1 Introduction

This thesis falls under the broad category of ‘systems and control’. In the thesis we study a number of control system design problems for dynamical systems from the viewpoint of control in the behavioral framework.

Roughly speaking, control system design deals with the problem of making a system (the to-be-controlled system) behave according to certain desired specifications. The result of this design problem is another system that, if connected to the to-be-controlled system, makes it behave according to the specifications. This system is called a controller. Starting from a to-be-controlled system, the procedure of obtaining a controller can be divided into five main steps. First step is to obtain a mathematical model of the to-be-controlled system. Such a mathematical model can take many forms.

For example, the model could be in the form of a system of ordinary and/or partial differential equations, together with a number of algebraic equations, relating the relevant variables of the system. The model could also involve difference equations, some of the variables could be related by transfer functions, etc. The usual way to obtain a mathematical model of a system is by applying basic laws that the system satisfies. Often, this method is called first principles modeling. For example, if one deals with an electro-mechanical system, the set of basic physical laws (Newton’s laws, Kirchhoff’s laws, etc.) that the variables in the system satisfy form a mathematical model. Another way to get a mathematical model is called system identification. In this case a mathematical model is obtained by doing experiments on the system: certain variables in the physical system are set to particular values from the outside, and at the same time other variables are measured. In this way, one attempts to estimate (‘identify’) the laws governing the system, thus obtaining a model. Very often, a combination of first principles modeling and system identification is used to obtain a model.

The second step in a control system design problem is to formulate desirable properties that we want the to-be-controlled system to satisfy. Very often, these properties can be formulated mathematically by requiring the mathematical model to have certain qualitative or quantitative mathematical properties. Together, these properties form the design specifications.

Often, due to physical constraints we have some restrictions on the controllers which are admissible to alter the behavior of the to-be-controlled system. For example,
a) if we want to visualize the interconnection of the to-be-controlled system and the controller as a feedback interconnection, then only those controllers are admissible whose laws governing the interconnection variable are independent from the laws governing the plant,

b) if certain components of the to-be-controlled system interconnection variables represent sensor measurements, then only those controllers are admissible in which these variables are not constrained, and

c) if certain components of the to-be-controlled system variables represent external disturbances which are not constrained by the plant, then only those controllers are admissible that leave these external disturbances unconstrained in the interconnection of plant and controller, etc..

Thus, the third step in a control system design problem is to identify the set of admissible controllers which we can use to modify the system behavior to achieve the given specification.

Obviously, for a given to-be-controlled system not all given specifications may be achievable with the given set of admissible controllers. Therefore, the fourth important step in a control system design problem is to check whether the given design specification is indeed achievable.

After 1) obtaining a mathematical model of the to-be-controlled system, 2) listing the design specifications, 3) specifying a set of admissible controllers, and 4) checking that the given design specification is achievable by using an admissible controller, the fifth and most important step in control system design is to obtain a mathematical model of a controller. It is the fourth and the fifth step in the control design problem that we deal with in this thesis: they deal with mathematical control theory, in other words, with the mathematical theory of existence and design of models of an admissible controller. The problem of getting from a model of the to-be-controlled system, and a list of design specifications to a model of an admissible controller is called a control synthesis problem. Of course, for a given model, each particular list of design specifications will give rise to a particular control synthesis problem.

In this thesis we will study control synthesis problems for design specifications like stabilization, regulation, $\mathcal{H}_\infty$-control, and robust stabilization. Of course, these problems have been studied before in the literature, in an input/output framework, where control is viewed as feedback. In this thesis we solve these problems in the behavioral framework. In the behavioral framework, controlling a system means intersecting its behavior with a controller behavior. The intersection is then called the controlled behavior, which is required to satisfy the design specifications. In terms of representations, control means that additional laws (e.g. in the form of differential
equations representing the controller behavior) are put on the plant variables. Thus, the plant and controller are interconnected by sharing their variables. In our context we do not distinguish between inputs and outputs, so the interconnection does not necessarily involve feedback.

1.1 Outline of the thesis

We now proceed to give a summary of the contents of this thesis. The material presented in this thesis is based on the following papers: Fiaz & Trentelman [[35], [11]], Trentelman, Fiaz & Takaba [[36], [37], [38]], Fiaz, Takaba & Trentelman [[8], [9], [10]]. Here we summarize the contents of the thesis.

Chapter 2. In this chapter we lay the mathematical foundation for the discussion in the subsequent chapters. We review some basic concepts from the behavioral theory for modeling and studying properties of dynamical systems. Notions like elimination, controllability, autonomy, stability, observability are dealt with in this chapter. We consider dynamical systems that can be modeled by differential equations and give characterizations of properties of dynamical systems in terms of the polynomial matrices arising from the differential equations. Most of the contents of this chapter can be found in Polderman & Willems [26].

