Chapter 3

Simulations of the instrumental response

It is nice to know that the computer understands the problem. But I would like to understand it too.

Eugene Wigner

ABSTRACT

One of the primary scientific goals of the LOFAR telescope is to measure fluctuations in the redshifted 21-cm signal from the Epoch of Reionization (EoR). These fluctuations are associated with the ionized bubbles produced by the first structures that formed in the Universe. Observing the EoR signal though, is challenging due to the instrumental and ionospheric effects that corrupt the signals received by the antennae as well as the presence of strong Galactic and Extragalactic foregrounds. Realistic simulations are essential to study the effects of all such corruptions with respect to the extraction of the EoR signal. In this chapter we present a method to model instrumental corruptions that are statistically similar to the complex gains and the ionosphere and we compare our models with real calibration parameter estimates from the initial LOFAR commissioning observations. These models are also particularly useful for the MWA and the SKA, since real-time solution of their calibration parameters during small time intervals is computationally unfeasible. We also consider errors on the dipole and station beam patterns, in particular with respect to instrumental polarization. We use the different instrumental effect models to generate a realistic, simulated observational data cube that can be calibrated using the standard calibration pipeline of LOFAR. Finally, we compare calibration parameters with the obtained solutions.

3.1 Introduction

Traditionally, cosmology focuses on two pivot points: the study of the CMB at \(z \sim 1100\) (400,000 years) and the structure and dynamics of the Universe at \(z\) up to 6 (few billion
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years). Both these epochs constrain the models of the universe to a high degree and allow for precision cosmology. In between those epochs, the Universe evolved under the influence of gravity from being predominately neutral after the recombination era (and during its Dark Ages) to the hierarchy of structures that we observe today (Furlanetto et al., 2006c). Understanding what happened in this epoch is crucial for the determination of the matter power spectrum at small scales and the nature of the first objects (e.g. stars, mini-quasars etc.), and thus remains one of the most exciting open questions of observational cosmology. Theory and simulations suggest that the ultra-violet radiation from the first objects that were formed, was sufficient to ionize the Universe by redshift \( z=6.5 \) (Iliev et al., 2009). A promising direct probe of the end of the “Dark Ages” and reionization is the study of the redshifted 21-cm line of the hydrogen spin-flip transition, which falls in the regime of low frequency radio astronomy. Although this idea is not new (e.g. Madau et al., 1997b), it was impossible till recently to go forward due to the lack of the necessary technology that is needed for the construction of a low frequency array with high sensitivity and resolution like supercomputing resources, receiver and beamformer hardware, fast interconnects and algorithms.

LOFAR is a next generation radio-interferometer, currently under construction in the Netherlands and other European countries and it is one of the most promising instruments to detect the EoR signal in the near future, mainly due to its large collecting area, long baselines and the archiving of the visibility data for reprocessing as well as its flexible software-based approach to signal processing. It operates in the largely unexplored spectral window between 30 (also capable going down to 10 MHz) and 240 MHz. The expected cosmological signal is two to five orders of magnitude fainter than a host of instrumental contaminants, ionospheric corruptions, Radio Frequency Interference (RFI) and Galactic and extra-Galactic foregrounds. Add to that our uncertainty regarding the characteristics of the EoR signal. The success of the experiment thus lies entirely on our understanding of the noise properties and systematics and how the signal can manifest itself differently from those.

In recent years there have been many attempts to describe the foreground removal (Jelić et al., 2008; Liu et al., 2009) and signal extraction (Bowman et al., 2006; Harker et al., 2009d,b) methodologies for upcoming EoR experiments. All those methods work under the assumption that the instrumental effects are well under control. Given the fact that the EoR signal is extremely weak, those assumptions have to be tested for their validity under different circumstances. A few papers (Labropoulos et al., 2010; Datta et al., 2009) have attempted to introduce instrumental corruptions in a simplistic manner. More specifically Labropoulos et al. (2010) discusses the general overview of the data model, leaving its implementation for simulations open. On the other hand, Datta et al. (2009) ignore several well know sources of errors as well as the noise response of the instrument. Mitchell et al. (2008) present the online calibration system for the Murchison Wide Field Array (MWA) and give a basic overview of the performance of the system.

An important question about the feasibility of any EoR experiment still remains: How well does that calibration process perform in practice? It is well known that the image-plane effects such as the ionospheric Faraday rotation and the primary beams of the stations, share partially similar mathematical forms. It is not clear how well the calibration process can distinguish between such degenerate parameters, given the fact that there is a limited number of independent measurements that can be used to constrain the
3.2 Statistical modeling of the complex gains

The complex gain factor ($G$ Jones in Labropoulos et al. 2010) incorporates the response of the antennae and the electronics in such a way, that the input signal is proportional to the correlator output for a given antenna pair, when a point source of unit flux is observed at the phase center. The actual behavior of the complex gains depends on the design of the receiving elements and electronics of every system. Thus, a general description cannot be devised. It can also depend on external sources, such as the temperature and humidity, since both those factors affect the performance and stability of electronic circuit components. Moreover, rapidly varying phase errors, such as those arising from the clock drifts reduce the correlation coefficient and thus result in loss of signal amplitude and hence sensitivity. The effect on the signal phases is lower due to the visibility averaging, as even in the low frequencies there are $\sim 10^7$ samples per averaging interval.

Time series analysis methods have been used in statistics and signal processing to understand the underlying context of sequences of data points and to make forward predictions (e.g. predictions of stocks values in econometrics). The classical approach is to use Fourier analysis to study such data sets, but this provides only the variance as a function of the sampling frequency and does not contain information about the time evolution of the process. Even with this limitation, the assumptions that justify the use of Fourier analysis (Nyquist sampling and infinite duration) do not always hold. To specialize this general remark in the problem of the temporal evolution of the gain solutions the data might not be sampled properly due to the fact that part of the data might be flagged.
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(e.g. for RFI mitigation), the variation of certain effects is comparable to the length of the observation and there might be more than one frequency that characterizes the time evolution of the parameters. Radio-astronomy is using concepts of time-series analysis. For example the CLEAN algorithm is a case of pre-whitening where strong signals are removed from the data in order to remove bias (Roberts et al., 1987).

In this chapter we shall consider a parametric model that is commonly used in time-series analysis in the context of modeling radio interferometric array gains. Using a parametric model allows us to apply likelihood techniques for the estimation of the model parameters. The basic concept in such models is that observations closely related in time are indeed closely related, in contrast to Gaussian processes like the thermal noise. This means that the points of the time-series (i.e. of the gain solutions) are dependent on the previous points in time and consequently the time ordering of the data is important. A useful extension of this modeling is the spatio-temporal analysis where the data might be neighboring in time as well in space i.e. similar baselines, frequencies etc. There are three broad-classes of models: Auto Regressive (AR) where the model has memory of past values, Moving Average (MA) and Integrated (I) models. All those models work for aperiodic, autocorrelated time-series and share a common disadvantage: the physical interpretation of the model parameters is unintuitive and hard to interpret. The basic assumptions of these models are that the processes are stationary, that is that their joint probability density function (PDF) does not change when shifted in time, and ergodicity. A plethora of other models can be constructed i.e. for vector data, forced behavior etc. but the underlying principles are the same. Finally, models where the standard deviation of the error depends on the error on previous time intervals can be used to describe highly non-linear processes which exhibit periods of rapid fluctuations alternating with relatively calm periods. We shall consider such modified models if the need arises in the future. This kind of modeling has also been used extensively in the realm of astronomy to study periodic phenomena (i.e. binary orbit radio pulsars and Cepheids), stochastic phenomena (i.e. accretion, jet variations) and extreme phenomena (i.e. gamma-ray bursts and solar flares). The main challenges in applying such techniques in astronomical data are the heteroscedasticity (different SNR for different samples) and the large time-scale properties of the processes i.e diurnal variation of the gain solutions. For a general review of the applications of ARMA models to astronomy we refer the reader to Rao et al. 1997.

In the following we are going to deal with causal models only. In those models the value at a time $t$ depends only on the past values and not on the future ones. We shall provide the conditions for causality later. An AR model is an all-pole infinite impulse response filter (IIR), that is a frequency response function that goes infinite (poles) at specific frequencies, but there are no frequencies where the response function is zero. The p-th order AR model can be written as (Therrien, 1992):

$$X_t = c + \sum_{i=1}^{p} \psi_i X_{t-i} + \epsilon_t,$$  \hspace{1cm} (3.1)

where $c$ is a constant, $\epsilon_t$ is white noise and $\psi_i$ are the parameters of the model. The MA model suggests that the current value of a time-series is a linear regression against previous unobserved white, Gaussian errors or shocks. Those errors are assumed to come
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Figure 3.1: The figure shows a typical ARMA(15,14) model used for the simulations. **Left:** The gain amplitude as a function of time for a HBA station. The XX and YY substation coefficients are presented in blue and red respectively. **Right:** The unwrapped phase of stations for the same station. Note the evolution of the mean of the process at certain scales.

From the same distribution and they propagate through the future values of the time series. The last property makes the solution of an MA model essentially more difficult than that of an AR model, as it requires iterative non-linear fitting. In any case after fitting the errors have to be independent. The MA model is a finite impulse response filter (FIR). The q-th order MA model can be written as (Therrien [1992]):

\[ X_t = \mu + \varepsilon_t + \sum_{i=1}^{q} \theta_i \varepsilon_{t-i}, \]

where \( \mu \) is the mean and \( \theta_i \) are the model parameters. Combining both models we get (Therrien [1992]):

\[ X_t = c + \varepsilon_t + \sum_{i=1}^{p} \psi_i X_{t-i} + \sum_{i=1}^{q} \theta_i \varepsilon_{t-i} \quad (3.2) \]

If the process exhibits non-stationarity, taking the difference of the time series at different intervals can help remove the non-stationarity and end up with an ARMA model. The ARMA model is quite robust and usually requires fewer parameters than either the AR and/or MA models, at the expense of being significantly more difficult to design and analyze. More complex models that present non-linearity can in principle be used (e.g. using local linearization or global spline approximations), but we shall refrain from more complex modeling unless the data suggests so. After fitting an ARMA model in either manner, we have to investigate the goodness of fit. This can be done using the likelihood ratio test.
3.2.1 Solution of the ARMA models

The ARMA model can be written as the following difference equation:

\[ \sum_{i=0}^{p} \psi_i X_{t-i} = \sum_{j=0}^{q} \theta_j \varepsilon_{t-j} \]

where we have set \( \psi_0 = 1 \) and \( \varepsilon_t \) is assumed to be a white noise process with unit variance. The correlation and cross-correlation functions satisfy same difference equations as the random process. Recall that the cross-correlation function \( R_{X\varepsilon} [l] \) is given by \( R_{X\varepsilon} [l] = IR[l] * R_{\varepsilon}[l] = IR[l] \), where \( IR[l] \) is the impulse response function and \( R_{\varepsilon} \) is the noise auto-correlation. Then:

\[ \sum_{i=0}^{p} \psi_i R_{t-i} = \sum_{j=0}^{q} \theta_j \varepsilon_{t-l} \]

(3.3)

The system of equations (3.3) is known as the Yule-Walker system. All one has to do is to evaluate the above equation for a set of \( l \) consecutive values and solve the set of simultaneous equations. It is imperative to note that the above system is non-linear because the coefficients of the AR and MA parts of the model enter the impulse response function IR in a rather complicated way. The coefficients also appear in the equation making it at least quadratic except for the trivial case. This system is very general as we used very few assumptions. The Gaussian process can be viewed as a special case of this model. If the system is causal, then the Yule-Walker equations provide both the AR and MA pa-
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The non-smoothed power-spectrum of the true ARMA(15,14) mode (red) and the Yule-Walker estimate (blue) using the “observed” correlations.

Figure 3.3: The non-smoothed power-spectrum of the true ARMA(15,14) mode (red) and the Yule-Walker estimate (blue) using the “observed” correlations.

rameters of the model. Its solution is well studied in the literature (Therrien, 1992) and is beyond the scope of our current analysis.

The weak point of the above solution is that the correlation matrices $R[l]$ and the order of the models need to be known a priori. The estimation problem without any information about the process is unsolvable or ill-posed. However, stationary, ergodic, invertible processes with Gaussian residuals have a solution (Papoulis, 1965). Even in the case where the order of the system is not determined one can derive a set of extended Yule-Walker equations by adding extra equations estimated from the data. This results in an over-determined system that can be solved in a least squares sense. In the absence of other constraints one has to solve the conditional expectation $E[X_{t+1} | X_t, ..., X_{t-n}]$. If the joint distributions are Gaussian, then the conditional expectation is linear. Kolmogorov and Wiener derived linear estimations to solve the above problem in the 1940s. The disadvantage of their method is that it needs to recalculate the values of $X_{t+m}$ for every time-step. Kalman and Box & Jenkins (1976) provide recursive algorithms to solve this problem that reduce the amount of memory needed and saves time. The estimation of the Kalman filter does not coincide with that of the conditional expectation for non-Gaussian errors. On the positive side though, the Kalman filter also works for time-dependent matrices.

In this work, we solved the Yule-Walker system of equations for the gain solutions obtained during a LOFAR observation of a field centered around 3C 196. The duration of the observation was $3 \times 8$ hours and the bright sources in the field provided enough signal-to-noise for accurate calibration. As we already mentioned, the order of the model is a-priori unknown without using external information. Thus, we need an objective method to assess the number of model parameters required. One way to select the order
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of the model is to solve for different orders of models and use a statistical information criterion. In the case where several observation epochs are available, the use of Akaike's Final Prediction Error (FPE) \cite{Ljung} is justified. According to this criterion the most probable model has the lowest FPE defined as:

$$FPE = \left(1 + \frac{d}{N}\right) \frac{1}{1 - \frac{d}{N}} \sum (y' - y)^2$$

where \(N\) is the number of data values, \(d = p + q\) is the total number of parameters of the ARMA\((p,q)\) model and the second factor is residual sum of squares between the different models \(y\) and \(y'\). We find that for our measured gain parameters we achieve the minimum of the FPE using a model of order \((15,14)\). In Figure 3.4 this is demonstrated: The error of the model reaches a minimum and then starts increasing with increasing order. The robustness of the ARMA model is evident, as it reaches an optimal solution fast and for a smaller order of the model. The Cramér-Rao bound of the ARMA model is given by \(\sigma^2_1 \left(1 + \frac{d}{N}\right)\), where \(\sigma^2_1\) is the innovation variance (the variance of the model minus the 1-step prediction). For an \(N\) of 150 samples which corresponds to 4500 seconds, since the averaging time of our observation is 30 seconds, we see that we are close to the Cramér-Rao bound and we have optimal efficiency. This result is robust against different stations as well as different observation nights. To justify the lack of need for using more complex statistical models (e.g. models whose parameters also change with time) than the static ARMA model we used the Likelihood Ratio test to assess whether such a complication would be necessary. We do not find statistically significant evidence for increasing the complexity of the model based on the data from this observation, however we need to test the model selection against more LOFAR data, when they become available.

