On the time optimal bang bang control of linear multivariable systems with small initial perturbations
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1. INTRODUCTION.

The last fifteen years control theory has found its feet and has already reached a certain degree of completeness. Two schools in particular, the one of Pontryagin in the U.S.S.R. and the one of Bellman in the U.S.A., have greatly stimulated the development of control theory. A well-known branch within the field of control theory on which a great deal of work has been done, is linear time optimal control theory. In the latter a system is considered that can be steered by means of one or more control components. The linear system has to be steered in such a way that a certain aim is achieved as quickly as possible. The control components may only take on values which are situated in a region previously given.

In this special field a fair amount of publications have appeared, both of a theoretical [4], [5] and of a numerical [12] nature (numbers between square brackets refer to publications to be found in the list of references). Much attention has been paid to control problems in which one control component occurs and also to the numerical methods of solution with regard to this. Relatively little, however, has been published on problems with more than one control component. The theory treated in this thesis deals with such control problems. This theory, which is original, shows how the control components act together to make the system achieve the ultimate aim — which must be near the initial position — as quickly as possible. We will first try to make this clear by means of a simple example, preceded by an introductory one.

Consider a system, such as a trolley, which moves along a horizontal track without friction. The position $x_1$ of the trolley, which has mass one, is described by Newton's law of motion

$$\frac{d^2 x_1}{dt^2} = u_1(t),$$

or

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_1(t),$$

where $t$ is the time and $u_1(t)$ is the external controlling force applied to the trolley. Suppose that the initial position at time $t = 0$ of the trolley along the track is $x_1 = -\epsilon$ ($\epsilon > 0$) and that the initial velocity $x_2 = \frac{dx_1}{dt}$ is zero. We consider the problem of bringing the trolley to $x_1 = 0$ where it has to arrive with velocity $x_2 = 0$ in minimal time. This has to be done by means of a (possibly discontinuous) controlling force $u_1(t)$ subject to the constraint

$$|u_1(t)| \leq 1.$$

The solution of the time optimal control problem introduced above is intuitively clear. The optimal control $u_1^*(t)$ is a maximal accelerating force followed by a maximal deceler-
ation until the mechanism stops exactly at the required position \( x_1 = 0 \). The critical time for the switch from acceleration to deceleration is precisely half-way the process. A simple calculation shows that the optimal trajectory in the \((x_1, x_2)\) plane consists of two parts of parabolas of which the formulas are

\[
\begin{align*}
x_1 &= \frac{1}{2} \omega^2 t^2 - \epsilon, \\
x_1 &= -\frac{1}{2} \omega^2 t^2.
\end{align*}
\]

These parabolas have been sketched in fig. (1.1).

![Fig. (1.1). The optimal trajectory in the \((x_1, x_2)\) plane.](image)

First the controlling force \( u_1 = +1 \) is applied to the trolley and the trolley moves along the parabola \( x_1 = \frac{1}{2} \omega^2 t^2 - \epsilon \). At the moment that \( x_1 = -\frac{1}{2} \epsilon \) (see figure), i.e. after \( \sqrt{\epsilon} \) units of time, the controlling force \( u_1 \) jumps from \( +1 \) to \( -1 \) and the trolley continues along the parabola \( x_1 = -\frac{1}{2} \omega^2 t^2 \) until it reaches the origin after \( 2\sqrt{\epsilon} \) units of time. Hence the optimal control is

\[
u_1^*(t) = \begin{cases} +1, & 0 \leq t < \sqrt{\epsilon}, \\ -1, & \sqrt{\epsilon} \leq t \leq 2\sqrt{\epsilon}. \end{cases}
\]

The minimal time of the process is \( 2\sqrt{\epsilon} \) units of time.