Chapter 3. In this chapter we discuss the notion of interconnection of systems from the behavioral perspective. We review different types of interconnections like full and partial interconnections and also their regularity. We also study several concepts of control in the behavioral framework starting from the viewpoint arising in the context of control as interconnection. We review the important control objective of stabilization. The material presented in this chapter can be found in Willems and Trentelman [50], Belur and Trentelman [2], Julius, et al [20], [19], Fiaz & Trentelman [11].

Chapter 4. In this chapter we discuss how in some situations the structural constraint of pre-specified input/output partition on the controllers achieving a given specification arises naturally. We show that in these situations not all regularly implementable specifications need to be physically realizable. We obtain necessary and sufficient conditions for a given specification to be regularly implementable by using controllers with pre-specified input/output structure. We use these results to obtain necessary and sufficient conditions for stabilization of
the plant by using controllers with pre-specified input/output structure. The conditions obtained are representation free and depend only upon the required input/output structure on the controller, the plant behavior and the given specification. The material presented in this chapter is based on the papers Fiaz & Trentelman [11] and Trentelman & Fiaz [35].

Chapter 5. Given a plant, together with an exosystem generating the disturbances and the reference signals, the problem of asymptotic tracking and regulation is to find a controller such that the plant variable tracks the reference signal regardless of the disturbance acting on the system. If a controller achieves this design objective, we call it a regulator for the plant with respect to the given exosystem. In this chapter we formulate the asymptotic tracking and regulation problem in the behavioral framework, with control as interconnection. The problem formulation and its resolution are completely representation free, and specified only in terms of the plant and exosystem dynamics. In the process of solving this problem, in this chapter we also discuss the behavioral version of the internal model principle. The material presented in this chapter is based on the papers Fiaz, Takaba & Trentelman [8], [9], [10].

Chapter 6. In this chapter we review the notion of rational representations of behaviors introduced recently in Willems & Yamamoto [51]. We characterize important properties of behaviors in terms of the rational matrices defining the behaviors. These characterizations will be used in the subsequent chapters. The material presented in this chapter is based on the papers Willems & Yamamoto [51], Trentelman, Fiaz & Takaba [36], [37], [38].

Chapter 7. In $\mathcal{H}_\infty$-control, the main desired property of the controlled system is that certain components (called the to-be-controlled variables) of the system’s manifest variables are small (in an appropriate sense), regardless of the values that certain other components (called the disturbances) take. In addition, the controlled system should be stable, in the sense that if the disturbance happens to be zero then the to-be-controlled variables should converge to zero as time tends to infinity. In this chapter we formulate the $\mathcal{H}_\infty$-control problem in the behavioral framework. To solve this problem, we use the theory of dissipative systems with respect to supply rates given by quadratic differential forms (QDF’s). $\mathcal{H}_\infty$-control problem in the behavioral framework was studied before in Trentelman & Willems [41]. In Trentelman & Willems [41] it was assumed that the to-be-controlled variables are observable from
the interconnection variables, and the interconnection of plant and controller need not be regular. In this chapter we consider the case where the to-be-controlled variables are only detectable from the interconnection variables, and the interconnection of plant and controller is regular. These results will be instrumental in solving the robust stabilization problem in chapter 8. The material presented in this chapter is based on the papers Trentelman, Fiaz & Takaba ([36], [37], [38]).

Chapter 8. Given a nominal plant, together with a fixed neighborhood of this plant, the problem of robust stabilization is to find a controller that stabilizes all plants in that neighborhood (in an appropriate sense). If a controller achieves this design objective, we say that it robustly stabilizes the nominal plant. In this chapter we formulate the robust stabilization problem in a behavioral framework, with control as interconnection. We use both rational as well as polynomial representations for the behaviors under consideration. Necessary and sufficient conditions for the existence of robustly stabilizing controllers are obtained using the theory of dissipative systems. In the process of solving this problem, in this chapter, we also discuss the behavioral version of the small gain theorem. We will also find the optimal stability radius, i.e. the smallest upper bound on the radii of the neighborhoods for which there exists a robustly stabilizing controller. This smallest upper bound is expressed in terms of certain storage functions associated with the nominal control system. The material presented in this chapter is based on the papers Trentelman, Fiaz & Takaba ([36], [37], [38]).

Chapter 9. This chapter contains the conclusions that can be drawn from the discussion so far and highlights the contributions made in this thesis. We also point out some directions for future research.

We conclude this chapter with a section on the notation used in this thesis and some preliminary background on polynomial and rational matrices.