Another application of the ARMA modeling is forward in time predictions. We can use the derived models to predict the values of the solutions a few steps ahead in time. Figure 3.5 shows the true and predicted values of the gain coefficient for 5, 10, 20 and 40 steps ahead in time which correspond to 2.5, 5, 10 and 20 minutes ahead in time. We see that we can predict quite accurately the value of the next solution during the next 2.5 to 5 minutes which means that there are no rapid temporal variations on the solved parameters. This information can be used during the calibration process as improved initial values for the optimization problem. Starting from an initial value close to the extremum, the algorithm will converge faster and closer to the true minimum. It is really encouraging to see this behavior in observed data, as it means that the stochastic part of the instrumental variation is finite and that the instrument demonstrates stability to relatively long time-scales.

ARMA models and their generalizations provide a robust tool for modeling the errors on gains and possibly the ionospheric parameters, if they are ported to the spatio-temporal domain. The models are generic enough to include most of the cases of interest i.e. the case of white Gaussian noise is the zeroth order model. Under certain conditions there exists an optimal estimator for the parameters of the model. The usage of such models is two-fold: they can be used for the estimation of the parameters of the models and thus they help in the understanding of the processes and they can also be used for forward predictions using efficient techniques like Kalman filters. Analyzing the time-
series of the observed gain solutions helps us study properties of the non-linear interplay between the system and the calibration process that would otherwise be impossible to study. We show a typical realization of an ARMA model used in simulating gains of a LOFAR station in Fig. 3.1. Due to rapid phase variations, the correlations could also be highly decorrelated (thus needing a shorter integration time at the correlator) as seen in Fig. 3.2.

In Figure 3.3 the true model power spectrum is compared with the one obtained by solving the Yule-Walker system of equations.

### 3.3 Station and dipole beams

Our dipole and beam modeling (E Jones in [Labropoulos et al. (2010)](#)) is described in detail in [Yatawatta (2007)](#). In this section we make a few general remarks about the beam properties. To calculate the High Band Antennas (HBA) electric fields, one needs to calculate the current distributions along the wire for every branch of the bow-tie dipole. The effect of an infinite ground plane is incorporated by using the method of images. For the Y dipoles, which are perpendicular to the X ones, the same equations hold, but one has to use an azimuth angle of $\phi + \pi/2$ instead of $\phi$. The LOFAR HBA dipole elements are organized in tiles of $4 \times 4$ dipoles, and each HBA micro-station uses 24 tiles in a near circular planar configuration. To calculate the station beam-shape one has to take into account the spatial distribution of the elements of each station as well as the orientation of the X and Y dipoles. Let $k_0$ be the wavenumber of the approaching photon. Then each dipole $n$ of the station experiences a geometrical delay $\tau_n = \frac{1}{\omega} k_0^T r_n$, where $\omega$ is the cyclic frequency of the photon and $r_n = (x, y, z)$ is the position vector of the element within the station. We assume that we have a narrow-band system, such that we can efficiently correct for the phase delay by using phase shifts between the dipoles, and we use a delay-and-sum beamformer. A more sophisticated beam-forming scheme, in which the data are weighted in certain ways in order to suppress the sidelobes, is not recommended as it can affect the statistical properties (which are not known beforehand) of the cosmic signal, through the errors in the weighting coefficients.

The fact that the array consists of dipoles and that the source tracking is done by beam-forming instead of steering large dishes has some profound implications (Carozzi & Woan, 2009). During earth-rotation synthesis the array is projected at different angles with respect to the source, and this leads to a distorted polarimetric measurement. The dipoles move in the three-dimensional space sampling all three components of the EM radiation under different projections. For a large FoV the station beam-shape also exhibits polarimetric distortions towards the edges of the FoV and this effect becomes stronger for lower elevation angles. Both effects suggest that source flux and polarization determination have to be done when the sources are close to the zenith, where the errors are as small as possible.

In Figure 3.6 we show the estimated degree of linear polarization for an intrinsically unpolarized background field as a function of elevation for a LOFAR HBA station beam. When the phase center is at the zenith there is no deviation from the initial state, as there is no projection and the station beam coincides with the element beam at that point. As the phase center moves further away from the station zenith the polarization distortion
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Figure 3.4: The normalized accuracy using AR, MA and ARMA models of different order. Note that the ARMA model yields a better fit with a smaller order.
becomes more and more significant. In Figure 3.7 we show the I,Q,U and V maps for a simulated field at $\delta = 52^o$ and for twelve hours of synthesis. The sources were initially unpolarized and the observed polarization is due to the projection of the stations with respect to the source. Close to the horizon only one of the two dipoles can be sensitive to a given direction.

The station beam (see Fig. 3.8) also has time and frequency varying sidelobes. This introduces additional complications due to strong sources seen through the sidelobes.

As a final remark we should mention that even though we are using an analytic model for the HBA bowtie-dipole pattern, our algorithm allows for several free parameters to be changed, namely the position and orientation of each dipole in three dimensions as well as the individual elements behavior (e.g. certain dipoles might fail randomly during the observation). Our beam modeling is currently lacking the inclusions of mutual coupling and cross-talk, and we are actively looking into modeling those effects as well, but for that we need information from forthcoming larger LOFAR data sets.
Figure 3.6: HBA station beam polarization effects due to projection. The degree of linear polarization $m_l$ as a function of the zenith angle. The effect becomes more significant towards the horizon and also on the axes parallel to the dipoles.
3.3 Station and dipole beams

Figure 3.7: A simulated map at $\delta = 50^\circ$. The sources were initially unpolarized and the polarization shown here without corrections, is due to distortion of the station beam.
3.4 Ionospheric modeling

We continue our discussion with the ionospheric modeling. We introduced the $Z$ Jones and $F$ Jones matrices for ionospheric phase delay and Faraday rotation, respectively in Labropoulos et al. (2010). LOFAR is able to probe ionospheric distortions on timescales of a few seconds and phase shifts corresponding to a few centimeters in path length. Understanding the effects of those small scale distortions is crucial in achieving the required high dynamic ranges, sensitivities and angular resolutions. LOFAR has a spatial resolution of $2\, \text{m}$, a temporal resolution of $1\, \text{s}$ and a Total Electron Content accuracy of $10^{-3}$ TECUs ($10^{16}$ electrons per $m^3$). The EoR experiment is set to utilize mostly the core of LOFAR. The station pencil beams of the core stations can overlap at ionosphere altitudes and cross each other, providing information about the 3D structure of the ionosphere as well (e.g. Koopmans 2010).

Our ionospheric modeling consists of two components: (1) two-dimensional large scale traveling ionospheric waves which introduce a phase

$$\phi_{ij} = 2A\sin(k(x_i - x_j)/2)\sin[\omega t - k(x_i + x_j)/2]$$

between station $i$ and $j$. $\omega$ is the cyclic frequency, $A$ is constant and $k$ is the wavenumber. The other component is (2) Kolomogorov turbulence with an inner scale of $2$ metres and an outer scale of 200 kilometres. We ignore ionospheric amplitude scintillations, that occur on relatively small scales but we plan to incorporate them in the future. The large scale wave characteristics are taken from Spoelstra (1996b) (Table III). By adding both components we create the TEC maps for every time step. The time evolution of the TEC is governed by the periods of the Traveling Ionospheric Disturbances (TID) and Gaussian random waves introduced in the turbulent part. The source position error due to ionospheric distortion using previous LOFAR observations is shown in Fig. 3.9.
Figure 3.9: The source position error estimate as a function of the detection SNR and the baseline length. Even for mild ionospheric conditions the source position error due to inaccuracies in the beam will be much larger (see e.g. Condon 1974).

Using the 4D (3-spatial plus time) TEC cubes we calculate the total ionospheric phase and Faraday rotation angle introduced by means of ray-tracing. The length of each ray transversing the ionosphere depends on the TEC values within the volume that it passes through. The ionospheric refraction index is defined as:

\[ n = \left( 1 - 81 \left[ m^3 s^{-2} \right] \frac{n_e}{\nu^2} \right)^{\frac{1}{2}} \]

where \( \nu \) is the frequency of the EM wave in Hz and \( n_e \) the number density of free electrons in \( m^{-3} \). The extra path that an ionospheric wave travels, is:

\[ P(\theta_{zenith}) = \int \sec(\theta_{zenith}) (n - 1) \, dh \]

where \( h \) is the ionospheric layer thickness and \( \theta_{zenith} \) the zenith angle of the sub-ionospheric point of a source at \( s^1 \).

The Faraday rotation angle \( \Omega \) in degrees is given by:

\[ \Omega = \frac{K}{\nu^2} H \cos \eta \cos \theta_{zenith} \cdot N_e \]

\(^1\text{At this point we must acknowledge a technicality. We are outputting data Measurement Set (MS) columns and we must follow the exact values that exist in the populated MS tables. This means that the ionospheric model has to provide data at exactly those Julian times and that the elevations and azimuths used to compute the Ionospheric Piercing Points (IPP) have to also coincide with the ones calculated using the CASA core library. To do that we implemented a small and fast astrometry library in C for use with the ionospheric modeling.} \]
where $K$ is a constant in $m^4 s^{-2} A^{-1}$, $\eta$ is the angle between the magnetic field and the line of sight and $H$ is the magnetic field strength in Ampères per meter. For the geomagnetic field we use the International Geomagnetic Reference field model\textsuperscript{2}. The model provides us with the horizontal, vertical, north and east components of the terrestrial magnetic field and thus we can calculate the value along the line of sight. A typical value of the field along the line of sight is 45 nT. The typical values of our model at 150 MHz are 560 radians in phase fluctuations, 0.7 arcminutes of refraction and 2.9 radians of Faraday rotation.

We partition the ionosphere in $N_l$ co-centric spherical layers. Starting from each source we create a large number of rays that pierce through the ionosphere and using the refractive index at each point we calculate the extra path travelled and the Faraday rotation angle. Since the ionosphere varies at small temporal and spatial scales we need to interpolate the piercing position of the ray at each layer which is a computationally intensive process. Every 10 seconds the 3D TEC distribution is updated in order to compute the next value of the $Z, F$ Jones matrices. Essentially we need to keep the full 4D TEC hypercube in the memory. The available memory size is the limiting factor of the portion of the ionosphere that we can simulate. For the ray-tracing we must assume that the transversal through the ionosphere is instantaneous and that there are no TEC changes within that time-scale. With those assumptions, we can calculate the exact, curved path of each ray.

A common statistic used to describe random fields is the structure function (Spoelstra, 1996a). The structure function is defined as the mean square difference of the function \(\varphi\) values taken at two different points separated by $r$:

$$D_{\varphi,2}(r) \equiv \langle |\varphi(x) - \varphi(x + r)|^2 \rangle$$

Structure functions are more useful than Autocorrelation Functions (ACF) as they remain finite because they treat local fluctuations. In general the ionospheric phase fluctuations are coming from a non-stationary process with a-priori unknown PDFs. Thus, we cannot assume ergodicity on time-scales larger than a few minutes. In figure 3.10 we show the phase structure function of the models as a function of the number of layers used.

### 3.5 Foregrounds and the EoR signal

In Sections 3.2, 3.3 and 3.4 we described the modeling of the LOFAR instrumental parameters and gave estimates on their expected values as we currently understand them. In this section we describe the sky models and the generation of the simulated data cubes using the ChopChop\textsuperscript{3} pipeline (Labropoulos and Yatawatta, 2009), which is a code developed by Labropoulos and Yatawatta to simulate realistic EoR observational data cubes.

The sky model that we use in this simulation consists of (a) discrete non-polarized point sources and (b) diffuse polarized Galactic synchrotron emission, non-polarized EoR signal and Galactic free-free emission. For the discrete point-sources we use the catalogue that was extracted from the WSRT 3C 196 field observation of Bernardi et al.\textsuperscript{2}IGRF 10, http://www.ngdc.noaa.gov/IAGA/vmod/igrf.html

\textsuperscript{3}http://code.google.com/p/chopchop
Figure 3.10: Smoothed spatial phase structure functions of the ionospheric model after de-trending to remove the effect of large-scale TIDs. The blue line shows the values of the East-West direction and the red line for the North-East direction.

The 3C196 field is centered on the bright (75 Jy at 138MHz), radio source 3C196 at RA 08\textsuperscript{h} 13\textsuperscript{m} 36.0623\textsuperscript{s} and Dec +48\textdegree 13’ 2.249”. Using the DUCHAMP\textsuperscript{4} source extractor, 330 sources can be extracted above the confusion limit of that observation. We also include 30000 fainter, unresolved sources using the two-point correlation function 0.02\gamma^{-0.8} used by Jelić et al.\textsuperscript{5} (2008). We assign random spectral indices to the sources in the range $-0.7 \pm 0.15$, assuming a flat PDF.

The list of point sources is used as the input of the MeqTree\textsuperscript{5} visibility prediction script. The visibilities are calculated using the Hamaker–Bregman–Sault measurement equation (Labropoulos et al., 2010):

$$V_{ij} = G_i \left( \sum_n K_i E_i Z_i F_i C_n F_j^\dagger Z_j^\dagger E_j^\dagger K_j^\dagger \right) G_j^\dagger + N_{ij}$$

where $G$, $E$, $Z$ and $F$ stand for the Jones matrices of the complex gains, the station beam pattern, the ionospheric phase and Faraday rotation respectively (See Chapter 2). The matrices that are inside the sum over the sources, are direction dependent effects. The values of the matrices are generated as described in the previous section and are stored as CASA Measurement Equation Parameter (MEP) tables. $K$ stands for the Fourier kernel, $N$ for the noise and $C_n$ stands for the coherency matrix of the $n$-th source:

$$C_n = \begin{pmatrix} I + Q & U + iV \\ U - iV & I - Q \end{pmatrix}_n$$

\textsuperscript{4}http://www.atnf.csiro.au/people/Matthew.Whiting/Duchamp/

\textsuperscript{5}http://www.astron.nl/meqwiki
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The 3C 196 field. The size of the crosses corresponds to the source brightness. Only the sources that are above the confusion limit are shown. The central source is 3C 196 with a flux of 75 Jy at 138 MHz.