Let us now consider another system, of which the equations are

\[
\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.
\]
The physical interpretation is not as simple as for the system above. The difference between the two systems is that in the latter example a control component $u_2(t)$ has been added, which acts directly on the velocity $\frac{dx_2}{dt}$. The problem will be the same as above: the initial position and velocity are $x_1 = -\varepsilon$ ($\varepsilon > 0$) and $x_2 = 0$ respectively and the system must be steered to $x_1 = x_2 = 0$ as quickly as possible by means of the two control components $u_1(t)$ and $u_2(t)$, which are constrained to

$$|u_1(t)| \leq 1, \quad |u_2(t)| \leq 1.$$ 

With the aid of control theoretical methods this problem can be solved. The solution is again intuitively clear: during the whole manoeuvre is $u_2(t) = +1$ and $u_1(t)$ first accelerates ($u_1 = +1$) and then decelerates ($u_1 = -1$). The switch from $u_1 = +1$ to $u_1 = -1$ is again half-way the process. A simple calculation shows that the minimal time is $-2+2\sqrt{\varepsilon}$ units of time and that the optimal control $u_1^*(t), u_2^*(t)$ is

$$u_1(t) = \begin{cases} +1, & 0 \leq t < -1 + \sqrt{\varepsilon}, \\ -1, & -1 + \sqrt{\varepsilon} \leq t \leq -2+2\sqrt{\varepsilon}, \\ u_2(t) = +1, & 0 \leq t \leq -2+2\sqrt{\varepsilon}. \end{cases}$$

Let us consider both examples from one point of view. A difference between these examples is that in the first one the time necessary for the manoeuvre (if the manoeuvre starts at $t = 0$ this time will be called the final time) is proportional to $\sqrt{\varepsilon}$ whereas in the second one the final time is about proportional to $\varepsilon$ for $\varepsilon$ sufficiently small (then $-2+2\sqrt{\varepsilon} \approx \varepsilon$). This difference appears to be very characteristic of the two systems.

Also if instead of $(x_1 = -\varepsilon, x_2 = 0)$ the initial position and velocity are

$$x_1 = \sigma_1 \varepsilon, \quad x_2 = \sigma_2 \varepsilon,$$

where $\sigma_1$ and $\sigma_2$ are constants which satisfy $\sqrt{\sigma_1^2 + \sigma_2^2} = 1$, then this different behaviour remains valid. In the case of the first system one finds that for almost all values of $\sigma_1$ and $\sigma_2$ (restricted to $\sqrt{\sigma_1^2 + \sigma_2^2} = 1$) the final time is about proportional to $\sqrt{\varepsilon}$ for $\varepsilon$ sufficiently small. In the case of the second system one finds that for all values of $\sigma_1$ and $\sigma_2$, again satisfying the restriction, the final time is about proportional to $\varepsilon$ for $\varepsilon$ sufficiently small.

A rough explanation of this difference is that in the first example the control $u_1$ can influence directly only the velocity in the $(x_1, x_2)$ plane of the system in the direction $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ of this plane, whereas in the second example the control $(u_1, u_2)$ can influence directly the velocity in the directions $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The last two vectors span the $(x_1, x_2)$ plane.

We will now generalize the examples mentioned. A system is described by the differential equation
(1.1) \[ \frac{dx}{dt} = Ax + Bu, \]

where the \( n \)-vector \( x \) denotes the state of the system and the \( r \)-vector \( u \) is the control; \( t \) is the time. The matrices \( A \) and \( B \) (of size \( n \times n \) and \( n \times r \) respectively) have constant elements. The components \( u_i(t), i = 1, \ldots, r \), of the control vector \( u(t) \) are restricted to

\[ |u_i(t)| \leq 1, \; i = 1, \ldots, r. \]

It is required to choose the control \( u(t) \) in such a way that the system is steered from the initial point \( x = cx^0 \) (\( x^0 \) is an \( n \)-vector with length one and \( c \) is a positive scalar) at \( t = 0 \) to the origin as quickly as possible.

For a large class of systems the time optimal control (the control we are looking for) shows a bang-bang behaviour, i.e. the control components are piecewise constant and take on only the values \(+1\) and \(-1\). Points on the time axis at which a control component jumps from \(-1\) to \(+1\) or vice versa are called switches. The time at which such a switch occurs is called a switching-time.

Under certain conditions concerning the system and the direction \( x^0 \) of the initial value, the following facts about the time optimal control, which is bang-bang, will be proved in this thesis, provided that \( c \) is sufficiently small.

