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THE UNIQUE SOLUTION OF THE INVERSE DIFFRACTION PROBLEM

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The problem of the determination of the values of a field on a surface from its values on a surface to which it has propagated is shown to have a unique solution if the field satisfies any linear elliptic partial differential equation.

Suppose that a scalar field $\psi$ is the solution of a linear second order elliptic partial differential equation

$$ L \psi = 0, $$

in a domain $D$ bounded by two closed surfaces $S_1$ and $S_2$, (see fig. 1). The equation (1) can for instance be the Helmholtz equation $(\nabla^2 + k^2 n^2)\psi = 0$, valid in a medium with variable index of refraction, or the time independent Schrödinger equation in the presence of an electromagnetic field, characterized by the vector potential $A$ and the scalar potential $\phi$:

$$ e^{2\pi i m n} \psi \nabla^2 \psi + \frac{eh}{im} A \cdot \nabla \psi + \left( \frac{e^2}{2m} |A|^2 - e\phi \right) \psi = E \psi. $$

The field outside $A$ is supposed to be uniquely determined by its values on $S_1$ and Sommerfeld’s radiation condition at infinity, i.e. It is assumed that the Dirichlet problem for the operator $L$ in the domain $D$ outside $S_1$ admits a unique solution. The inverse diffraction problem then requires to determine the values of $\psi$ at $S_1$ from its values at $S_2$ (or perhaps on some part of $S_2$), to which the field has propagated. For a review see Hoenders [1].

For the solution of this problem we need the following corollary of a remarkable theorem derived by Beckert [2]:

**Theorem** Let $P$ denote a continuously differentiable surface lying entirely within a $n$-dimensional domain $E$ with boundary $\partial E$. The dimension $s$ of $P$ satisfies $1 \leq s \leq n - 1$, and $P$ does not separate $E$. Let $u$ be a solution of the elliptic partial differential equation

$$ - \frac{\hbar^2}{2m} \nabla^2 u + \frac{eh}{im} A \cdot \nabla u + \left( \frac{e^2}{2m} |A|^2 - e\phi \right) u = E u. $$

The functions $a_{ik}$ are two times Hölder continuous differentiable, $c$ and $f$ are Hölder continuous, and $b_i$ are one time Hölder continuous differentiable.

Suppose that the interior Dirichlet problem $L \psi = 0$, with $\psi = \mu(x)$ if $x \in \partial E$, and if $\mu(x)$ denotes an arbitrary $L^2$ function is solvable. Then the set of functions $$ \{ \psi_n(x) \}, x \in \Gamma, $$ generated by a suitable set of boundary values $\{ \mu_n(x) \}, x \in \partial E$ or $x \in$ any subset of $\partial E$, $n = 1,2, \ldots$ is together with its normal derivatives dense in the Hilbert space of all $L^2$ functions on $\Gamma$, i.e. Any $L^2$ functions $h(x)$ and $(\partial/\partial n)h(x), x \in \Gamma$ can be approximated simultaneously in the mean arbitrarily closely by a set of solutions $\{ \psi_n \}$ of $L \psi_n = 0$ generated by an appro-
appropriate set \( \{ \mu_n \} \) of surface distributions on \( \partial E \).

We will need Green's formula, which reads as

\[
\int_{\partial E} [\phi(L\psi) - \psi(M\phi)] \, d\tau = \int_{\sigma} \left[ (\phi \partial \psi/\partial n - \psi \partial \phi/\partial n) + b \psi \phi \right] \, d\sigma,
\]

if \( M \) denote the adjoint operator to \( L \):

\[
M\phi = \sum_{i,k} \frac{\partial}{\partial x_k} \left( a_{ik}(x) \frac{\partial}{\partial x_i} \phi(x) \right) - \sum_i \frac{\partial}{\partial x_i} (b_i(x) \phi(x)) + c(x) \phi(x),
\]

\( \sigma \) denotes the boundary of the domain \( \tau \), and

\[
a = \sum_{i,k} a_{ik} n_i, \quad b = \sum_i e_i n_i,
\]

\[
e_i = b_i - \sum_{k=1}^n \frac{\partial a_{ik}}{\partial x_k},
\]

and \( n_i \) denote the cartesian components of the normal \( n \) or \( \sigma \).

Let \( S_1' \) and \( S_2' \) be the surfaces drawn in the figure. Suppose that \( DCE \) and that \( \phi \) is a solution of \( M(\phi) = 0 \) if \( x \in E \), with

\[
\phi(x) \simeq 0,
\]

\[
\frac{\partial}{\partial n} \phi(x) \simeq \delta_1(x - y), \quad \text{if} \; x \; \text{and} \; y \in S_1',
\]

and \( \delta_1(x - y) \) denotes a regularisation of the \( \delta \)-func-
tion. The existence of such a function is ascertained by the corollary to Beckert's theorem, stated above. Eqs. (1) and (4) lead to:

\[
\int_{S_1'} \left\{ a(\phi \partial \psi/\partial n - \psi \partial \phi/\partial n) + b \psi \phi \right\} \, d\sigma
\]

\[
= - \int_{S_2'} \left\{ a(\phi \partial \psi/\partial n - \psi \partial \phi/\partial n) + b \psi \phi \right\} \, d\tau,
\]

or, on using eqs. (7) and (8) we derive from (9):

\[
a(x) \psi(x)\bigg|_{x \in S_1'}
\]

\[
= - \int_{S_2'} \left\{ a(\phi \partial \psi/\partial n - \psi \partial \phi/\partial n) + b \psi \phi \right\} \, d\sigma.
\]

Hence, the values of the field on \( S_1' \) can be determined from the known values of the r.h.s. of eq. (10): \( \phi \) and \( \partial \phi/\partial n \) are known by construction, and \( \psi \) is assumed to be known on \( S_2' \). The knowledge of \( \psi \) on \( S_2' \) together with Sommerfeld's radiation condition enables us to determine \( \partial \psi/\partial n \), so that the r.h.s. of eq. (10) is a known quantity.

The proof of Beckert's theorem as well as the corollary used in this letter will be given in a forthcoming paper, which also contains some explicit examples.

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References