Figure 3.11: The 3C 196 field. The size of the crosses corresponds to the source brightness. Only the sources that are above the confusion limit are shown. The central source is 3C 196 with a flux of 75 Jy at 138 MHz.

The indices $i, j$ indicate the LOFAR stations for which the specific correlation is calculated.

The next step is to include the diffuse emission. We use different methods for predicting the visibilities of the diffuse Galactic emission (DGE) and the cosmological signal for reasons that we will explain shortly. For the DGE we need to generate data at the $uvw$ points that are visited during the observation. Using a new methodology for generating the DGE [Jelić et al. 2010], we create maps of the polarized synchrotron and free-free emission for a large FoV of ten square degrees with a resolution of one arc-minute. We then take the 3D Fourier transforms of those maps and interpolate the value of the coherency matrix at each $uvw$ point of our observation using a bi-linear interpolation method. This method takes automatically into account the $w$ term due to the curvature of the sky. The image plane effects are added directly on the images before calculating the Fourier transform using the same values for their parameters as the ones used for the point sources. Each frequency channel is completely independent from the other ones and this yields an embarrassingly parallel computation. The code is programmed using threads, with each thread corresponding to a single channel and can run on shared-memory SMP (Symmetric Multi-Processor) computers as well as on clusters. The predicted components are then written to the relevant column of an CASA Measurement Set.

For the cosmological signal we use a different approach. Currently, cosmological

---

http://casa.nrao.edu
3.5 Foregrounds and the EoR signal

Figure 3.12: The evolution of the EoR power spectrum of the EoR brightness temperature maps as a function of frequency (redshift). Before and after reionization significant fraction of the power is on large scales. During the EoR power is shifted towards scales that correspond to the sizes of the ionization bubbles.

Simulations are limited to approximately 200 Mpc comoving boxes (Thomas et al., 2009). Generating maps for a large FoV that extends outside the HPBW (Half Power Beam Width) of LOFAR is computationally not feasible at the moment. In order to overcome this restriction we use the following remedy: we estimate the 2D power spectrum of the signal using the brightness temperature cube from Thomas et al. (2009). We then fit a fourth-order polynomial to the rotationally averaged 2D power-spectrum in order to get an analytical expression and then generate an isotropic random field following the same power spectrum for every frequency slice. We can extrapolate to larger scales by using the analytical form of the power spectrum. The generated signal has the same rms statistics as a function of frequency as the expected signal. Of course, the phase information of the real signal can be quite different as they are correlated with the matter distribution, but that is not important if one is interested in a statistical detection as in the context of this thesis. Moreover, the EoR signal itself as well as the rest of the diffuse emission is not used for the calibration process. Assuming that the global and local sky models can lead to an accurate enough calibration, we are interested in the detection of excess rms fluctuations in the noise signal as function of frequency. For that goal, the actual properties of the signal are not crucial as long as they are distinguishable in their frequency behavior from the noise properties. We plan, though, to improve our modeling of the signal on large scales in the near future using proper radiative transfer simulations. The power-spectral density of the EoR signal as function of frequency is shown in Figure 3.12. In order to reduce the substantial computational overhead, we do not apply the instrumental corruptions to the cosmological signal. The EoR signal is very weak compared to the noise, the foreground and the point-sources and any error caused on it by those artifacts does not significantly affect the calibration process. The effects of the calibration errors on the EoR signal are discussed in the next chapter.

Finally, we concatenate the tables containing the extragalactic, Galactic and cosmological signal visibilities and add white noise of the order of 0.5 K at 150 MHz that corresponds to a typical LOFAR EoR observation of four hours using ten second averaging and one MHz of bandwidth, scaled down to a number that corresponds to 400 hours of observation. We should note that when adding noise on the uv plane the noise has to follow $n(u, v) = \bar{n}(-u, -v)$. 
Figure 3.13: The phase due to clock/ionospheric delay for a single LOFAR station. The mostly random temporal variation and the strong frequency dependence is clearly seen.

3.6 The standard calibration pipeline

In this section we describe the standard off-line calibration pipeline as it is currently implemented in the MeqTrees software. First, it is important to emphasize the differences of the LOFAR EoR observations compared to a normal LOFAR interferometric observation. The LOFAR EoR KSP observes the same fields multiple times (for about 100 nights). The long repetitive observing enables us to know in great detail the invariants: the discrete sources in the sky and the average beam shape. Moreover, normal interferometric observations require a long time duration to fill the uv plain by earth rotation (e.g. WSRT 12 hours). This is needed for a compact PSF yielding good images. In contrast, the EoR observations are aimed at producing residual visibility/image cubes where all celestial sources have been subtracted (though we also need to have a good uv coverage). The variability of the sources has to be measured as well. Variability can spoil the EoR signal subtraction as it can lead to inaccurate calibration and leave residuals that are stronger than the cosmic signal.

We also intend to exploit known frequency dependencies as much as we can. First, we assume that we accurately the spectral characteristics of certain bright discrete sources in our field. Secondly, the phase due to ionospheric delay, the Faraday rotation as well as the phase due to clock delay will have a strong frequency dependence (see Figure 3.13), but in principle these parameters cannot be disentangled. The LOFAR beams are made from a multitude of narrowband beamformers. However, we will know exactly their frequency dependence as well as the frequency dependence of the element beam. The original data is divided into spectral bands of a few hundred channels each. The time duration of each of these observations is planned to be about six hours. In order to make
the data size manageable in calibration, we divide the whole time duration in smaller parts of about 15 minutes.

The calibration is done in several stages. We refer to the HBS (Hamaker–Bregman–Sault) measurement equation given in Labropoulos et al. (2010)

- We first solve for the delay/gain errors due to the receiver front-end, clock and the ionosphere. Note that because \( Z \) is a scalar, we solve for the product \( G_i Z_i \) together which we call \( \tilde{G}_i \). We also solve simultaneously for the Faraday rotation \( \theta \)

\[
F = \begin{bmatrix}
\cos(\theta) & \sin(\theta) \\
-\sin(\theta) & \cos(\theta)
\end{bmatrix}
\]

- Once we have the solutions for \( \tilde{G}_i \), we correct the visibilities, to get \( \tilde{V}_{ij} = \tilde{G}_i^{-1} V_{ij} \tilde{G}_j^{-\dagger} \). Afterwords, we subtract the strong, discrete sources from the corrected visibilities by solving for a direction dependent gain \( J \) for each of these sources. This yields a residual data cube suitable for further EoR studies.

- We also make snapshot images of the corrected visibilities, before and after subtraction of strong sources. The snapshots enable us to study and update our models for image plane effects such as the beam \( E \). These images are also corrected for ionospheric phase and Faraday rotation. The corrected snapshot images are combined to produce full synthesis images. These full images enable us to accurately estimate the fluxes and positions of even the fainter sources, leading to updates of the local sky model database.

- Once we have updated the sky and beam models, we repeat the whole procedure until convergence.

### 3.7 Results

This section briefly discusses the results of the simulation pipeline, using the 3C 196 field as an example. In Fig. 3.17, we show the dirty (not deconvolved) image of the sky, as seen by LOFAR during a 4 hour synthesis observation. This image was produced without any corruptions. With perfect calibration of the corrupted data, we should recover this image. The polarized diffuse foregrounds simulated for the same field is shown on Fig. 3.15. We show this separately because addition of the strong point sources would make this hardly visible, due to their amplitude.

Finally, in Fig. 3.17 we show the corrupted image, with all instrumental and ionospheric corruptions added. This image includes point sources as well as the polarized diffused foreground plus the EoR signal as the sky model. As can be seen from this image, the phase errors due to instrument and ionosphere smear the sky completely. We shall present calibration the post-processing of this observation and the recovery of the EoR signal in the following chapter (Chapter 4).
Figure 3.14: Simulated image of the 3C 196 field (not deconvolved) at 150 MHz, seen by LOFAR during a 4 hour synthesis observation. There are no instrumental corruptions added.

Figure 3.15: Simulated diffused foregrounds in the 3C 196 field at 150 MHz.
3.7 Results

Figure 3.16: Simulated 3C 196 field image with all instrumental corruptions and diffused foregrounds added.

Figure 3.17: The final map obtained after calibration.
3.7.1 Solution analysis

By using simulations, we can directly compare the original instrumental parameters with the ones recovered after calibration. The combined effect of all the instrumental parameters leads to an effective Jones matrix $J$ per direction. We compare the initial values of this matrix with the ones recovered.

Traditionally, Fourier analysis and correlation are used to analyze the behavior of correlated time-series. Cross-correlation is very sensitive to phase differences but it does not provide information about changes in the frequency content with time. To alleviate those restrictions we use semblance techniques combined with the use of the complex Continuous Wavelet Transform (CWT) ([Mallat, 2008] [Percival & Walden, 2006]).

The cross wavelet transform is defined as $W_{12} = W_1 \bar{W}_2$ ([Torrence & Compo, 1998]), where $W_i$ is the CWT vector of time-series $i$. The semblance $S$ is defined as:

$$S = \cos^n \left( \tan^{-1} \left( \frac{\Im (W_{12})}{\Re (W_{12})} \right) \right),$$

for $n$ an odd integer number. The semblance is essentially a measure of phase correlation. Values close to 1 imply strong positive correlation and values close to -1 imply strong negative correlation.

Figure 3.18 presents the results of the CWT and semblance analysis for the case of good calibration, where the data are corrected to an accuracy level of 0.1 per cent. The original and recovered parameters are highly correlated on large time-scales. On smaller scales the correlation is smaller due to the calibration (solver) noise. In Figure 3.19 the same analysis is applied to a worse calibration scenario where the data are corrected to an accuracy level of one per cent. The correlation is reduced, especially at intermediate scales. Figure 3.20 is the same as Figure 3.18 but with an extra number of 80 more scales shown. Correlation is very high on large scales.

Finally, we compare the first four standardized moments of the difference between the two calibration cases: a case with Jones matrix errors of the order of 0.1 per cent and a case with errors of the order of one per cent. We see that in the worst case the mean of the difference between the original and solved parameters is comparable to data values and is not exactly zero. This is a hint of bias on the calibration solutions, as we would expect the mean to be zero (the mean of the ARMA model is zero on large time-scales and calibration errors (solver noise) should be Gaussian, especially in the large number regime). In both cases we see slight positive skewness. The kurtosis shows a different trend. In the first case the difference is platykurtic (negative excess kurtosis, wide peak, thin tails) while in the latter case it becomes leptokurtic (positive excess kurtosis, sharp peak, fatter tails). In both cases this effect is mild and its hard to deduce its cause.

<table>
<thead>
<tr>
<th>Standardized Moments of Differences</th>
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<tbody>
<tr>
<td><strong>Error Level [%]</strong></td>
</tr>
<tr>
<td><strong>Mean</strong></td>
</tr>
<tr>
<td><strong>Variance</strong></td>
</tr>
<tr>
<td><strong>Skewness</strong></td>
</tr>
<tr>
<td><strong>Kurtosis</strong></td>
</tr>
</tbody>
</table>
Figure 3.18: The original cumulative Jones and its solution via the calibration process described in this section. For each time-series we also show the complex continuous wavelet transform and then we present the semblance. Red color signifies strong correlation and blue color strong negative correlation. As we can also verify visually, the large scale correlation is indeed high.
Simulations of the instrumental response

Figure 3.19: Same as the previous figure but focusing on the first 20 scales.
Figure 3.20: Same as the previous figure but with higher calibration error amplitude.
3.8 Conclusions

In this chapter we have refined the LOFAR EoR data model introduced in [Labropoulos et al. (2010)]. More specifically we introduced a generic statistical model for the complex gains based on the theory of time-series. The model is generic enough to include the different contributions to the gain errors and contains the simple case of Gaussian random errors as a trivial case. Our beam modeling includes beam-forming using a delay-and-sum beamformer as well as polarization distortion due to the beam. To model the ionosphere we use a combination of wedges and 3D turbulent fluctuations in the TEC. We perform proper ray-tracing to take ionospheric the phase and Faraday rotation angle into account.

We also developed a parallel algorithm to predict the visibilities that correspond to wide fields of diffuse emission in a relatively fast way. This is done by computing the 3D Fourier transform of the emission on a fine grid and then interpolating the value on the uvw plane coordinates that are used during the observation.

Using the above models that are statistically as well as physically similar to true components of our data model, we can simulate realistic interferometric observations using LOFAR for the EoR experiment. We use the output of these simulations in the next chapter to study the EoR signal extraction from LOFAR observations.

This is the first detailed simulation and processing of LOFAR EoR data cubes, using the standard EoR calibration pipeline. The realism of the simulations can be further increased by better beam and ionospheric modeling, more realistic sky models and generation of large number of observations in a Monte Carlo sense. We have ignored the effects of RFI and we used a simplified ME for reconstructions. In Chapter 4, we show that if the calibration errors are following a Gaussian distribution, we asymptotically achieve statistical efficiency. Using the above results, EoR detectability seems at least promising, and we currently see no major show-stopper as far as the calibration is concerned. An information theoretic approach to study the efficiency of calibration using Least-Squares and the Expectation maximization algorithm is studied in Kazemi et al., (in prep.). However, we must note that we used an initially accurate LSM for the calibration and that the signal-to-noise ratio was also close the desirable values. The validity of those assumptions will be tested during the Million Sources (MS$^3$) survey that will be performed before the final LOFAR commissioning.

Acknowledgments

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Chapter 4

Optimal data-model inversion

Statistics are like a bikini. What they reveal is suggestive, but what they conceal is vital.

Aaron Levenstein

ABSTRACT
The data model describing the response of the LOFAR telescope to the intensity distribution of the sky is characterized by the non-linearity of the parameters and the large level of noise compared to the desired cosmological signal. The application of the primary calibration needs to take those issues into account. In this chapter, we discuss the implementation of a statistically optimal map-making process and its properties. The basic assumptions of this method are that the noise is Gaussian and independent between the stations and frequency channels and that the dynamic range of the data can been improved significantly during the off-line LOFAR processing. These assumptions match our expectations for the LOFAR EoR experiment.

