Dirac structure is power-conserving, that is, the total power entering (or leaving) the Dirac structure is zero.

We next discuss a special case when the Dirac structure is constant. We first start with the space of power variables \( \mathcal{F} \times \mathcal{F}^* \), where the linear space \( \mathcal{F} \) is called the flow space and \( \mathcal{F}^* \) is the corresponding dual space called the effort space. The elements of the flow space (referred to as flow variables) are denoted as \( f \in \mathcal{F} \), the elements of the effort space (referred to as effort variables) are denoted as \( e \in \mathcal{F}^* \), and the duality product \( <e | f> \) is defined to be the incoming power. In the previous discussion on state modulated Dirac structures, the flow space and the effort space can be identified as the tangent and co-tangent space at the point \( x \) of the manifold \( \mathcal{X} \). We next state the following definition.

**Definition 2.3.** Given an \( n \) dimensional linear space \( \mathcal{F} \) and the corresponding dual space \( \mathcal{F}^* \), a constant Dirac structure on \( \mathcal{F} \times \mathcal{F}^* \) is an \( n \) dimensional subspace \( D \subset \mathcal{F} \times \mathcal{F}^* \) such that \( D = D^\perp \), where \( \perp \) denotes the orthogonal complement with respect to the indefinite bilinear form \( \ll, \gg \) given as
\[
\ll (f^a, e^a), (f^b, e^b) \gg := <e^a | f^b> + <e^b | f^a>, (f^a, e^a), (f^b, e^b) \in \mathcal{F} \times \mathcal{F}^*.
\]

In other words, \( D \perp \) is defined as
\[
D^\perp := \{(\tilde{f}, \tilde{e}) \in \mathcal{F} \times \mathcal{F}^* | \ll (\tilde{f}, \tilde{e}), (f, e) \gg = 0, (f, e) \in D\}.
\]

A Dirac structure is known to admit several equivalent representations as shown in [28], [87]. We shall now briefly recall few of them which will be used later in chapter 6.

**Definition 2.4.** (Kernel and Image Representations) Locally around any point \( x \in \mathcal{X} \), we can find \( n \times n \) state dependent matrices \( E(x), F(x) \) such that the Dirac structure can be represented in kernel representation as
\[
D(x) := \{(f, e) \in T_x\mathcal{X} \times T^*_x\mathcal{X} | F(x)f + E(x)e = 0\}, \] (2.22)
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or equivalently in image representation as

\[ D(x) := \{(f, e) \in T_x \mathcal{X} \times T^*_x \mathcal{X} | f = E^\top(x) \lambda, e = F^\top(x) \lambda, \lambda \in \mathbb{R}^n \}, \]  

where the matrices \( F \) and \( E \) satisfy

\[ E(x)F^\top(x) + F(x)E^\top(x) = 0, \]
\[ \text{Rank}[F(x) : E(x)] = n. \]

**Definition 2.5.** (Hybrid input-output representation)[17] Let \( D \) be given as in (2.22). Suppose that locally around some point \( x \in \mathcal{X} \), the \( \text{Rank}(F) = n_1 (\leq n) \). Let \( F \) be partitioned as \( F = [F_1 | F_2] \) where \( F_1 \) is a full rank matrix (locally around \( x \)) of dimension \( n \times n_1 \). Correspondingly partition \( E = [E_1 | E_2] \), \( f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \) and \( e = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \). Then, the Dirac structure \( D \) is defined locally around \( x \) as

\[ D(x) := \left\{ \left( \begin{array}{c} f_1 \\ f_2 \end{array} \right), \left( \begin{array}{c} e_1 \\ e_2 \end{array} \right) \mid \left( \begin{array}{c} f_1 \\ e_2 \end{array} \right) = J(x) \left( \begin{array}{c} e_1 \\ f_2 \end{array} \right) \right\}, \]

where \([F_1 | E_2]\) is an invertible matrix and \( J = -[F_1 | E_2]^{-1}[F_2 | E_1] \) is skew-symmetric.

In the case of Dirac structures defined on a constant linear space, the matrices \( E, F \) in the definitions 2.4, 2.5 are constant.