1.2 Notation and properties of polynomial and rational matrices

We now devote a few words to the notation used in this thesis. We use standard symbols for the fields of real and complex numbers $\mathbb{R}$ and $\mathbb{C}$; $\mathbb{C}^-$ and $\mathbb{C}_+$ will denote the open left half plane and closed right half plane, respectively. We use $\mathbb{R}^n$, $\mathbb{R}^{n\times p}$, etc., for the real linear spaces of vectors and matrices with components in $\mathbb{R}$. $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^p)$ denotes the set of infinitely
often differentiable functions from $\mathbb{R}$ to $\mathbb{R}^v$, and its subspace consisting of functions with compact support is denoted by $\mathcal{D}(\mathbb{R}, \mathbb{R}^v)$, or sometimes simply by $\mathcal{D}$. The space of all measurable functions $w$ from $\mathbb{R}$ to $\mathbb{R}^v$ such that \( \int_{-\infty}^{\infty} \|w\|^2 dt < \infty \) is denoted by $L_2(\mathbb{R}, \mathbb{R}^v)$. The $L_2$-norm of $w$ is \( \|w\|_2 := (\int_{-\infty}^{\infty} \|w\|^2 dt)^{1/2} \). If the domain and co-domain are obvious from the context, we denote $L_2(\mathbb{R}, \mathbb{R}^v)$ simply by $L_2$.

We use $\text{rowdim}(S)$ to indicate the row dimension of a matrix $S$, or just $\text{dim}(S)$ if $S$ is a column vector or a square matrix. $I_n$ denotes the identity matrix with $\text{dim}(I_n) = n$. Similarly, $0_{m \times n}$ denotes the zero matrix with $m$ rows and $n$ columns. We use $\text{diag}(d_1, d_2)$ to denote the diagonal matrix

\[
\begin{bmatrix}
d_1 & 0 \\
0 & d_2
\end{bmatrix},
\]

again suitably generalized to more than two arguments. We use the notations $\det(S)$ and $\lambda_{\text{max}}(S)$ to denote the determinant of a square matrix $S$ and its largest eigenvalue, respectively. Given a matrix $M \in \mathbb{R}^{n \times n}$, the Moore-Penrose inverse $M^\dagger$ of $M$ is the unique $n \times m$ matrix that satisfies the following properties: $MM^\dagger M = M$, $M^\dagger MM^\dagger = M^\dagger$, $(MM^\dagger)^\dagger = MM^\dagger$, and $(M^\dagger M)^\dagger = M^\dagger M$.

$\mathbb{R}[\xi]$ denotes the ring of polynomials in the indeterminate $\xi$ with real coefficients, and $\mathbb{R}(\xi)$ denotes its quotient field of real rational functions in the indeterminate $\xi$. We use $\mathbb{R}[\xi]^n$, $\mathbb{R}[\xi]^{n \times n}$, $\mathbb{R}(\xi)^n$, $\mathbb{R}(\xi)^{n \times n}$, etc. for the spaces of vectors and matrices with components in $\mathbb{R}[\xi]$ and $\mathbb{R}(\xi)$, respectively. Elements of $\mathbb{R}[\xi]^{n \times n}$ are called real polynomial matrices, elements of $\mathbb{R}(\xi)^{n \times n}$ are called real rational matrices.

$R \in \mathbb{R}^{m \times n} \ (R \in \mathbb{R}^{n \times m})$ denotes a matrix $R$ with $m$ rows (n columns) and the number of columns (rows) depending on the context, i.e., we use $\bullet$ when it is unnecessary to specify the number of columns (rows), and we use $R \in \mathbb{R}^{n \times \bullet}$ when it is unnecessary to specify both the number of rows and columns, again suitably generalized to polynomial and rational matrices.

We call a polynomial $p \in \mathbb{R}[\xi]$ monic if the coefficient of its highest order term is 1. For any $a, b \in \mathbb{R}[\xi]$, we abbreviate the greatest common divisor of $a, b$ to $\gcd(a, b)$. We call two monic polynomials $a, b \in \mathbb{R}[\xi]$ coprime if $\gcd(a, b) = 1$. We now come to some properties of polynomial and rational matrices. A square, nonsingular real polynomial matrix $R$ is called Hurwitz if all roots of $\det(R)$ lie in the open left half complex plane $\mathbb{C}^-$. It is called anti-Hurwitz if all roots of $\det(R)$ lie in the closed right half complex plane $\mathbb{C}^+$. $U \in \mathbb{R}[\xi]^{p \times p}$ is called unimodular over $\mathbb{R}[\xi]$ if $U^{-1}$ exists and $U^{-1} \in \mathbb{R}[\xi]^{p \times p}$. This is equivalent to $\det(U)$ being equal to a non-zero constant. Unimodular polynomial matrices play a ubiquitous role in this thesis. We shall use them here for the construction of the Smith-McMillan form (Smith form) of a rational matrix (polynomial matrix).
Proposition 1.2.1. Let $M \in \mathbb{R}(\xi)^{n_1 \times n_2}$. There exist $U \in \mathbb{R}[\xi]^{n_1 \times n_2}$, $V \in \mathbb{R}[\xi]^{n_2 \times n_2}$, both unimodular, $\Pi \in \mathbb{R}[\xi]^{n_1 \times n_1}$, and $Z \in \mathbb{R}[\xi]^{n_1 \times n_2}$ such that