4.1 Introduction

The main scientific goal of LOFAR is the detection of the redshifted 21-cm signal from the Epoch of Reionization, which is the epoch when the first radiating sources in the Universe formed (Madau et al. 1997b). Despite our lack of direct observations to date of this epoch, except a number of drop-out sources at $z \gtrsim 6.5$ (Oesch et al. 2009) and a gamma-ray burst at $z = 8.2$ (Salvaterra et al. 2009), we have a number of indirect observational constraints. We know from Lyman-$\alpha$ forest data that the universe is highly ionized at redshifts $\lesssim 6$. At higher redshifts the Sloan Digital Sky Survey quasars clearly indicate an increase of the Intergalactic Medium (IGM) neutral fraction, demarcating the tail of the reionization process (Fan et al. 2003, 2006; the amount of neutral hydrogen
at $z \gtrsim 6$ is quite uncertain (e.g. [Mesinger & Dijkstra 2008]). Moreover, the WMAP polarization data show that the optical depth for Thomson scattering $\tau$ of the CMB off the electrons released during reionization is about 0.09 ([Dunkley et al., 2009b]), which puts a strong integral constraint on the EoR process but says very little about the details of the reionization process. This evidence is not enough if one wishes to uncover the answers to key questions about the sources of reionization, its timeline and its effect on subsequent structure formation.

It is widely recognized that the redshifted 21-cm line of the hyperfine transition of neutral hydrogen is the most promising probe of the detailed evolution of the EoR. This line follows, in principle, the reionization process as it evolved in space and time. Unfortunately, the strength of the expected signal is very weak and it is normally masked by bright foregrounds.

The detection of the EoR signal fits marginally within the sensitivity limits of the upcoming generation of low-frequency radio arrays like LOFAR and the MWA. In these experiments, the signal extraction is also hampered by the level of thermal noise plus instrumental and ionospheric distortions. Therefore, the initial aim of the first generation of EoR experiments is to detect the cosmic signal statistically. At the frequency range covered by these instruments, dipoles and phased arrays are sufficient to receive radio waves in the upper VHF band (30-300 MHz) and more complex antenna designs like dishes with feed horns etc. are not necessary.

Phased dipole-array designs, however, pose several challenges that one does not encounter in classical dish-based radio telescopes ([Jeffs et al., 2008], [Maaskant et al., 2006]). The beam of phased arrays varies much more than the beams of large dish antennas ([Bhatnagar et al., 2008]). The variations in the beam pattern are more pronounced and less predictable (i.e. some elements or sub-arrays might fail while observing). This requires much more sophisticated direction-dependent calibration. In addition, the ionospheric scintillation and refraction is much more significant at low-frequencies ([Koopmans, 2010]). The next generation radio arrays also have intrinsically large Fields of View (FOV) and approximating the sky as a plane, thus ignoring its curvature, leads to undesired distortions in the maps, a problem that was addressed in a limited manner during the analysis of interferometric data by current telescopes. Finally, the unique spatial and frequency sampling of the new generation of low-frequency instruments mean that the amount of data generated will exceed the amount of data generated by classical interferometers by two to three orders of magnitude. The time required to analyze the data can in many cases exceed the total integration time by an order of magnitude as well.

One might therefore ask the question: What are the prospects and limitations of EoR imaging or power-spectrum estimation with a phased array? Even if the array is perfectly calibrated the result will be noise-limited at diffraction-limited resolution. The fidelity of the map is difficult to assess even in this simple case, due to the improper sampling of the uv plane. This leads to a number of questions regarding the calibration:

- How can we correct, to the desired level of accuracy, for the systematic errors in the presence of noise?
- How can we achieve this in a computationally efficient way, given the large number of visibilities that the experiment produces.

1[^1]: [http://www.mwatelescope.org](http://www.mwatelescope.org)
4.1 Introduction

- What is the theoretical limit on calibration accuracy (expressed via the information theoretic Cramér-Rao bound) and can we reach it in practice?
- Is the detection of the EoR signal feasible under the various instrumental distortions, incomplete uv-plane sampling and large noise power?

The large amount of data (\(\sim 1-2\) petabytes) and the complex imaging problem lead to certain limitations in the selection of the data analysis methods. In the case of the MWA, where the amount of correlated elements is 512, the correlator output is so high that it is necessary to perform calibration and imaging in real time. In contrast this is not a problem for LOFAR as it utilizes fewer correlated elements and the data can be stored for further processing. This enables us to use iterative methods and revisit the data for a number of iterations.

During the history of synthesis imaging a handful of shortcuts like the w-projection (Bhatnagar et al., 2008), faceted imaging and the CLEAN approach to deconvolution (Högstrom, 1974; Clark, 1980; Voronkov & Wieringa, 2004) have been used to alleviate the computational burden of interferometric data processing. The validity and application of those methods is hard to assess mathematically and their validity is often shown only through practical experience and rules of thumb.

In any case, the properties of the noise have to be understood to a very high level of accuracy. The noise of a map is a combination of thermal noise from the receivers, the sky noise, residual calibration errors and confusion noise. This is the “effective” noise (see also Wijnholds, 2010) and we shall refer to this whenever we mention the term noise, unless stated otherwise explicitly.

In array signal processing, the problem of estimating the parameters of multiple sources emitting signals that are received is addressed. There are several estimators proposed in the literature, but in this chapter we focus on the Maximum Likelihood (ML) estimator. The method was pioneered by R. A. Fisher in 1922. The maximum likelihood estimator (MLE) selects the parameter value which gives the observed data the largest possible probability density in the absence of prior information, although the latter can be easily incorporated to make the analysis fully Bayesian. For small numbers of samples, the bias of maximum likelihood estimators can be substantial, but for fairly weak regularity conditions it can be considered asymptotically optimal (Mackay, 2003). Thus, for large samples the MLE has been shown to achieve the Cramér-Rao lower bound and yield asymptotically efficient parameter estimates (Stoica & Nehorai, 1990). With large numbers of data points (of the order of \(10^9\) contrasted to \(10^6\) estimated parameters), as in the case of the LOFAR EoR KSP, the bias of the method tends to be very small (Central Limit Theorem).

In this chapter, we assess how accurate a regularized ML inversion of the calibrated uv data-set can reconstruct the sky intensity as function of frequency and how accurate the EoR signal can be retrieved from the residual (foreground-subtracted) data-cube.

In general, it is not feasible to estimate the size of the data needed in order to obtain a good enough level of approximation of the likelihood function to a multivariate Gaussian. We assume that errors associated with each are independent, but they have different variances and memory of previous values over time and/or frequency. Many of these problems can be overcome by Markov Chain Monte Carlo (MCMC) or nested sampling of the posterior (Skilling, 2004; Feroz et al., 2009). This process could even be
iterated, but we expect convergence after a couple of iterations, since we start with an already good approximation of the model parameters after reprocessing (Chapter 3). With the addition of extra regularization terms, a deconvolved image can be obtained, from which the foreground and point-sources can be subtracted or filtered, to leave only the EoR signal and the noise.

4.2 Map-making

All the planned EoR detection experiments are characterized by a large volume of data, produced by a new generation of radio arrays. The data requires elaborate analysis, which can be done by following these basic steps: (i) produce maps from the time-ordered visibilities after correcting for the instrumental errors (Chapter 3), (ii) combine the maps into 4/5D (three spatial dimensions plus frequency and polarization) data cubes, (iii) remove the foregrounds and finally (iv) study the residuals with the goal of extraction information about the reionization process. In this chapter, we focus mostly on the map-making step rather than on the primary calibration. After the calibration and provided that it performs in a satisfactory manner, the data model takes the form of a set of linear equations. The traditional approach to solve such a problem is through a brute force, direct matrix inversion. However this is not feasible in the context of modern arrays as the number of data-points is huge (i.e. of the order of $10^9$ visibilities per frequency channel and per station beam for LOFAR) and the complexity of the inversion operation is $O(N^3_{\text{vis}})$. Even if we had the required computing power to perform a direct inversion, the numerical errors of the computation would magnify the noise of the produced maps by a large factor as we will demonstrate in a following section. We can circumvent this in a memory efficient way by an iterative algorithm. In this section we discuss the underlying mathematical principles of map-making and in the subsequent sections we specialize in the case of the LOFAR EoR experiment.

In this section we shall present three flavors of likelihood estimators. The Least Squares estimator (LSE) (i) has minimum variance amongst all linear unbiased estimators of a given parameter and is known as the best linear unbiased estimator (BLUE). If the random variables have a normal distribution, then the Least Squares estimator of that parameter is the Maximum Likelihood estimator, has a normal distribution and is the (ii) MVUE (Minimum Variance Unbiased Estimator). However, this might not always be the case. It is known that when a transformation is applied to one parameter (i.e. square root), then the estimator of that parameter need not be unbiased. This is important because e.g. estimating the real and imaginary part of a parameter is not equivalent to estimating the amplitude and phase. The mean-square error (MSE) of an MVUE estimator $\hat{x}$ is given by $\text{var}(\hat{x}) + \text{bias}(\hat{x})$. A biased estimator can have lower MSE because its variance can be significantly lower. Finally, we discuss the Asymptotic Likelihood estimator (ALE) (iii) which deals with statistical inference as the sample size approaches infinity. EoR experiments plan to increase the total integration time and consequently the number of samples. It is thus important to take the asymptotic properties of parameter estimation into account. The ALE is shown (Plackett, 1950) to be asymptotically Normal, unbiased and have the minimum asymptotic variance.

We shall start with the Best Linear Unbiased estimator. This estimator has by defini-
tion the minimum variance among all unbiased linear estimators estimator of a parameter. We also discuss the minimum variance unbiased estimator

The observed visibilities can be written in the form of the following narrowband model:

$$V_{obs} = A(p) v A^H(p) + \sigma^2_w I, \quad (4.1)$$

where the $2 \times 2$ matrix $A(p) = G(t,f) \prod J_m(p)$ is the ordered product of the Jones matrices of the uv plane ($G$) and image plane ($J_m$) effects (Chapter 2), $I$ is the identity matrix, $v$ are the true underlying visibilities of the sky and $\sigma^2_w$ is the standard deviation of spatially-white noise. It is therefore logical to cast the relationship between the sky brightness distribution (or its Fourier transform) and the observed visibilities as a linear algebra system. This is achieved by using the column-wise Khatri-Rao product ($\otimes$) and we thus get (Boonstra, 2005):

$$\hat{V}_{obs} = \left[ A^T(p) \otimes A(p) \right] \text{vec}(v) + \text{vec}(\sigma^2_w I). \quad (4.2)$$

The optimal map-making is essentially the estimation of the map $v$ given our measured data $\hat{V}_{obs}$ and the properties of the noise. To be statistically optimal, the map choice has to maximize the posterior probability of a deduced set of map parameters which is

$$P(v|\hat{V}_{obs}) = \frac{P(\hat{V}_{obs}|v) P(v)}{P(\hat{V}_{obs})} \quad \text{(or posterior = likelihood \cdot prior \over norm. constant)}.$$  

Without assuming a prior explicitly, $v$ follows a uniform distribution, which means that the ratio $\frac{P(v)}{P(\hat{V}_{obs})}$ is constant. There are four cases that we must distinguish: (i) the noise level is known and Gaussian, (ii) the noise level is unknown and Gaussian, (iii) the noise level is unknown and non-Gaussian and finally (iv) that the noise is known and non-Gaussian. We shall concentrate on the Gaussian noise case with either known or unknown noise level. This is because the amount of data is so large that the central-limit theorem justifies this choice and each visibility can be assumed to have a Gaussian error on its complex and imaginary parts (Thompson et al., 2001). In this case the parameter probability density function (PDF) approaches Gaussianity as well. Moreover, it is also safe to assume that visibilities are independent samples of the sky power spectrum and that there is scale mixing introduced due to the convolution of the visibilities with the image plane effects. For this we ignore the effects of RFI and mutual coupling, which affect short baselines more. We can thus write:

$$P(v|\hat{V}_{obs}) \propto P(\hat{V}_{obs}|v) = \exp\left(-\frac{1}{2} (\hat{V}_{obs} - A(p)v)^T C_N^{-1} (\hat{V}_{obs} - A(p)v) \right), \quad (4.3)$$

which is the likelihood function in the case of Gaussian noise and also the posterior in case of a flat prior on the model parameters. The normalizing constant is given by the inverse of $P(\hat{V}_{obs}) = \sum_k P(\hat{V}_{obs}|v_k) P(v_k)$. This means that the best (minimum variance) unbiased (in the sense of the minimal difference between the expectation of the estimator
and then true parameter being estimated) estimator of \( \mathbf{v} \) is the one that maximizes the above probability function given the noise covariance matrix \( \mathbf{C}_N^{-1} \) and the data model equation. We should remark that this approach implies implicit belief in the data model. However, this might not always be reasonable. For example, the number of the Jones matrices is deduced from our knowledge of the physical processes along the signal path [Hamaker 1999]. Those processes are not always easily identifiable or independent i.e. the number of operational elements affect both the beam-shape and the Point Spread Function (PSF). If the matrix \( \mathbf{A} \) (we drop the explicit dependence of \( \mathbf{A} \) on the calibration parameters \( \mathbf{p} \) from hereon) has full-rank then the best linear unbiased estimator (BLUE) is equivalent to the least-square solution [Feller 1968] and has the form:

\[
\hat{\mathbf{V}}_{\text{BLUE}} = \left( \mathbf{A}^T \mathbf{C}_N^{-1} \mathbf{A} \right)^{-1} \mathbf{A}^T \mathbf{C}_N^{-1}.
\] (4.4)

This is the Gauss-Markov theorem [Plackett 1950]. In the case where the noise between all telescopes is the same, we arrive at the solution of a weighted, deterministic least-squares problem. The inverse of the noise covariance matrix plays the role of a metric in the multi-dimensional parameter space. The second factor of the estimator gives the dirty “map” and the first factor which is enclosed in square brackets, is the deconvolution step [Boonstra 2005]. It can be viewed as an inversion of a beam-forming operation that weighs the dirty map parameters according to the instrumental corruptions. The CLEAN algorithm attempts to solve the deconvolution problem by iteratively removing models of point-sources from the measured data. Its computational expense scales as the number of map elements. However CLEANing yields less optimal solutions (e.g. Starck et al. 2002).

We could incorporate more priors to the inversion process. For example if we assume a Gaussian prior for the underlying data [Zaroubi et al. 1995] then:

\[
\hat{\mathbf{V}}_{\text{BLUE}} = \left[ \mathbf{C}_N^{-1} + \mathbf{A}^T \mathbf{C}_N^{-1} \mathbf{A} \right]^{-1} \mathbf{A}^T \mathbf{C}_N^{-1}.
\] (4.5)

We can get this solution by further Wiener filtering of the Maximum Likelihood solution without priors as well [Egmark & Efstathiou 1996 Bouchet et al. 1998]. Note that the Wiener filtering is related to the Tikhonov regularization for a certain selection of weights [Kitaura & Enßlin 2008].