We next define the composition of two constant Dirac structures arising on linear spaces with partially shared variables.

**Definition 2.6.** Consider the Dirac structures \( D_1 \subset \mathcal{F} \times \mathcal{F}^* \times \mathcal{F} \times \mathcal{F}^* \) and \( D_2 \subset \mathcal{F} \times \mathcal{F}^* \times \mathcal{F} \times \mathcal{F}^* \). Then, the composition of \( D_1 \) and \( D_2 \) denoted as \( D_1 \circ D_2 \) is defined as

\[ D_1 \circ D_2 := \{(f_1, e_1, f_2, e_2) \in \mathcal{F}_1 \times \mathcal{F}^*_1 \times \mathcal{F} \times \mathcal{F}^* \mid (f_1, e_1, f, e) \in D_1 \text{ and } (-f, e, f_2, e_2) \in D_2 \}. \]

Therefore, we can deduce that if \((f_1, e_1, f_A, e_A) \in D_1 \) and \((f_B, e_B, f_2, e_2) \in D_2 \), then \( D_1 \circ D_2 \) is given by the power conserving relationship

\[ f_A = -f_B, e_A = e_B. \]  

(2.26)

It can be shown that the composition of two Dirac structures, \( D_1 \subset \mathcal{F}_1 \times \mathcal{F}^*_1 \times \mathcal{F} \times \mathcal{F}^* \) and \( D_2 \subset \mathcal{F} \times \mathcal{F}^* \times \mathcal{F} \times \mathcal{F}^*_2 \) yields another Dirac structure \( D_1 \circ D_2 \subset \mathcal{F}_1 \times \mathcal{F}^*_1 \times \mathcal{F} \times \mathcal{F}^*_2 \). The proof can be found in [23].

The interconnection given by the power conserving relationship (2.26) is called the canonical interconnection. Another commonly used interconnection which also defines a power conserving relationship is the gyrative interconnection given as

\[ f_A = e_B, f_B = -e_A. \]  

(2.27)
We can see that the standard feedback interconnection in (1.22) with $e_C = y_C = 0$ is the same as the gyrative interconnection. Now, the composition of two Dirac structures $D_1, D_2$ by this gyrative interconnection also results in a Dirac structure. In particular, the gyrative interconnection of $D_1$ and $D_2$ gives the Dirac structure $D_1 \parallel I \parallel D_2$, where $I$ is the gyrative (or symplectic) Dirac structure defined as

$$f_{IA} = -e_{IB}, \quad f_{IB} = -e_{IA},$$

(2.28)

and is interconnected to $D_1, D_2$ via the canonical interconnections

$$f_{IA} = -f_A, e_{IA} = e_A; \quad f_{IB} = -f_B, e_{IB} = e_B.$$

Finally, we can see the following. Since the composition of two Dirac structures is a Dirac structure, it readily follows that the interconnection of a number of Dirac structures defined by a power conserving relationship is a Dirac structure.

We next consider port-Hamiltonian systems and show that they can be defined with respect to an underlying Dirac structure.

### 2.1.3 Dirac structure representation of port-Hamiltonian systems

From a network modeling perspective, a port-Hamiltonian system model can be described by a set of energy-storing elements $x_1, \ldots, x_n$ which are coordinates for some $n$-dimensional manifold $\mathcal{X}$ having a total energy $H : \mathcal{X} \to \mathbb{R}$, a set of energy-dissipating or resistive elements and a set of external ports interconnected to each other by a power-conserving interconnection (see Figure 2.1). This power-conserving interconnection is described by a Dirac structure $\mathcal{D}$ on the space $\mathcal{F} := T_x \mathcal{X} \times \mathcal{F}_R \times \mathcal{F}_P$ where $T_x \mathcal{X}$ is the tangent space at $x$, denoting the space of flows $f_S = -\dot{x}$ connected to the energy-storing elements, $\mathcal{F}_R$ is the linear space of flows $f_R$ connected to the resistive elements and $\mathcal{F}_P$ is the linear space of external flows $f_P$ which can be connected to the environment. Dually, we have $\mathcal{F}^* := T^*_x \mathcal{X} \times \mathcal{F}_R^* \times \mathcal{F}_P^*$ where $T^*_x \mathcal{X}$ is the cotangent space at $x$, denoting the space of efforts $e_S = \nabla H(x)$ connected to the energy-storing elements, $\mathcal{F}_R^*$ (dual space of $\mathcal{F}_R$) denotes the space of efforts $e_R$ connected to the resistive elements and $\mathcal{F}_P^*$ (dual space of $\mathcal{F}_P$) denotes the space of efforts $e_P$ to be connected to the environment. Typical examples of such flow-effort pairs would be currents and voltages in electrical circuits or forces and velocities in mechanical systems. The underlying Dirac structure $\mathcal{D}$ of a port-Hamiltonian system is defined locally around some point $x \in \mathcal{X}$ in the kernel representation as