1°. All control components together have precisely \((n-1)\) switches. For the validity of this property it is not necessary that all the eigenvalues of the matrix \( A \) are real. A known fact is \([5]\) that if these eigenvalues are real then each control component separately has at most \((n-1)\) switches.

2°. Simple criteria exist which give the number of switches per control component. These numbers are \( p-1 \) or \( p \) (and the sum of these numbers is \( n-1 \)). The quantity \( p \) is the smallest natural number in such a way that \( p \geq \frac{n}{r} \). If one knows the switching-times of a control component, the starting-sign of this component (i.e. \( \pm \)) has to be known in order to determine this control component completely as a function of time. There are also simple criteria which give the starting-signs of the different control components.

3°. The final time and the switching-times are analytic functions of \( \sqrt{c} \), where \( p \) has been defined above. (We assume that \( x^0 \) is fixed and that \( \sqrt{c} \) is real). These functions have the value zero if their argument is zero, and if they are developed in a Taylor series with respect to their argument, the coefficients of the linear terms are not equal to zero. Hence for \( c \) sufficiently small the final time of the manoeuvre is about proportional to \( \sqrt{c} \).

Consider for instance \((4,1)-, (4,2)-, (4,3)-\) and \((4,4)\)-systems, which means that of these four systems \( n = 4 \) and \( r = 1, 2, 3 \) and \( 4 \) respectively. The approximate dependence of the final time on \( c \) is proportional to \( c^{1/4} \), \( c^{1/2} \), \( c \) and \( c^2 \) respectively.
It follows that for $p > 1$ (i.e. $r < n$) we must know the switching-times and the final time very exactly in order to steer the system from $x(0)$ to the origin. An error of order $\epsilon$ in one of the switching-times or the final time — such an error is small with respect to the duration of the whole process if $\epsilon$ is sufficiently small — causes that the final situation of the system has again a distance of order $\epsilon$ from the origin (the distance from a point $x$ with components $x_1, \ldots, x_n$ to the origin is defined to be $\left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$). In this way we can conclude that for $p > 1$ and $\epsilon$ sufficiently small the criterion of time optimality is not a fortunate one, if we want to steer the system to the origin exactly. This is a mathematical formulation of the generally accepted engineering principle that in order to reduce small initial perturbations one should not use bang-bang control (which is moreover often difficult to realise for $t_f$ sufficiently small). Instead of this a (continuous) linear feedback control is frequently used.

It also follows that for $\epsilon$ sufficiently small the systems move with an approximately constant velocity in the $x$-plane during the time interval between $t = 0$ and the first switching-time along a more or less straight line (i.e. in the direction of $B\nu$). Hence the system reaches a distance of order $\frac{\epsilon^p}{t_f}$ from the origin which is large (if $p > 1$) with respect to the initial perturbation which had the distance $\epsilon$.

The three properties mentioned above are formulated more precisely in the "main theorem" in chapter 6. The whole thesis is concerned with this main theorem and its proof. The chapters 9 up to and including 13 are the heart of the proof and an outline of these chapters and also of chapters 7 and 8 is given at the end of chapter 6.

The main theorem — formulated under the condition that $\epsilon$ is sufficiently small — will probably also be valid for many systems which are governed by nonlinear differential equations of the form

$$\frac{dx}{dt} = g(x,t) + B(x,t)u$$

provided that these nonlinear equations can be linearized to the form (1.1).

In the proof of the main theorem use is made of the concept of formal power series which is introduced in chapter 2. In that chapter some definitions and (known) results concerning the theory of functions are given. In chapter 3 a new function theoretical result is derived. The important aim of these two chapters is a generalization of the implicit function theorem, which is crucial for the proof of the main theorem.