In light of the above discussion, the BLUE is unsuitable for the solution of noisy, ill-posed conditioned systems\(^2\), such as the one we are dealing with. To remedy this, we assume that the signal is a realization of a Gaussian random vector. What follows can be viewed as a linear least-squares analog of Bayesian estimation. Suppose that both \( \hat{\mathbf{V}}_{\text{obs}} \) and \( \mathbf{v} \) are jointly distributed Gaussian random vectors whose source spectral matrix \( \mathbf{S} \triangleq \mathcal{E} \left[ \mathbf{y}_v \mathbf{y}_v^H \right] \) is unknown and whose components have finite expected squares. \( \mathbf{y}_v \) is the output of the array element for a given frequency \( v \). The log-likelihood function is [Trees 2002]:

\[
\mathcal{L} (\mathbf{p}, \mathbf{v}) \sim -\ln \det \mathbf{V}_{\text{obs}} - \frac{1}{N_{\text{data}}} \sum_{N_{\text{data}}} \mathbf{Q}^T \mathbf{v}_{\text{obs}}^{-1} \mathbf{Q},
\] (4.6)

\(^2\)Even in the presence of a large number of data, rows of the array response vector can be dependent and moreover the difference in magnitude of the data can be very large.
where $Q$ is the measured electric signals of each station. The estimation problem requires to maximize the above equation with respect to the variance of the noise, the source parameters and measured coherencies. The minimization problem is greatly simplified by employing the result of Bohme (1984). The maximizing arguments are presented explicitly in that paper and it is important to note that those expressions reduce the dimensionality of the estimation problem by $n^2 + 1$. The drawback is that this method does not consider Hermitian positive (semi)-definite matrices as it should.

The assumption of the spatially-white Gaussian noise still holds. In this case the Asymptotic Likelihood Estimator (ALE) is obtained by maximization of the above likelihood function (Trees, 2002):

$$
\hat{V}_{ALE} = \left[ A^T (p) A (p) \right]^{-1} A^T \left[ \hat{V}_{obs} - \sigma^2 \mu I \right] A \left[ A^T (p) A (p) \right]^{-1}, 
$$

(4.7)

where $(A^T A)^{-1} A^T$ is the Moore-Penrose pseudo-inverse. If the number of map elements is too large the Moore-Penrose pseudo-inverse is singular and thus a high resolution image of a crowded field cannot be constructed. From the above equation we also derive the AML estimator for the calibration parameters. They have been derived in Trees (2002):

$$
\hat{\sigma}^2 = \frac{1}{m - n} \text{Tr} \left[ \left( I - A A^T A \right) \hat{V}_{obs} \right]_{p=\hat{p}},
$$

$$
\hat{p} = \arg \min_p \left( A \hat{V}_{obs} A^T + \hat{\sigma}^2 I \right)_{\sigma=\hat{\sigma}},
$$

where $\hat{\sigma}$ and $\hat{p}$ are the estimated variance of the noise and parameter vector respectively. Tr denotes the trace of a matrix, $n$ is the number of sources (pixels) and $m$ is the number of data points. If we assume that the noise is zero-mean (not necessarily Gaussian) and that it is independent from the signal, then we obtain the expression of the Minimum Variance linear estimator (MVAR):

$$
\hat{V}_{MVAR} = C_v A^T \left[ A^T (p) C_v A (p) + C^{-1}_N \right]^{-1}.
$$

(4.8)

When the covariance of the signal is non-singular then this is the Maximum a Posteriori (MAP) estimator, which is the Bayesian equivalent of the Wiener filtering approach that we discussed earlier. We reiterate that in the presence of Gaussian errors all the above estimators are equivalent. In this case, we show that the local MSE approaches the Cramér-Rao bound in the asymptotic regime and thus justifies the approach of currently planned EoR detection experiments to increase the number of measured visibilities by long observation runs and large number of baselines. There are many other approaches that can be used for the estimation of ML solutions like the Expectation Maximization and Space Alternating Generalized Expectation-maximization algorithm (Yatawatta et al. (2008), Kazemi et al., in prep.), and the spatial ARMA process (see discussion in Chapter 3) and each of them has different accuracy and convergence properties. However, what we must focus our attention on is that any estimator has to perform satisfactorily and be computationally efficient. In this chapter we concentrate on the accuracy of the estimation. The computational issues are discussed in chapter 5. Geometrically, the signal subspace should be independent from the noise sub-space. In the presence of noise and errors on the calibration parameters this independence is not always achieved. We discuss those issues in the following sections.
Table 4.1: Simulation parameters.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coordinates of the 3C196 field centre (J2000.0)</td>
<td>α = 8h13m36.0, δ = 48°13'03&quot;'</td>
</tr>
<tr>
<td>Galactic coordinates</td>
<td>l ≃ 171°, b ≃ 33°</td>
</tr>
<tr>
<td>Number of spectral bands</td>
<td>128</td>
</tr>
<tr>
<td>Frequency coverage (MHz)</td>
<td>120 - 184</td>
</tr>
<tr>
<td>Width of each band (MHz)</td>
<td>1</td>
</tr>
<tr>
<td>Frequency resolution (MHz)</td>
<td>0.5</td>
</tr>
<tr>
<td>Time resolution (sec)</td>
<td>30</td>
</tr>
<tr>
<td>FoV</td>
<td>0.0</td>
</tr>
<tr>
<td>Noise at 150 MHz (mK)</td>
<td>840 mK</td>
</tr>
<tr>
<td>Obs. duration (hrs)</td>
<td>4</td>
</tr>
</tbody>
</table>

4.3 Sky realizations

The sky realization used in this work is created by using the ChopChop simulation pipeline described in Chapter 3. The cosmological EoR signal is modeled using the 200 Mpc$^2$ squared simulation produced by Thomas & Zaroubi (2008). For the foregrounds, we use a polarized model described in Jelić et al. (2008). Except for the diffuse emission we include 350 resolved and approximately 20,000 unresolved and unpolarized point-sources which contribute to the confusion noise. The data model includes a set of corruptions, namely station gains which are modeled as Autoregressive Moving Average (ARMA) processes whose parameters are estimated from real LOFAR solutions, a 3D ionospheric model with wedges and turbulence, as well as an analytical model for the station beam taking into account the polarization distortion due to projection (see Chapter 3). However, we currently ignore cross-talk and mutual coupling between the elements. Mutual coupling tends to introduce correlations between the receiver noise of two elements and has to be corrected very well at the station level. However, here we are dealing with station correlations and the stations are much further apart than the individual elements. The simulation is described in detail in Chapter 3. A summary of the “mock” LOFAR-EoR observation specifications is given in table 4.1.

4.4 Pixel size, choice of domain and regularization

A linear problem is said to be well-posed when the following two conditions hold: (ia) for every observed datum there is a solution or (ib) the solution is unique and (ii) the solution is stable under perturbations. If any of these conditions do not hold the problem is ill-posed. The condition number is a measure of how well-posed a system is. It is defined as the ratio between the largest and smallest eigenvalues of the inversion matrix:

$$
\kappa = \frac{\lambda_{\text{max}}}{\lambda_{\text{min}}}
$$

(4.9)
Physically, it is a measure of the "magnification" of perturbations due to numerical or systematic errors and thus it is an intrinsic property of the problem even in the case where the noise is insignificant. There are some profound implications that result from this. If the problem is ill-posed, additional constraints are needed in order to obtain a proper solution. There are numerous types of extra constraints that can be applied and depending on the type or the strength with which they are applied, they can give different results. Trying to estimate more parameters than the data allows, only makes the ill-posed problem worse. We shall shortly discuss the method of regularization and then return to the discussion of the issues raised above. Regularization is commonly used to remedy the uninvertibility of matrices in linear systems. That said, there are certain caveats that one must be careful of. If the dynamic range of the image is very large (i.e. there are very bright sources together with very faint ones), the weakest sources can be affected by the artifacts arising from the imperfectly removed dirty beams of the strongest sources (Fig 4.3). We shall demonstrate that shortly.

The regularized solution for the BLUE estimator (all the above mentioned ML estimators are statistically equivalent in the presence of Gaussian errors) that we employ in this Chapter and the next is given by

\[
\hat{V}_{\text{BLUE}} = \left[ A^T C_N^{-1} A + \alpha R^T R \right] A^T C_N^{-1}
\] (4.10)

where \( \alpha > 0 \) is the regularization parameter and the positive-definite matrix \( R^T R \) is the penalty operator. Since \( A \) is non-linear one can replace it with the Fréchet derivative. For a non-quadratic penalty functional, \( R^T R \) can be replaced by the Hessian or another linearization. The standard Tikhonov penalty functional (Tychonoff, 1943), for \( R = I \), is then given by

\[
\frac{1}{2} \| v \|^2
\] (4.11)

When the regularization parameter is small, filtering of the noise is inadequate and the solution is highly oscillatory. On the other hand, when the regularization parameter is large, the noise components are partially filtered out. In that case though, components of the solution are suppressed too. In an image with very high dynamic range this will suppress very bright point-sources, and thus leads to results of questionable fidelity. Hence, if the dynamic range can be improved (e.g. through accurate point-source subtraction), better regularized solutions can be obtained.

There are several ways of selecting the regularization parameter. In the realm of simulations one can apply different methods and conduct a Monte Carlo study based on the simulated data. However, a more systematic approach is necessary when dealing with data. Our analysis is based on the analysis of the regularized solution total error. Under the assumption of a known data-model, we determine the attributes and the asymptotic rate of convergence of the regularized solution. The regularized solution error is given by:

\[
e_{\alpha} = V_{\text{obs},\alpha} - V_{\text{true},\alpha} = e_{\alpha}^{\text{trunc}} + e_{\alpha}^{\text{noise}}
\]

The above definition states that the error between the regularized solution and the true data consists of the truncation error \( e_{\alpha}^{\text{trunc}} \), which describes the loss of information of
Figure 4.1: Foreground maps generated using different regularization methods and resolutions. The top row shows the reconstruction using diffusion operators and the bottom row using Tikhonov functionals. The maps at the left column are at a resolution of $80 \times 80$ pixels and on the right at $160 \times 160$ pixels:
4.4 Pixel size, choice of domain and regularization

Figure 4.2: The same map as Figure 4.1 but for an image size of 240 × 240 pixels and 320 × 320 respectively.

Figure 4.3: The Tikhonov method over-corrects around the position of bright point-sources as it tries to suppress the oscillatory points in the image. In this image the difference between the true map and the map obtain through the ML inversion is shown. There are clear artifacts shown at the positions of point-sources.
Figure 4.4: The choice of the regularization parameter as a function of frequency. We see that the parameter is relatively stable.

the solution due to regularization and the noise amplification error $\epsilon_{\text{noise}}$. Our aim is to choose a parameter such that both errors converge to zero as the noise level (norm of the noise) tends to zero. Using the Singular Value Decomposition (see also Chapter 5) of the regularized solution one can prove that the noise error is less or equal than $\sqrt{\alpha^{-1}} \delta$, where $\delta = \|n\|$ is the error level and $n$ is the noise vector. Similarly the truncation error converges to zero as $\alpha$ tends to zero. This means that we can obtain a map with minimal errors, if we choose $\alpha = \delta^m$, with $0 < m < 2$.

This is an a priori regularization parameter selection rule in the sense that it depends on information which we already know about the solution. An a posteriori selection of a parameter depends only on the data. One such selection method arises from the Morozov discrepancy principle (Anzengruber & Ramlau, 2010), which selects the largest regularization parameter $\alpha$, for which $\|A\alpha - V_{\text{obs}}\| \leq \delta$. In this case $\alpha = \delta / \|V_{\text{obs}}\|$.

The other regularization method that we will consider in this paper is a method that penalizes non-smooth solutions and can be viewed as a generalization of Tikhonov regularization. For very large and ill-conditioned systems such as those arising during the processing of interferometric observations, it is often impractical to implement regularization by filtering as this requires the computation of the decomposition of a very large matrix. However, the Tikhonov solution can be written in a variational form with respect to $v$:

$$v_\alpha = \arg \min \|A\alpha - V_{\text{obs}}\|^2 + \alpha P,$$

(4.12)

where $P \equiv v^T R^T R v$ is a penalty functional. Penalty functionals can also incorporate information about the solution i.e. the non-negativity of intensity in the imaging problem. In this chapter we consider penalty operators $\hat{L}$ of the diffusion type:

$$\hat{L} = -\sum_{i=1}^{2} \frac{\partial}{\partial p_i} \left( \rho \frac{\partial}{\partial p_i} \right),$$

(4.13)
where $\rho$ is the local curvature, so that
\[
P = \langle \tilde{L}v, v \rangle.
\]

Diffusion operators can be viewed as a local averaging operators. They depend on the local, fine-scale geometry of the data and introduce a notion of smoothness on the data. In chapter 5 we will discuss an iterative regularization algorithm for applying diffusion-type regularization, that is a generalization of total variation regularization that avoids the need of line searches which arises in the application of other iterative regularization methods. The benefit of this algorithm is that it is easy to implement on hardware accelerators. However, iterative methods are generally impractical for large-scale problems.

For a more detailed discussion of regularization we refer the reader to the tutorial by Neumaier (1998).

Figures 4.1 and 4.2 show the comparison of the Tikhonov and diffusion regularization methods for final maps for different map resolutions. We see that Tikhonov regularization is particularly harsh on suppressing extreme values. The MLE is performed on the uv-plane and we present the direct Fourier transform of the visibilities (dirty map). To get statistically optimal results the dynamic range of the data has to be 100:1 or larger. In Figure 4.4 we plot the optimal regularization parameter $\alpha$ for each map as a function of frequency.

From the above discussion we see that working on the Fourier (uv) domain is more convenient as the condition number of the map-making matrix (in this case the map refers to the true underlying visibilities). The matrices are smaller and the number of independents measurements is roughly given by the maximum baseline divided by the station diameter, which for the LOFAR EoR experiment is of the order of $80 \times 80$. Those parameters are constrained by $10^9$ visibilities and thus we have an over-constrained
problem. Figure 4.5 shows the change of the condition number of the deconvolution matrix as a function of the number of the map elements for each linear dimension. Each visibility contributes to the value of a single pixel in the image domain as it is described through the deconvolution matrix. The foreground removal performs better and there are no deconvolution artifacts (i.e. side-lobe confusion noise) when it is done in the uv plane, since there is no spatial scale error introduced due to different uv sampling for every frequency (Harker et al. 2009a).