$$\mathcal{D} := \{(f_S, e_S, f_R, e_R, f_P, e_P) \in \mathcal{F} \times \mathcal{F}^* \mid F_S f_S + E_S e_S + F_R f_R + E_R e_R + F_P f_P + E_P e_P = 0\}$$

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and equivalently in the image representation as

\[
\mathcal{D} := \{(f_S, e_S, f_R, e_R, f_P, e_P) \in \mathcal{F} \times \mathcal{F}^* \mid f_S = E_S^T(x)\lambda, \\
e_S = F_S^T \lambda, f_R = E_R^T \lambda, e_R = F_R^T \lambda, f_P = E_P^T \lambda, e_P = F_P^T \lambda\}
\]

(2.30)

where the state dependent matrices \(E_S(x), F_S(x), E_R(x), F_R(x), E_P(x), F_P(x)\) satisfy the relations:

\[
E_S F_S^T + F_S E_S^T + E_R F_R^T + F_R E_R^T + E_P F_P^T + F_P E_P^T = 0,
\]

(2.31)

\[
\text{Rank}\{F_S \mid F_R \mid F_P \mid E_S \mid E_R \mid E_P\} = \dim(\mathcal{F}).
\]

(2.32)

Now, the port-Hamiltonian system dynamics can be written in the kernel form (2.29) as

\[
-F_S(x)\dot{x} + E_S(x)\nabla H(x) + F_R(x)f_R + E_R(x)e_R + F_P(x)f_P + E_P(x)e_P = 0,
\]

(2.33)

which we call the implicit port-Hamiltonian system. Next, from the power conserving property of a Dirac structure (also evident from (2.30)) and (2.31), we can see that a port-Hamiltonian system satisfies the lossless energy-balance property

\[
\dot{H} = e_R^T f_R + e_P^T f_P.
\]

(2.34)

The implicit port-Hamiltonian system (2.33) generally consists of a mixed set of differential and algebraic equations (DAE’s) (refer to [87, chapter 4]). The port-Hamiltonian system (2.14)-(2.16) is in its explicit form (without algebraic constraints) or the so-called input-state-output form and is a special case of
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(2.33) where

\[ F_S = \begin{bmatrix} I_n \\ 0 \\ 0 \end{bmatrix}, \quad E_S = \begin{bmatrix} 0 \\ J(x) \end{bmatrix}, \quad F_R = \begin{bmatrix} g_R(x) \\ -g_R(x) \end{bmatrix}, \quad E_R = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \]

(2.35)

We next define a resistive structure as follows.

**Definition 2.7.** A resistive structure \( \mathcal{R} \), is defined locally around some point \( x \in \mathcal{X} \) as

\[
\mathcal{R} := \{(\hat{f}_R, \hat{e}_R) \in \mathcal{F}_R \times \mathcal{F}_R^* \mid R_F(\hat{f}_R) - R_E(\hat{e}_R) = 0\}, \tag{2.36}
\]

where \( R_E, R_F \) are state dependent matrices satisfying the relation,

\[
R_E R_F^\top \geq 0, \tag{2.37}
\]

\[
\text{rank}[R_E \mid R_F] = \text{dim}(\mathcal{F}_R). \tag{2.38}
\]