Suppose we are given the functions $f_i(z_1, \ldots, z_k; \omega)$, $i = 1, \ldots, k$, of the complex variables $z_1, \ldots, z_k$ and $\omega$ which satisfy $f_i(0, \ldots, 0; 0) = 0$ and which are analytic in a neighbourhood of the origin $z_1 = \ldots = z_k = \omega = 0$. The implicit function theorem states that the equations

$$f_i(z_1, \ldots, z_k; \omega) = 0, \quad i = 1, \ldots, k,$$

have one and only one analytic solution $z_1(\omega), \ldots, z_k(\omega)$ in a neighbourhood of $z = 0$ if
the Jacobian of the functions \( f_{i} \) with respect to the variables \( x_{j} \):

\[
\frac{\partial (f_{1}, \ldots, f_{k})}{\partial (x_{1}, \ldots, x_{k})}
\]

is not equal to zero at \( x_{1} = \ldots = x_{k} = \omega = 0 \). That the Jacobian is not equal to zero is a sufficient, but not a necessary condition for the equations \( f_{i} = 0 \) to have an analytic solution. In chapter 3 this condition will be weakened. Note that if the Jacobian is \( \neq 0 \) then the solution mentioned is unique; this is in general not true if only the weaker conditions are satisfied. It is possible in that case that more than one analytic solution exists.

In chapter 4 a short review is given of known control theoretical facts. For a more extensive introduction one can consult [4].

In chapter 5 the class of systems is introduced for which the main theorem is valid. The relation to generally known classes of systems, such as controllable systems, is shown.

In chapter 14 something is said about possible extensions of the main theorem. The main theorem has been formulated under sufficient conditions but simple examples indicate that these conditions can be weakened.

The main theorem offers a numerical method to calculate the time optimal control; one only has to construct the power series mentioned in 3.4. Some numerical results obtained by this method, especially for \((4,2)\) systems, have been given in chapter 15. Such a \((4,2)\) system stems from a problem in ship manoeuvring, which is as follows. A ship follows a rectilinear course and must be steered to a parallel course as quickly as possible. There are two control components: one rudder at the stern and one at the bow of the ship.

This way of determining the time optimal control seems to be applicable for systems with \( n \) and \( r \) rather large. However, the condition "\( \epsilon \) must be sufficiently small" may not be satisfied if one deals with practical time optimal control problems.

Some notations and conventions.

The \( n \)-dimensional real vector space is denoted by \( \mathbb{R}^{n} \). Let \( x \) be a (column) vector in \( \mathbb{R}^{n} \), then the transpose of \( x \) is denoted by \( x' \). The equality \( x = 0 \) means that all components of the vector \( x \) are zero. The norm of \( x \), denoted by \( \|x\| \), is defined by - if we suppose that \( x_{1}, \ldots, x_{n} \) are the components of \( x \)-

\[
\|x\| = \left( \sum_{i=1}^{n} x_{i}^{2} \right)^{\frac{1}{2}}.
\]

To distinguish between vectors we sometimes write \( x^{1}, x^{2}, \ldots \); the numbers 1, 2, \ldots are indices and not exponents. The innerproduct of two vectors \( x^{1} \) and \( x^{2} \) is written as \( (x^{1}, x^{2}) \).

If we are given a square matrix \( A \), then \( \text{det}(A) \) denotes the determinant of \( A \). If this determinant is zero, \( A \) is called singular and otherwise it is called regular or non-singular.
If $a$ is a scalar, then, if $a \neq 0$, \( \text{sgn}(a) = a/|a| \), where $|a|$ is the absolute value of $a$. In case $a = 0$, the sgn-relation is not defined.

We will not distinguish between the notations $a_1, \ldots, k$ and $a_{j, \ldots, k}$. In both cases these notations denote the same quantity.

The real intervals $a < t < b$, $a \leq t \leq b$, $a < t \leq b$ and $a \leq t < b$ will be denoted by $(a,b)$, $[a,b]$, $(a,b]$ and $[a,b)$ respectively.

In this thesis the following abbreviations will be used:

- **r.h.s.** right-hand side,
- **l.h.s.** left-hand side,
- **f.p.s.** formal power series,
- **c.p.s.** convergent power series,
- **t.o.c.** time optimal control,
- **a.e.** almost everywhere.

In this thesis some paragraphs are headed by "Remark" or "Observation". Remarks are essential for understanding the investigation; observations are not strictly necessary and can be omitted.