4.5 Results

In this section we discuss the results from the data inversion procedure. The sky realization that we use is a realization of the LOFAR 3C196 field (see Section 4.3). This field is centered on the bright (75 Jy at 138 MHz) quasar 3C 196 and is a very useful field to study for advancing our understanding of LOFAR. It is located at a moderate declination of 48 degrees and is populated by approximately 20 bright point sources of the order of a Jansky as well as hundreds of fainter ones. This field has also been studied extensively by the Westerbork Synthesis Radio telescope (Bernardi et al. 2010), which is one of the most stable interferometers available.

4.5.1 Properties of the noise

We mentioned that the errors affect the separation of the noise and signal subspace. This is particularly important for the calibration step because the successful estimation of the relevant maximum likelihood estimator depends on the ability to separate those two subspaces. For a more thorough discussion we refer the reader to Trees (2002) and Stoica & Nehorai (1990). We also used several assumptions for the noise in constructing the estimators of Section 4.2. It is therefore imperative to discuss the noise properties before continuing. Ideally, the noise is Gaussian and spatially-white and each telescope, as well as each snapshot, should have independent noise. When we refer to noise, we mean the effective noise (Wijnholds, 2010), which is a combination of calibration residuals, the confusion noise and the thermal noise. We shall discuss each of these contributions shortly. The noise of the dirty map is equal to the average thermal noise per baseline. The map-making process involves a deconvolution step which is effectively weighing and re-arranging of the data. This essentially means that the noise on the image is pixel-correlated. Even in the simplified case where all the stations behave in the same manner and thus have identical gains and beams, the noise on the image depends on the row sums of \((\bar{A}^T \bar{A} \odot A^TA)^{-1}\), where \(\odot\) stands for the Hadamard product (Wijnholds, 2010). If this product is not constant (i.e. in the case of non-symmetric uv coverage), the noise is distributed in a more complex way.

The thermal noise is given by the radiometer equation and scales as the inverse of the number of data points, 

\[ \sigma_{\text{img}} = \frac{T_{\text{sys}}}{\sqrt{N_{\text{tel}}(N_{\text{tel}}-1)\Delta\tau BW}}, \]

where \(\Delta\tau\) is the averaging time and \(BW\) is the bandwidth. Increasing any of the parameters in the denominator decreases the noise, at least theoretically. If the number of sources is large enough, at some point the source distribution will occupy the whole image and the map starts to resemble diffuse...
Figure 4.6: A map of the thermal noise at 150 MHz for a field at declination 48 and an integration time of 300 hours.

Figure 4.7: The confusion noise resulting from a crowded field with 60,000 sources with fluxes between 1 and $10^{-6}$ Jansky. Bright sources have been removed to the noise level from the uv data. The map is shown at 150 MHz and corresponds to an observation with the LOFAR core.
emission. There is a maximum number of separable sources, given a certain map resolution, and this sets the classical confusion limit (Condon, 1974). Another complication is that if the number of pixels is large and each pixel is occupied by a point-source, then the problem becomes ill-posed as for each source we need to estimate its flux and position inside the pixel. Since the LOFAR EoR experiment, will use around 40 to 48 High Band Antenna stations during the EoR experiment the maximum number of sources that the LOFAR core can separate is approximately 500. This is because an array with \( N_{tel} \) elements provides \( N_{tel}^2 \) correlations. Each source can be described with four parameters (intensity and position in (l,m,n,) coordinates) relative to the central source and also the noise power needs \( N_{tel} \) parameters (Wijnholds, 2010). Using longer baselines within the Netherlands the number can be increased to 6200 to 8200 sources. The equivalent theoretical limit for the maximum number of sources that the MWA can separate is approximately 250,000! However, these computations need to be adjusted for the redundancy of the array configuration. Redundant baselines translate into linearly dependent measurements and therefore linearly dependent rows in the deconvolution matrix.

The covariance of the calibration residuals on the image is given by:

\[
\text{cov}(\mathbf{v}) = \left( \frac{\partial \mathbf{v}}{\partial \mathbf{p}} \right) \mathbf{C}_p \left( \frac{\partial \mathbf{v}}{\partial \mathbf{p}} \right)^T,
\]

where \( \mathbf{C}_p \) is the covariance of the calibration parameters. We need to estimate the partial derivative with respect to the calibration parameters and this can be done analytically in the case of Gaussian noise. The analytical equations are derived in Wijnholds, 2009:

\[
\begin{align*}
\frac{\partial \mathbf{v}}{\partial \vartheta}^T &= -2D^{-1} \text{Im} \left\{ J^T \odot J G^2 J C N J^T \right\} G^2 \\
\frac{\partial \mathbf{v}}{\partial \sigma}^T &= -D^{-1} \left( J^T J \odot J^T J \right)^{-1} I \\
\frac{\partial \mathbf{v}}{\partial g}^T &= -2D^{-1} \text{Re} \left\{ J^T \odot J G^2 J C N J^T \right\} (2I - G) G,
\end{align*}
\]

where \( \mathbf{G} \) is the uv-plane effect Jones matrix, \( D \equiv [J^T G^2 J] \odot [J^T G^2 J] \) is the deconvolution matrix, \( \vartheta \) are the image-plane array response vector parameters, \( g \) are the uv-plane parameters and \( \sigma \) is the noise variance. The bar above a matrix denotes complex conjugation and \( \odot \) the Hadamard product (element-wise multiplication).

Calibration residual errors follow the beam-forming redistribution and thus the patterns of the dirty map. Moreover, the image covariance decreases with an increase in the number of data points in a way similar to the thermal noise, provided that the estimator is unbiased. In the LOFAR EoR experiment we are going to observe the same windows of the sky for about 100 nights. In that case we will be able to determine the PDF of the parameter vector \( \mathbf{p} \). We know that the PDF of the gains is non-Gaussian and we expect a
similar behavior for the ionospheric phase and the beam error to some extent. In this case we have to calculate numerically the global Cramér-Rao bound that does not depend on the specific measurements. The analytical derivation of such a bound is not possible.

4.5.2 Correlation between the original and the recovered maps

As an initial test we study the correlation of the map obtained via the Fourier transform of the corrected visibilities with the dirty map of the true model that we used for the simulation. In Figure 4.8 we present the correlation coefficient of those maps as a function of frequency. We see that the plot is not trended, which means that the inversion process does not introduce frequency dependent errors. We also see that the correlation coefficient is relatively high ($\sim 0.86$), which means that the inversion performs well. It also has a low change from channel to channel which implies that the foregrounds are probably smooth along the frequency direction. We then compare the correlation between two adjacent channels in the true cube and in the reconstructed cube. A fundamental assumption in the EoR signal extraction method introduced by Jelić et al. (2008); Harker et al. (2009a) is that the foregrounds are strongly correlated in adjacent channels. In Figure 4.9, we show that the residual calibration errors do not affect the validity of this assumption.

In Fig. 4.9, we show the correlation of adjacent frequency channels for the original and reconstructed maps. We also plot the Kullback-Leibler divergence (KLD), which is defined as:

$$D_{KL} = \sum_k I_{\nu}(k) \log \frac{I_{\nu}(k)}{I_{\nu + \Delta \nu}(k)},$$

where $I_{\nu}(k)$ is the intensity histogram of the map for a given frequency $\nu$. The KLD is a measure of the information distance between two models and is a measure of the information lost when assuming that due to strong correlation of the sky signal, adjacent channels carry almost the same information. To calculate the KLD we estimate the bootstrap PDF for each map at a given frequency and we generate 100 maps in Monte Carlo sense that follow that PDF. Both the KLD and the correlation between adjacent channels drop by a small amount in the reconstructed maps and this indicates that the inversion step is leaving some chromatic error on the final maps. However, this error is rather random and does not affect the signal extraction significantly.

4.5.3 EoR signal extraction

The ultimate benchmark of the EoR experiment is set by the ability to extract the cosmological HI signal. In this section we perform a simple extraction of the signal using a polynomial fitting method in the uv-plane (Harker et al., 2009a). To do that we need to select scales on the uv-plane that are sampled for a large fraction of the available observation bandwidth and with high SNR. Figure 4.10 shows the mask that we use for selecting visibilities. We achieve the best fit with a polynomial of 7th or 8th order which is higher than what Harker et al. (2009b) and Jelić et al. (2008) suggest. This is due to the higher effective noise level that is present in the current simulations. In the aforementioned work the authors did not include the effects of calibration residual or the confusion noise and they assumed that the properties of the noise are known to a very high precision. In
Figure 4.8: The correlation coefficient between the recovered and true dirty maps.

our case that does not hold true. The effective noise has different components that are
affected in different ways by the data model and they also have different spatial, tem-
poral and frequency behaviors. One way to quantify the noise properties is to use the
difference of successive narrow-channel data. If the frequency resolution is high enough
we can assume that the astrophysical signal does not change significantly and that the
difference is purely determined by the noise. However, it is unfeasible to generate sim-
ulated data at such a high frequency resolution. To remedy this, we estimate the noise
for each of the 128 channels of the simulation using the true underlying maps. We note
that for the real observations, we will obtain very narrow frequency channels (10 kHz)
from which we can accurately assess the noise level as function of frequency, time and
baseline. We then estimate the PDF of the effective noise using bootstrapping. The result
is shown in Figure 4.11.

4.5.4 Cramér-Rao lower bound and the power-spectrum

In order to demonstrate the potential gain in estimation accuracy of the ML estimator
we need to evaluate numerically the asymptotic covariance. Equation 4.5.1 provides the
derivatives of the observed visibilities. In order to calculate the CRB, which states that the
variance of an unbiased estimator \( \hat{\vartheta} \) is bounded by the inverse of the Fisher information
matrix (FIM) \( F(\vartheta) \), the FIM has to be computed. The elements of the inverse CRB that is
the FIM are given by the Bangs formula (Stoica & Moses, 2005):

\[
F = \left( \frac{\partial \vec{V}_{obs}}{\partial m} \right)^T \left( \vec{V}_{obs} - \hat{V}_{obs} \otimes \hat{V}_{obs} \right) \left( \frac{\partial \vec{V}_{obs}}{\partial m} \right)
\]

\[
m = \begin{bmatrix} g & \theta & \sigma \end{bmatrix}
\]

where \( m \) is the instrumental parameter vector and \( g, \theta \) and \( \sigma \) describe uv-plane, image-
plane effects and noise. The CRB is obtained by applying block-matrix inversion identi-
ties:
Figure 4.9: The correlation coefficient between the adjacent channels of the original (top) and reconstructed (bottom) dirty maps. The correlation coefficient is reduced after reconstruction and the same pattern can be seen for the Kullback-Leibler divergence.
Figure 4.10: The sampling of the uv-plane by the LOFAR core along frequency after 6 hours of synthesis. The top set of figures show the average number of visibilities per uv cell for 8, 16, 32 and 64 MHz of total bandwidth (the instantaneous bandwidth of LOFAR is 48 MHz). We assume that the data are delivered at 0.1 MHz resolution. The color-bar shows the number of visibilities per grid point. The bottom set of figures shows the area in which less than 5% of the data along frequency is lost due to the scaling of the uv coverage with frequency, compared to the total bandwidth. The black points represent regions where the visibilities and their Fourier conjugates occupy the same place, while the grey points represent true visibility measurements. This distinction is made because the Fourier conjugates do not contribute to the SNR.
4.5 Results

Figure 4.11: Signal extraction using polynomials on the uv-plane. In the figure above the rms as a function of frequency is shown. The red dots correspond to the measured rms of the residuals after foreground subtraction. The green line shows the rms of the effective noise measured by differencing adjacent channels. The blue line is the evolution of the rms of the cosmic signal and the magenta line the recovered signal.

\[
\text{CRB} = F_{\theta\theta} - F_{\theta g} F_{gg}^{-1} F_{g\theta} \\
\left( F_{\theta\theta} - F_{\theta g} F_{gg}^{-1} F_{g\theta} \right) \left( F_{\theta\sigma} - F_{\theta g} F_{gg}^{-1} F_{g\sigma} \right)^T \\
F_{\sigma\sigma} - F_{\sigma g} F_{gg}^{-1} F_{g\sigma}
\]

The computation of F seems straight-forward but in practice it is more complicated. The matrix is usually rank-deficient and thus a reparametrization might be needed.

In Figure 4.12 we present the variance of the noise and the CRB as a function of SNR. In the case of interferometry the SNR scales with the number of data points as \(N^{-\frac{1}{2}}\). Thus we can move to the asymptotic convergence regime by either integrating more or increasing the number of stations. We have assumed that we observe for 4 hours per night using 0.5 MHz of bandwidth and 30 seconds of averaging. That means that within each integration time interval we accumulate \(\sim 10^4\) visibilities per frequency. The blue line shows the CRB computed from the parameters used for the simulation of Chapter 3. We then estimated the standard deviation using a varying number of visibilities to generate maps. We see that we approach the CRB for a number of visibilities that approaches \(\sim 10^9\), which corresponds to 400 hours of integration. However, we must raise attention to the following issues: In this work we have ignored RFI and mutual coupling. These effects can add coherent contamination to the observed visibility and that would result in a Fisher information matrix with less than full-rank. In that case the MLE estimator would not be unbiased, although it might exhibit the same asymptotic behavior. The effect is more prominent for dipoles and tile correlation that are relatively close to each other, but for stations that are separated by a large enough distance this should not be a problem. This issue must be addressed through new, more sophisticated simulations. A Monte-Carlo type simulation is also needed in order to calculate the global FIM.
Figure 4.12: Bounds on the standard deviation of the signal power. Thirty seconds of averaging and 0.5 MHz of bandwidth are assumed.

4.6 Conclusions

The extraction of the EoR signal presents two fundamental difficulties: on the one hand the SNR is extremely low, even for the new generation of radio telescopes and on the other hand the problem of imaging is not well posed. Furthermore, data-processing is limited by the computational power available at a given time. The new generations of radio interferometers and especially the SKA are highlighting these issues. In this Chapter we presented a Maximum-Likelihood estimation framework for post-processing the visibility data after calibration and bright source removal have been performed. To stabilize the solution we used Tikhonov and diffusion penalty functionals. A regularization parameter of the order of $10^{-4}$ is needed to stabilize the data, and though it is relatively small it is indeed necessary to producible sensible maps. The advantage of such a method is that it provides the MLE of the visibilities after a matrix inversion. The method can also be formulated in an iterative manner, that in principle requires a small number of iterations. Iterative methods that require very large number of iterations are impractical due to their computational overhead, given the huge data sizes involved. The computational issues related to the inversion are discussed in Chapter 5.