Then, locally around some point \( x \), we define the composition of a Dirac structure and a resistive structure (similar to that between two Dirac structures) as

\[
\mathcal{D} \circ \mathcal{R} := \{(f_S, e_S, f_P, e_P) \in T_x \mathcal{X} \times T_x^* \mathcal{X} \times \mathcal{F}_P \times \mathcal{F}_P^* \mid \exists (f_R, e_R) \text{ s.t } (f_S, e_S, f_R, e_R, f_P, e_P) \in \mathcal{D}; (-f_R, e_R) \in \mathcal{R}\}. \tag{2.39}
\]

The composite structure \( \mathcal{D} \circ \mathcal{R} \) in (2.39) is obtained by closing the resistive ports \((f_R, e_R)\) of the Dirac structure (2.29) (equivalently (2.30)) with the resistive relationship (2.36) and the resultant structure represents what is called a port-Hamiltonian system with dissipation (PHSD) (please refer to [87]). We next state the following proposition which give the kernel and image representation for a PHSD.

**Proposition 2.8.** The composite structure \( \mathcal{D} \circ \mathcal{R} \) is represented in its kernel form as

\[
\mathcal{D} \circ \mathcal{R} := \{(f_S, e_S, f_P, e_P) \in T_x \mathcal{X} \times T_x^* \mathcal{X} \times \mathcal{F}_P \times \mathcal{F}_P^* \mid F_S f_S + E_S e_S + F_P f_P + E_P e_P \in \text{im}\{F_R R_E^\top - E_R R_F^\top\}\}, \tag{2.40}
\]

and in its image form as

\[
\mathcal{D} \circ \mathcal{R} := \{(f_S, e_S, f_P, e_P) \in T_x \mathcal{X} \times T_x^* \mathcal{X} \times \mathcal{F}_P \times \mathcal{F}_P^* \mid f_S = E_S^\top \lambda, e_S = F_S^\top \lambda, f_P = E_P^\top \lambda, e_P = F_P^\top \lambda, \lambda \in \ker\{R_E F_R^\top + R_F E_R^\top\}\}, \tag{2.41}
\]

where the matrices \( F_S, E_S, F_P, E_P, F_R, E_R \) satisfy the relations (2.31), (2.32) and the matrices \( R_E, R_F \) satisfy (2.37), (2.38).
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Proof. We first prove the kernel representation. From (2.36) and (2.37) we obtain that \( \hat{f}_R \in \text{im}(R_E^\top) \) and \( \hat{e}_R \in \text{im}(R_E^\top) \). Next, from (2.39) we get that \( (-f_R, e_R) \in \mathcal{R} \) and therefore \( -f_R \in \text{im}(R_E^\top) \) and \( e_R \in \text{im}(R_F^\top) \). Finally, by eliminating \((f_R, e_R)\) from (2.29) we obtain that

\[
F_S f_S + E_S e_S + F_P f_P + E_P e_P \in \text{Im}\{F_R R_E^\top - E_R R_F^\top\},
\]

which proves (2.40).

We next prove the image representation of PHSD. From (2.30), we obtain that

\[
f_S = E_S^\top \lambda, e_S = F_S^\top \lambda, f_R = E_R^\top \lambda, e_R = F_R^\top \lambda, f_P = E_P^\top \lambda, e_P = F_P^\top \lambda.
\]

(2.42)

Now, from (2.39) we see that \((-f_R, e_R) \in \mathcal{R} \) and hence using (2.36) and (2.42) we obtain that

\[
\{R_E F_R^\top + R_F E_R^\top\} \lambda = 0
\]

which means that \( \lambda \in \ker\{R_E F_R^\top + R_F E_R^\top\} \). By eliminating \((f_R, e_R)\) from (2.30), we obtain the image representation of a PHSD as (2.41).