$$UMV = \Pi^{-1}Z, \quad \Pi = \text{diag}(\pi_1, \pi_2, \ldots, \pi_{n_1}),$$

$$Z = \begin{bmatrix}
\text{diag}(z_1, z_2, \ldots, z_r) & 0_{r \times (n_2-r)} \\
0_{(n_1-r) \times r} & 0_{(n_1-r) \times (n_2-r)}
\end{bmatrix}$$

with $z_1, z_2, \ldots, z_r, \pi_1, \pi_2, \ldots, \pi_{n_1}$ non-zero monic elements of $\mathbb{R}[\xi]$, the pairs $z_k, \pi_k$ coprime for $k = 1, 2, \ldots, r$, $\pi_k = 1$ for $k = r+1, r+2, \ldots, n_1$, and where $z_{k-1}$ is a factor of $z_k$ and $\pi_k$ is a factor $\pi_{k-1}$, for $k = 2, \ldots, r$.

In the above proposition

$$\Pi^{-1}Z = \begin{bmatrix}
\text{diag}(\frac{z_1}{\pi_1}, \frac{z_2}{\pi_2}, \ldots, \frac{z_r}{\pi_r}) & 0_{r \times (n_2-r)} \\
0_{(n_1-r) \times r} & 0_{(n_1-r) \times (n_2-r)}
\end{bmatrix}$$

is called the Smith-McMillan form of $M$. We say that $U$ and $V$ bring $M$ to Smith-McMillan form. In general, the unimodular matrices $U$ and $V$ are not unique.

Of course, $r = \text{rank}(M)$, and in the special case that $M$ has full row rank (i.e. $r = n_1$) the zero rows are absent. Similarly, when $R$ has full column rank then the zero columns are absent. Since for a given matrix, the column rank and the row rank are equal, we shall specify full row rank or full column rank only to indicate whether the matrix is wide or tall, respectively.

The roots of the $\pi_k$’s (hence of $\pi_1$ disregarding the multiplicity issue) are called the poles of $M$, and the roots of the $z_k$’s (hence of $z_r$, disregarding the multiplicity issue) the zeros of $M$.

In the above proposition, if $M$ is a polynomial matrix, the $\pi_k$’s are absent (they are equal to 1). We then speak of the Smith form

$$UMV = Z = \begin{bmatrix}
\text{diag}(z_1, z_2, \ldots, z_r) & 0_{r \times (n_2-r)} \\
0_{(n_1-r) \times r} & 0_{(n_1-r) \times (n_2-r)}
\end{bmatrix}.$$ 

Here the polynomials $z_i$ for $i = 1, 2, \ldots, r$ are called the invariant polynomials of $M$.

Any real polynomial matrix can be written as a finite sum $X(\xi) = \sum_{k=0}^{N} X_k \xi^k$. The real matrix $\left( \begin{array}{ccc} X_0 & X_1 & \ldots \end{array} \right)$ is called the coefficient matrix of $X(\xi)$, and is denoted by $\tilde{X}$.

A proper real rational matrix $G$ is called stable if all its poles are in $\mathbb{C}^-$. A square, nonsingular real rational matrix $M$ is called minimum phase, if all its poles and zeros are in $\mathbb{C}^-$. We denote by $\mathbb{R}(\xi)_{S}$ the ring of all proper stable real rational functions. $\mathbb{R}(\xi)_{S}$ and $\mathbb{R}(\xi)_{S}^{n \times m}$ denote the spaces of vectors and matrices with components in $\mathbb{R}(\xi)_{S}$. 
**Definition 1.2.2.** A proper, stable real rational matrix $G$ is called *left prime* over the ring $\mathbb{R}(\xi)$ if it has a proper, stable right inverse, i.e. if there exists a proper, stable rational matrix $H$ such that $GH = I$. A proper, stable real rational matrix $G$ is called *co-inner* if $G(\xi)G^T(-\xi) = I$.

Equivalent characterizations of left primeness can be found in Willems & Yamamoto [51].

If $G$ is a proper rational matrix and has no poles on the imaginary axis, then its $\mathcal{L}_\infty$ norm is defined as $\|G\|_\infty := \sup_{\omega \in \mathbb{R}} \|G(i\omega)\|$. If $G$ is proper and stable, then $\|G\|_\infty = \sup_{\lambda \in \mathbb{C}^+} \|G(\lambda)\|$, the $\mathcal{H}_\infty$-norm of $G$. 