Larger maps require also the processing of much larger data volumes. In the case of the LOFAR EoR experiment, the use of the LOFAR core means that the resolution requirements are moderate. Making maps or corrected uv datasets with 80-120 elements in each linear direction is not an issue. However, higher resolution images becoming less accurate because the condition number of the deconvolution matrix explodes rapidly
with increasing resolution.

In the case of LOFAR Core imaging, the ML method performs well and the problem is still tractable. Unfortunately, that cannot be said for large, wide-field surveys with LOFAR and the SKA. The method performs well when the dynamic range of the data has been reduced to $10^{2-3}$, after the brightest sources have been removed. Thus, accurate calibration is important and one needs to estimate the calibration parameters to an accuracy of $10^{-3}$.

We have also shown that the MLE becomes asymptotically optimal. By extrapolating the points of Fig. 4.12, we can see that integrating $O(10)$ more with LOFAR, one can get optimal estimates. This should in principle be within the grasp of the future SKA EoR experiment.

The next step would be to use and adaptive grids for parts of the image plane with low and high SNR. In any case, working on the uv plane tends to yield more accurate results than when working on the image plane.

Another issue of importance is the use of long-baselines. With LOFAR one can use longer baselines and this has a number of advantages. Long baselines enable us to sample the sky at smaller higher spatial frequencies and thus resolve and remove many more point sources from the image. On top of that, the RFI, the ionosphere and other near-field nuisances are uncorrelated for separated stations. It is still rather unclear to what degrees these effects are affecting the LOFAR data but in any case the presence of longer baselines is a reassuring factor.

4.6.1 Future work.

The next step and forerunner of the LOFAR EoR analysis is a fully fledged simulation generating a mock LOFAR EoR data set corresponding to 100 nights of observing. Longer baselines and far-sidelobe contamination of the maps as well as the effects of RFI and the ionosphere at such scales need to be introduced as well. After generating this data set, it has to be fed blindly into the standard LOFAR-EoR pipeline and processed in the same way as the actual LOFAR observations. This can enable us to test the calibration pipeline, fine-tune its computational performance and test the various signal extraction strategies in a realistic setting. Given our current experience and the results of the previous section we foresee no show-stopper, but this has to be verified by the actual data. From such a data set we can extract the observed statistics of the calibration parameters and compare them with the input ones and moreover estimate the global FIM.

In this paper we used two regularization methods: Tikhonov and diffusion operators. The Tikhonov method requires an accurate selection of the regularization parameter $\alpha$ in order to avoid over-amplification of the noise. It is easy to fold in the uv-plane sampling function of the interferometer in the regularization process. On top of that, we can even use only the spatial scales that are sampled well for most of the frequencies over a finite bandwidth in order to avoid the spatial-frequency scale mixing introduced by a varying uv coverage. If $P_H$ is the projector to the sub-space of spatially band-limited functions (equivalent to selecting a mask (spatial bandpass filter) on the uv-plane) then, this can be incorporated via the constraint

\[(I - P_Hv) = 0.\]
By using the above constraint all the available visibilities are taken into account even if they are redundant.

Based on the above discussion on band-limited regularization a similar remark can be made for the diffusion method. The diffusion method preserves the relationship between smoothness and sparsity in a “Fourier” basis. In this scenario the result depends on both the properties and geometry of the data space as well as the ME. We can use properties of the ME (via eigenvectors, multi-scale basis etc) to modify the data in such a way that the diffusion operator is linear. In the future, we plan to investigate such a regularization problem. The problem can be formulated as a discretized diffusion partial differential equation where the discretization is based on a finite differences method that provides certain filtering properties, well-posedness, mean conservation etc. A different method would be a dual approach based on representations and basis functions.
Chapter 5
Computational Issues

If you were plowing a field, which would you rather use: Two strong oxen or 1024 chickens?

Seymour Cray

ABSTRACT
LOFAR (LOw Frequency ARray) is a new and innovative effort to build a radio-telescope operating at the multi-meter wavelength spectral window. The electric signals from the LOFAR antennas are digitized, transported to a central digital processor, and combined in software in order to map the sky. One of the most exciting applications of LOFAR will be the search for redshifted 21-cm line emission from the Epoch of Reionization (EoR). It is currently believed that the Dark Ages, the period after recombination when the Universe turned neutral, lasted until around the Universe was 400,000 years old. During the EoR, objects started to form in the early universe and they were energetic enough to ionize neutral hydrogen. The precision and accuracy required to achieve this scientific goal, can be essentially translated into accumulating large amounts of data. In this Chapter we review the computational challenges and describe some aspects of the pipeline.

5.1 Data size
One of the most challenging aspects of the LOFAR EoR experiment is the large dynamic range between the different components of the sky signal. Discrete sources can be of the order of $10^{-5}$ Jy/beam while the Galactic diffuse emission as well as the confusion amount to 5 mJy/beam and 3 mJy/beam respectively. The noise in the data is of the order 10 $\mu$Jy/beam, while the desired cosmic signal of the order of 1 $\mu$Jy/beam, where we assumed a synthesized beam resolution of 3 arcminutes at 150 MHz. Even after

\[^{1}\text{We have assumed a PSF of three arcminutes, so that 1} \mu\text{Jy/beam corresponds to 2 Kelvin.}\]
very accurate foreground removal the EoR signal is still buried deep in the noise. To reach statistically detectable EoR signals, a long observation run of at least 400 hours is required.

The LOFAR EoR Key Science Project plans to observe up to five independent windows in the sky in order to support the statistical detection of the cosmic signal. For each window we plan to use three independent station beams and cover a bandwidth of 64 MHz with a resolution of 10 kHz. This yields 6400 channels. At each time step of 10 seconds, $\sim 1200$ full-polarization visibilities will be recorded. The total number of time-steps will be 3000-4000 per day. This will result in a recorded visibility dataset of the order of one to two petabytes, including calibration and flagging meta-data.

After the standard calibration (see introduction) the observed visibilities and ME parameters will be used for the inversion step described in Chapter 4. The numerical complexity of the algorithm is $O(N^3)$, similar to other statistically optimal algorithms (like the maximum a-posteriori and the asymptotic likelihood methods). However, the latter algorithms are more efficient in parallelizing the data processing as they can treat snapshots of data independently and then combine the results. The two main parallelization axes are the frequency axis and the observational window axis. Parallelization over those two is trivial. This means that one has to deal with $10^9$ visibilities per channel, with 192,000 channels for 5 windows and 6 beams.

5.1.1 Background

This immense amount of data is affected by instrumental corruptions, which will be determined, to first order, during the initial processing. This involves finding a good initial solution of the parameters for all instrument and sky effects using a modified SELFCAL loop and a simple model for for the sky (e.g. bright calibrator sources). Solving for the parameters is a highly non-linear process, bound to converge to secondary minima, if not carried out carefully.

The data model can be written as a set of linear equations $v = A(p)s + n$, where $v$ is the observed data vector, $A(p)$ is a sparse matrix describing the instrumental effects, $s$ is the true underlying sky signal and $n$ is a vector representing uncorrelated, spatially white noise. The Maximum Likelihood solution to this problem is (Chapter 4):

$$s_{ML} = \left[A^\dagger(p)C_{noise}^{-1}A(p)\right]^{-1}\left[A^\dagger(p)C_{noise}^{-1}\right]v.$$  

$C_{noise}^{-1}$ stands for the inverse covariance matrix of the noise. Solving this equation is essentially a linear algebra problem, but the solution is non-trivial because $v$ is a vector of $10^9$ double-precision, complex numbers.

5.2 Regularization

The system of linear equations that we described in the previous section cannot be inverted directly. The resulting system matrix is singular for most practical cases and thus regularization has to be used in order to get an approximate solution that is close to the real one. In Chapter 4 we discussed two regularization methods: Tikhonov and diffusion.
operators. Due to the immense data size we are obliged to choose an implementation of each method that requires the smallest number of transfers from the disk to memory and from the host memory to the GPU memory. We will describe those two methods briefly:

5.2.1 Tikhonov

Using a Tikhonov functional the solution becomes:

$$A^T C^{-1}_{\text{noise}} v = \left( A^T C^{-1}_{\text{noise}} A + \alpha I \right) s$$

The matrix can be rewritten in block matrix notation as:

$$\begin{pmatrix} A^T (N^T)^{-1} \sqrt{\alpha} I \\ \sqrt{\alpha} I \end{pmatrix} \delta K = \begin{pmatrix} A^T (N^T)^{-1} \sqrt{\alpha} I \\ 0 \end{pmatrix} v$$

or equivalently:

$$B^T B \delta K = B^T \begin{pmatrix} v \\ 0 \end{pmatrix}$$

The matrix $B$ has the structure:

$$B = \begin{pmatrix} n_{11} a_{11} & n_{11} a_{12} & \cdots & n_{11} a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ n_{mm} a_{m1} & n_{mm} a_{m2} & \cdots & n_{mm} a_{mn} \\ \sqrt{\alpha} & \sqrt{\alpha} & \cdots & \sqrt{\alpha} \end{pmatrix}$$

$B$ is neither completely dense nor sparse. The upper part is dense but the lower part is sparse. We use a column oriented transformation (Householder transformation, Golub & Loan [1996]) to estimate a transformation of the matrix $B$ to an upper triangular matrix. The Gram-Schmidt methods is related to the Householder methods but it is less stable numerically. The first step of the factorization involves the annihilation of the non-zero elements of the first column below the main diagonal. All elements of this sub-matrix are affected at every iteration and the process continues until all elements below the main diagonal have been annihilated. Using the Householder reflections we can factorize the matrix in an $R$ matrix which has the same size as $A$ and into a $Q$ matrix which is square with as many rows as $A$. Then:

$$(H_1, \ldots, H_n) v = Qv = Rs$$

The above algorithm is conceptually simple and dominated by matrix-vector multiplies. However, the number of computations per memory element fetched from global memory is quite low. To further improve performance, an algorithm in which several Householder transforms may be applied in a single operation was sought (Kerr et al., 2009).

The effect of the Tikhonov method on the eigenvalues can be seen in Figure 5.1. Regularization suppresses the lowest eigenvalues and thus the new system has a reduced
condition number. The correct choice of the regularization parameter is crucial as it can affect the cosmic signal in a non-trivial way. In Figure 5.2, we show the eigenspectrum of a simulated LOFAR map at a frequency of 150 MHz. The two red circles mark the inflection points. The first inflection point corresponds to the transition from the astrophysical signal subspace to a subspace representing both the EoR signal and the noise. The second inflection point corresponds to the true noise subspace. Figure 5.3 shows the reconstruction of the map using a different number of eigenmodes. By filtering the large singular values that correspond mostly to the foreground emission, we can filter the foregrounds. The rest of the eigenmodes correspond to the EoR signal and effective noise. In Figure 5.4, we compare the recovery of the rms of the cosmic signal as a function of frequency using the SVD and the polynomial fit method of Jelić et al. (2008). The SVD method gives the correct level of the rms but has a higher error. This is because the EoR signal subspace cannot be completely separated from the subspace of the noise.

**Figure 5.1:** (top) The eigenvalues of the $\mathbf{A}^T \mathbf{A}$ for a simulated LOFAR EoR observation. (bottom) the eigenvalues of above matrix plus the regularization matrix.

**Figure 5.2:** The singular value spectrum of a simulated LOFAR map.
5.2 Regularization

Figure 5.3: Simulated LOFAR map reconstruction using 2, 5, 10 and 16 modes

Figure 5.4: Comparison of the polynomial extraction method of Jelić et al. (2008) with the SVD method.

5.2.2 Iterative regularization for lagged diffusivity

In recent years a new class of partial differential equation-based techniques has emerged in image restoration problems (Vogel & Oman, 1996). Rudin et al. (1992) introduced the popular technique of Total Variation (TV). TV can be posed as a variational problem, resulting to a highly non-linear Euler-Lagrange equation. However, this method is quite unstable and convergence is slow. Vogel & Oman (1996) proposed a linearization tech-
nic technique for this problem, which essentially resolves to the solution of a linear equation at each step. The algorithm is very robust and linearly convergent. Radio astronomical imaging using ML techniques can greatly benefit from such a method because it preserves sharp features, like point sources superimposed on a diffuse background, much better. An outline of the algorithm, adapted for the case of synthesis imaging, is presented here:

\[
\begin{align*}
i &:= 0 \\
\mathbf{s}_0 &:= \text{initial guess} \\
\text{begin fixed point iterations} \\
\hat{L}_i &:= \hat{L}(\mathbf{s}_i); \text{discretized diffusion operator} \\
\mathbf{g}_i &:= \mathbf{A}^T(\mathbf{A}s_i - \mathbf{v}_o + \alpha \hat{L}_i\mathbf{s}_i); \text{gradient} \\
\mathbf{H} &:= \mathbf{A}^T\mathbf{A} + \alpha \hat{L}_i; \text{approximate Hessian} \\
\mathbf{d}_{i+1} &:= -\mathbf{H}^{-1}\mathbf{g}_i; \text{quasi-Newton step} \\
\mathbf{s}_{i+1} &:= \mathbf{s}_i + \mathbf{d}_{i+1}; \text{update solution} \\
\text{increment iteration counter}
\end{align*}
\]

At each iteration a linear diffusion equation is solved to obtain the new iterate, based on the result of the previous step. Notice that a global line-search (like a line-search in many optimizations) is not needed to ensure convergence. Furthermore, from our numerical experiments we have observed that the methods need a very small number of iterations (\(~3\)) and adapts very well to data-parallel implementations, such as those relevant for GPUs.

5.3 Computational burden

The currently, less complex, processing-pipeline requires access to a 1000 CPU cluster for more than a 1 year, and analyzing the results requires a further 10-100 Tflop/s processing power for 1 year to perform ML inversions. However, the linear equations describing the LOFAR data-model lend themselves perfectly to be solved, not on classical CPUs, but on Graphical Processor Units (GPUs). We have implemented basic simulation, inversion and analysis codes on a mini-cluster of 3 NVIDIA-Tesla S870 units and in several tests we obtain GPU/CPU speed up ratios of 30-85 in the relevant linear operations, including I/O, similar to test by other groups.