Now, the linear resistive relationship in (2.17) can be equivalently expressed as the resistive structure

\[
\mathcal{R} = \{(\hat{f}_R, \hat{e}_R) \in \mathbb{R}^p \times \mathbb{R}^p \mid \hat{f}_R = S_R \hat{e}_R, \; S_R = S_R^\top \geq 0\}.
\]

(2.43)

By comparing (2.43) with (2.36)-(2.37), we identify \( R_F = I \) and \( R_E = S_R \). Subsequently, we compute

\[
F_R R_E^\top - E_R R_F^\top = \begin{bmatrix} g_R(x) S_R & 0 \\ 0 & -I \end{bmatrix}.
\]

(2.44)

Now, using Proposition 2.8 we obtain the composition of the Dirac structure (2.35) with the resistive structure (2.43) as

\[
\begin{bmatrix} I \\ 0 \end{bmatrix} (\dot{x}) + \begin{bmatrix} J(x) \\ -g^\top R(x) \end{bmatrix} \nabla H(x) + \begin{bmatrix} g(x) \\ 0 \end{bmatrix} f_P + \begin{bmatrix} 0 \\ I \end{bmatrix} e_P = \begin{bmatrix} g_R(x) S_R \\ 0 \\ -I \end{bmatrix} \tilde{\lambda},
\]

(2.45)

for some \( \tilde{\lambda} \in \mathbb{R}^p \). After eliminating \( \tilde{\lambda} \) we obtain the system

\[
\dot{x} = [J(x) - R(x)] \nabla H(x) + g(x) f_P,
\]

(2.46)

\[
ed_P = g^\top (x) \nabla H(x),
\]

(2.47)

which is the explicit port-Hamiltonian system with dissipation (PHSD) where \( R(x) = g_R(x) S_R g_R^\top(x) \). This is exactly the same system which was introduced in chapter 1 in the equations (1.1)-(1.2) with \( f_P = u \) and \( e_P = y \).

We thus saw in this subsection that a port-Hamiltonian system with dissipation (PHSD) results from the power conserving interconnection (refer to (2.39)) of a Dirac and a resistive structure. We shall use these basic concepts later in chapter 6 to prove some interesting results.
2.2 Immersion and Invariance (I & I) method

In this section, we discuss the Immersion and Invariance (I & I) principle which was first articulated in [7] and since then has become an increasingly wide tool for control of nonlinear systems. The I & I principle has been adopted in different contexts of stabilization, adaptive control, observer design, tracking control (refer to [6] for a tutorial account of this method) etc, for mechanical and electro-mechanical systems. Our main objective is to explain how the I & I theory can be used for designing observers.

2.2.1 Reduced order observer design for nonlinear systems using I & I principle

We present a general framework for constructing reduced-order observers for nonlinear systems by invoking concepts from the Immersion and Invariance (I & I) principle. The main underlying idea of the I & I methodology in the context of observer design would be to cast the observer design problem as a problem of rendering invariant and attractive, an appropriately selected manifold in the extended state-space of the plant and the observer. We make this more clear now by providing the mathematical formulation, some of which, is taken verbatim from [6].

We first start with the concept of a reduced-order observer for a linear system which is based on the ideas of the classical Leunberger observer (refer to [52]). Consider a system described by the linear dynamics

\[
\begin{align*}
\dot{x} &= A_1 x + A_2 y, \\
\dot{y} &= A_3 x + A_4 y,
\end{align*}
\]

where \( x \in \mathbb{R}^n \) is the unmeasured part of the state, \( y \in \mathbb{R}^p \) is the measured part of the state. Now, consider the vector subspace

\[
V := \{(x, y, \zeta) \mid \zeta = Ty + x \} \subset \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^n.
\]

Let us define the dynamics of \( \zeta \in \mathbb{R}^n \) as

\[
\dot{\zeta} = (TA_3 + A_1)(\zeta - Ty) + (TA_4 + A_2)y
\]

and let

\[
z = \zeta - Ty - x
\]

be the distance from the subspace. Then upon differentiating \( z \) we obtain

\[
\dot{z} = (TA_3 + A_1)z.
\]

If the matrix \( T \) is designed such that the eigen values of \( (TA_3 + A_1) \) are in the negative left half plane then it ensures that \( z \to 0 \) asymptotically. Hence, the
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dynamics of \((x, y, \zeta)\) reaches the subspace \(\mathcal{V}\) and on account of being invariant with respect to the subspace, it actually stays on it. Hence \(\zeta\), which has the same dimension as \(x\), is a reduced-order observer for \(x\) and the asymptotic estimate of \(x\) is given by \(\zeta - Ty\).

We next extend the notion of a reduced-order observer to the nonlinear case. It is natural to expect that, for the nonlinear case, the subspace \(\mathcal{V}\) would be replaced by a manifold. Moreover, it might also happen that the dimension of the observer (that is \(\zeta\)) is greater than the dimension of \(x\). In general, \(\zeta\) can have a dimension greater than or equal to \(x\). We make these ideas more clear by considering the following nonlinear system dynamics

\[
\begin{align*}
\dot{x} & = f_1(x, y), \\
\dot{y} & = f_2(x, y),
\end{align*}
\]

where \(x \in \mathbb{R}^n\) is the unmeasured part of the state, \(y \in \mathbb{R}^p\) is the measured part of the state. The system is assumed to be forward complete, that is, the trajectories starting at time \(t = 0\) exist for all times \(t \geq 0\). Next, an observer for the system (2.50)-(2.51) is defined as follows.

**Definition 2.9.** The dynamical system

\[
\dot{\zeta} = \Upsilon(\zeta, y)
\]

with \(\zeta \in \mathbb{R}^q\), \(q \geq n\) is called a (reduced-order) observer for the system (2.50)-(2.51), if there exists mappings \(\beta : \mathbb{R}^q \times \mathbb{R}^p \to \mathbb{R}^q\) and \(\phi : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^q\) that are left invertible with respect to their first arguments and such that the manifold

\[
\mathcal{M} := \{(x, y, \zeta) \in \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q : \beta(\zeta, y) = \phi(x, y)\},
\]

satisfies the following properties:

i) All trajectories of the augmented system (2.50)-(2.52) that start on the manifold \(\mathcal{M}\) at some time \(t = T\), stay there for all future times \(t \geq T\), that is, \(\mathcal{M}\) is positively invariant.

ii) All trajectories of the augmented system (2.50)-(2.52) that start in a neighborhood of \(\mathcal{M}\) converge asymptotically to \(\mathcal{M}\), that is, the manifold is locally attractive. Equivalently, the distance of the state vector \((\zeta, x, y)\) to the manifold \(\mathcal{M}\), that is, \(\text{dist}\{\zeta, x, y\}, \mathcal{M}\) locally asymptotically goes to zero, where \(\text{dist}\{\tilde{x}, S\} := \inf_{\tilde{s} \in S} \{\text{dist}(\tilde{x}, \tilde{s})\}\) for any set \(S\).

By left invertibility of a mapping \(\Theta(y, y) : \mathbb{R}^l \times \mathbb{R}^p \to \mathbb{R}^q\) with respect to \(\tilde{y}\), it means that there exists another mapping \(\Theta^L : \mathbb{R}^q \times \mathbb{R}^p \to \mathbb{R}^l\) such that
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\[ \Theta^L(\Theta(\tilde{y}, y), y) = \tilde{y}, \text{ for all } \tilde{y} \in \mathbb{R}^l, y \in \mathbb{R}^p. \]  If the manifold \( \mathcal{M} \) satisfies the above properties then, an asymptotic (local) estimate of \( x \) is given by

\[ \hat{x} = \phi^L(\beta(\zeta, y), y), \]

where \( \phi^L \) denotes the left inverse of \( \phi \). Further, if the second condition holds globally, that is, for every \((\zeta(0), x(0), y(0)) \in \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q\), the distance to the manifold, \( \text{dist}\{(\zeta(t), x(t), y(t)), \mathcal{M}\} \rightarrow 0 \) as \( t \rightarrow \infty \), then the observer in (2.52) is a global (reduced-order) observer for the system (2.50)-(2.51).

The following remark is in order.

**Remark 2.10.** For the case of the linear system (2.48)-(2.49), the mappings become \( \beta(\zeta, y) = \zeta - Ly \) and \( \phi(x, y) = x \) and subsequently \( \hat{x} = \zeta - Ly \).

Next, a general tool for constructing reduced-order observers for nonlinear systems is presented, complying to the notion of the observer given in Definition 2.9.

### 2.2.2 Reduced-order Observers

From Definition 2.9, it can be seen that for constructing the observer, one needs to select the functions \( \beta, \phi \) and \( \Upsilon \) such that the conditions (i) and (ii) of Definition 2.9 are satisfied globally. The following discussion is contained in [6, Chapter 5].

Consider the system (2.50), (2.51) and assume that there exists a \( C^1 \) mapping \( \beta(\zeta, y) : \mathbb{R}^q \times \mathbb{R}^p \rightarrow \mathbb{R}^q \) such that for all \( \zeta, y \) the function \( \beta(\zeta, y) \) is left-invertible with respect to \( \zeta \) and \( \det(\nabla_\zeta \beta) \neq 0 \). Next, let

\[ z = \beta(\zeta, y) - \phi(x, y) \]

be the off-the-manifold coordinate and it can be seen that if \( z = 0 \), then the system reaches the manifold \( \mathcal{M} \), and if \( \mathcal{M} \) is also positively invariant, the system stays on the manifold. The task would now be to make \( z \rightarrow 0 \) as \( t \rightarrow \infty \) and make \( \mathcal{M} \) invariant. Upon differentiating \( z \) along the system dynamics, we obtain

\[ \dot{z} = \nabla_\zeta \beta \dot{\zeta} + \nabla_y \beta f_2(x, y) - \nabla_x \phi f_1(x, y) - \nabla_y \phi f_2(x, y). \quad (2.54) \]

Now, upon selecting the function \( \Upsilon(\zeta, y) \) in (2.52) as

\[ \Upsilon(\zeta, y) = -(\nabla_\zeta \beta)^{-1} \{ \nabla_y \beta f_2(\hat{x}, y) - \nabla_y \phi \left.f_2(\hat{x}, y) \right|_{x=\hat{x}} - \nabla_x \phi \left.f_1(\hat{x}, y) \right|_{x=\hat{x}} \}, \quad (2.55) \]

where

\[ \hat{x} = \phi^L(\beta(\zeta, y), y), \quad (2.56) \]
and substituting in (2.54) we obtain

\[
\dot{z} = \nabla_y \beta \{ f_2(x, y) - f_2(\hat{x}, y) \} + \{ \nabla_x \phi \bigg|_{x=\hat{x}} f_1(\hat{x}, y) - \nabla_x \phi f_1(x, y) \} \\
+ \{ \nabla_y \phi \bigg|_{x=\hat{x}} f_2(\hat{x}, y) - \nabla_y \phi f_2(x, y) \}.
\] (2.57)

Firstly, it can be seen that \( z = 0 \) is an equilibrium point of (2.57). This follows upon substituting \( z = 0 \) in (2.56) which implies that \( \hat{x} = x \). Hence, the manifold \( M \) is invariant. Now, by suitably designing the functions \( \beta, \phi, \phi^L \), if the dynamics (2.57) is proved to have a globally asymptotically stable equilibrium at \( z = 0 \), then the system (2.52), (2.55) will be a globally asymptotically stable observer for (2.50)-(2.51) with the global asymptotic estimate of \( x \) given by \( \phi^L(\beta(\zeta, y), y) \).

It can be seen that for any given mappings \( \beta \) and \( \phi \), the function \( \Upsilon \) in (2.55) makes the manifold \( M \) invariant. Thus, the task to build an observer is equivalent to computing the functions \( \beta, \phi, \phi^L \) such that the \( z \) dynamics becomes globally asymptotically stable at \( z = 0 \). Usually, the computation of these functions would involve solving a set of partial differential equations (PDE)’s which is usually a difficult task and there is no general procedure for computing the functions. However, the I & I methodology has been successfully demonstrated for some well know mechanical and electro-mechanical systems (refer to [6]). In the next chapter, we consider a special class of mechanical systems and construct velocity observers for them using the I & I concepts.
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