More specifically, forming \(\mathbf{A}^T\mathbf{C}_{\text{noise}}^{-1}\mathbf{A}\) requires \(10^{15}\) complex multiply-add (CMAD) operations. \(\mathbf{A}^T\mathbf{C}_{\text{noise}}^{-1}\mathbf{v}\) requires \(~2 \times 10^{10}\) CMADs and the solution of the ML equation requires \(~5 \times 10^{14}\) CMADs. Each CMAD requires 4 flops on a GPU. This amounts to \(~60\) Pflop per channel and \(~1.2\) Zetaflop for the whole data set. Our numerical tests have indicated that the total processing time per channel is 7 seconds and if we extrapolate to the total number of channels for the LOFAR EoR KSP, the relevant time is \(~100\) days.
5.4 Calculation accelerators

Commodity GPUs are inexpensive resources for delivering very high computing throughput for certain classes of applications. GPUs are sold primarily as an integrated component in display adapters for desktop personal computers. High-throughput GPUs are primarily aimed for the video game market, but in the last couple of years there is an increased interest in using them for numerical computations. This fact has allowed GPU vendors to exploit micro-architecture parallelism for increased performance without constraint by the application and without requiring much architectural infrastructure to facilitate parallel execution. Simultaneously, GPU execution models have grown fast, in response to the needs of graphics programmers, thereby enabling a wide range of computing tasks. GPU vendors have consequently developed graphics-agnostic programming models such as NVIDIA’s Compute Unified Device Architecture (CUDA) and Open Compute Layer (OpenCL) to facilitate general purpose computing on GPUs. Nevertheless, fully exploiting the peak performance capacity of GPUs has remained a challenge. Algorithms with very high arithmetic intensity, very little need to synchronize between execution paths, and very few scatter operations (collect data from many addresses in memory to process in a single procedure call) typically perform well on GPUs without the need for careful optimization, but many computing tasks do not follow these idealized constraints. The final data processing of the EoR KSP resolves to a numerical linear algebra problem and can thus take advantage of such hardware accelerators.

To solve the equations that results from the ML inversion, described in Chapter 4, we use the QR decomposition method. Several algorithms for fast QR decomposition exhibit a high degree of parallelism, but have low arithmetic intensity and are highly coupled between execution paths, requiring synchronization between elements after small numbers of arithmetic operations. As a result, attempts to exploit GPUs to accelerate QR decomposition have only been moderately successful achieving 4-5x speedup.

We have carried a series of benchmarks using NVIDIA Tesla GPUs and the various LAPACK/BLAS implementation for GPUs.

![Figure 5.5: The CUDA execution model: The host computer invokes a kernel than runs on the GPU. The kernel is executed as a collection of threads running on the GPU. The threads are organized in grids of thread blocks.](image-url)
5.4.1 Benchmarks

The test systems is equipped with an Intel Q6600 Core 2 Quad processor with 4 MB of cache per core. The physical RAM amounts to 8 GB with a bus speed of 667 MHz (DDR2). The disk is a Western Digital WDC WD1600AAJS-0 with 8 MB of cache. We measured the read speed of the disk to be 118 MByte/s at a radial distance of 10% of the maximum radius of the rotating rigid platter of the disk. The IO performance variation with radius is typical for a SATA disk, but we did not go further away than a tenth of a radius in order to have consistent measurements. The S1070 was connected via a PCI Express x16 Gen. 2.0 bus. The OS was Centos Linux 5.3 and the kernel version was 2.6.18-128.1.10.el5 without any patches other than those of the maintainer. We used CUDA v2.3 and the 190.18 driver. We measured the values of the Host-to-Device, Device-to-Host and Device-to-Device memory transfer bandwidth and we found them to be 1322 MByte/s, 1295 MByte/s and 74245 MByte/s consistently.

5.4.2 Test runs

We identify two key operations that we need to perform:

- Sparse vector-matrix multiplications
- Matrix decompositions (and eigenvalue estimation)

For the first type of operations we used a slight modification of the code by N. Bell and M. Garland described in the NVIDIA Technical Report NVR-2008-004 [Bell & Garland, 2008]. We used an input matrix of $1,748,122 \times 62,729$ double-precision elements with 6,804,304 nonzero values. We measured a performance of 6.7 Gflop/s and 14.4 Gflop/s with and without the usage of texture memory respectively. The matrix we chose was a worst case scenario as we expect even sparser matrices from LOFAR data.

For the matrix decompositions we used several codes. More specifically we used the code by Garland, the MAGMA\textsuperscript{2} project code, CULA\textsuperscript{3} tools and a custom written code performing LU factorization using the CUBLAS library.

At the moment the CULA library supports only single precision (unlike MAGMA) and a limited number of LAPACK of functions. We noticed that the U and V matrices in the SVD are different from those in Matlab. We subsequently used the LAPACK code by M. Garland. Finally, we used the MAGMA library. We were especially interested in the implementations of the block algorithm for Cholesky decomposition of positive definite matrices (*POTRF, BLAS lvl. 3). For a matrix of $16000^2$ elements we get a performance of 251 and 64 Gflop/s for single and double precisions floats respectively. The normalized errors are $1.27 \times 10^{-16}$ and $9.43 \times 10^{-8}$. The speedup versus the quad-core Q6600 CPU was 4.1 and 2.0 respectively. We also tested a matrix of $32000^2$ elements in single precision so that the full amount of the device RAM is allocated. The relevant numbers are 234.6 Gflop/s and $1.32 \times 10^{-7}$. The average efficiency for single and double precision compared to the peak performance of the GPU is 35 and 20 per cent respectively.

We also implemented our own version of LU factorization using the CUDA BLAS library.

\textsuperscript{2}http://icl.cs.utk.edu/magma
\textsuperscript{3}http://www.cula tools.com
in order to get hands-on experience on the programming environment. We experienced some issues with name mangling, but we managed to overcome those using the NVIDIA forums. We looked into the following issues as well, but without solid conclusions, given the limited testing time:

- Zero-copy: We actually transfer the data once from the disk to the memory and then perform the aforementioned operations. Zero-copy might prove useful especially if we manage to hide the latency.
- Asynchronous memory transfer
- Use of the multi-processor cache
- OpenMP: Using different GPUs to simultaneously process the real and imaginary part or process different chunks of data concurrently. We would like to look into a hybrid OpenMP/MPI solution in the future where the data are processed on several accelerators within a cluster.

- Portland Group compilers: S. Yatawatta tested the compiler for porting naive C code to run optimized on a GPU. Seems to work somewhat better, but the compiler is work in progress. Later versions should provide double precision, function calls, asynchronous transfer etc.

5.4.3 Conclusions
The results we presented above are just initial estimates. Careful fine-tuning of the kernels has to be done in order to tailor them for our specific needs. More specifically we still need to exploit ways to use the shared memory of the devices in a coalescent way, as well as tune the number of threads and blocks. Due to the large amount of transfers from the disk to memory and the memory to the device careful syncing of the threads after async. transfers from the disk is crucial. Nonetheless, we are quite satisfied with the performance and stability of the test system, reaching 20 per cent of the peak performance per node.
Chapter 6

Conclusions

Our investigations have always contributed more to our amusement than they have to knowledge.

Will Rogers

The Epoch of Reionization demarcates the phase transition of the Universe, during which the hydrogen gas turned from the neutral to the ionized state. The last two decades have seen an increased theoretical effort to understand the interplay between the physical processes that characterize the EoR. Nonetheless, observational evidence is still scarce and indirect. The LOw Frequency ARray (LOFAR) is one of the currently designed instruments with the aim to probe the end of the Dark Ages and cosmic reionization and will reach the final stages of its construction at the end of this year (2010).

LOFAR adds a number of new capabilities that were not available in current instruments, such electronic beam-forming, multiple beams and distributed, as well as centralized signal processing. Moreover, a host of instrumental and environmental issues have to be addressed. At the operational frequencies (10-240 MHz) of LOFAR the ionospheric effects as well as the Galactic foregrounds are strong. The diffuse emission from the Galaxy adds another complication to the imaging problem. Furthermore, the station beams vary rapidly and due to the close proximity of the elements and the receivers effects like mutual coupling and cross-talk have to be accounted for. The EoR experiment plans to observe in the frequency range above 110 MHz, that lies just above the FM band. This frequency range is quite populated by terrestrial and air communication channels due to its low atmospheric absorption. Low-power, broadband RFI is particularly difficult to address. Thus, radio frequency interference mitigation is important. Atomic clocks complicate the efficiency of the system. The effective noise which is the sum of the thermal, receiver noise, confusion noise and calibration noise is also several times stronger than the signal. Finally, the sheer data volume of 1-2 petabytes poses a computational challenge and constrains the selection of a proper algorithm.

Extracting the EoR signal from the corrupted observational dataset requires understanding of the data at a fundamental level. This thesis focuses on two aspects of the
LOFAR EoR Key science project: The first part describes the LOFAR EoR data model and the various components of the detailed LOFAR EoR simulation pipeline that is used to simulate realistic LOFAR EoR observations with all the sky components and instrumental corruptions added. The pipeline is generic enough to accommodate any effect and any telescope configuration. The second part, describes the LOFAR EoR KSP reprocessing step, which is a maximum likelihood inversion of the visibilities after calibration. This will give the final data product of the EoR KSP that will be used to remove the foregrounds and extract the cosmic signal. In the following we summarize the basic conclusions of this thesis and discuss the future prospects.

6.1 Prospects for the EoR Experiment

The success of the LOFAR EoR experiment relies on the detailed understanding of the astrophysical, terrestrial and instrumental distortions of the observed signal. Calibration is the name of the game. Without accurate calibration LOFAR, and for that matter, any of the new generation radio arrays will not be able to achieve better sensitivity than, for example, the Very Large Array at 74 MHz. The foreground emission is several orders of magnitude stronger than the signal and even errors of the order of one per cent or less on the foreground removal (due to calibration, imaging or extraction algorithm) can be comparable with the desired signal strength. In order to understand the observed visibilities and the effects of errors on the signal recovery, the LOFAR EoR team has developed an end-to-end pipeline that creates realistic sky maps with all the sky components included, and then distorts them according to our current understanding of the systematics of LOFAR. Then, the data are processed almost blindly in a manner similar to LOFAR EoR calibration pipeline and finally the signal is extracted from the corrected data.

The pipeline has three main modules: (i) Sky realizations: The cosmic signal is generated with the algorithm described in the thesis work of Rajat Thomas (2009). Diffuse Galactic and discrete extragalactic foregrounds are added using the results of the thesis work of Vibor Jelic (2010). This involves full-polarization realizations of the foreground contaminants. (ii) Systematics: The complex LOFAR instrumental response is simulated (Chapter 2 and 3). This includes ionospheric distortions (this thesis) and RFI (thesis work of A. Offringa, in prep.) (iii) Processing: Finally, the data are calibrated using the standard LOFAR-EoR calibration pipeline. The calibration solutions are used as an input to the maximum likelihood inversion (Chapter 4). The output of this step is used to extract the signal [Jelić et al., 2008; Harker et al., 2009a,c]. A flow chart is shown in Figure 6.1.

In Chapter 2 we introduced the physics-based data model for the LOFAR EoR Key Science Project. We provided a brief description of the physical connections between the Hamaker–Bregman–Sault formalism [Hamaker et al., 1996] and instrumental/propagation parameters. We use the above physical model to generate mock observations in Chapter 3. In that chapter we also introduced a generic statistical ARMA model for the complex gains based on the theory of time-series. The model is generic enough to include the different contributions to the gain errors and contains the simple case of Gaussian random errors as a trivial case. Our beam modeling includes beam-forming using a delay-and-sum beamformer as well as polarization distortion due to the beam. To model the ionosphere we use a combination of wedges and 3D turbulent fluctuations in the TEC.
6.1 Prospects for the EoR Experiment

Using the above models, that are statistically as well as physically similar to true components of our data model, we are able to simulate realistic interferometric observations using LOFAR for the EoR experiment. This is the first detailed simulation and processing of LOFAR EoR data cubes, using the standard EoR calibration pipeline.

In Chapter 4 we describe the regularized maximum likelihood inversion step, that is the base-level approach that the LOFAR EoR KSP will use to produce sky maps. As an input we used the simulations generated in Chapter 3. The method includes all image plane and uv-plane effects and the likelihood maximization translates essentially into a linear algebra problem. Diffuse emission is treated naturally within this framework. The total noise which is the superposition of thermal, confusion and calibration noise was used as the effective noise.

In Chapter 5 we briefly discussed the computational issues associated with the simulation and inversion steps, as implemented on Graphics Processin Units.

6.1.1 Ability of LOFAR to detect the EoR

To determine the theoretical limits and statistical efficiency of the regularized maximum likelihood inversion, we used a Cramér-Rao bound analysis. We have shown (Chapter 4) that the estimator becomes asymptotically efficient in the case of map-making with the LOFAR core and that imaging would require more than ten times better sensitivity. This is within the grasp of the SKA. Based on our current simulations and understanding of the astrophysical and instrumental processes the LOFAR EoR experiment should be capable to detect the EoR signal, provided that the calibration can increase the dynamic range of the data with an error of less than 0.1 per cent.
6.1.2 The next generation calibration algorithms

Before the 1980s the standard paradigm in calibration was to rely on the stability of the instrument. In the following 20 years the SELFCAL algorithm was introduced that iterated between the calibration and imaging steps, but mostly uv-plane effects were considered. This is the second generation of calibration. From 2000 to the present, direction-dependent effects were considered in calibration software like AIPS++/CASA, MeqTrees, BBS etc. This is the third generation of calibration. However, the increasing complexity and sophistication of the new instruments and the demand for more demanding scientific goals requires astronomers to continuously devise new statistical algorithms to process the data. This is the fourth generation of calibration (Jan Noordam, private communication) and leaves room for research in the mathematical, computational and engineering aspects of this problem. The next generation of calibration algorithms will most likely involve new procedures (computational and statistical) to calibrate and analyze the data and is expected to becomes of major importance for the next generations of interferometers such as SKA.

6.2 Suggestions for future work

In this section we highlight several of the most important future research directions:

- LOFAR EoR dry-run: An immediate future goal is to generate a full, single window LOFAR EoR data set that corresponds to the full duration of LOFAR EoR observations. This will be used to study the signal extraction in the presence of day-to-day statistical variation in the noise and instrumental parameters.

- Calculate the global Fisher Information for the given data model using MCMC/nested sampling to assess the level of correlation and degeneracy between calibration parameters (including ionosphere, beam, instrument, etc).

- Study advanced regularization techniques in the deconvolution/imaging of the visibilities that take into account the data model and the observational specification such as spatial and spectral frequency coverage.

- We implicitly assumed the validity of the data model. However, the validity of the ME can be checked against the data in a Bayesian framework and attempts could be made to model the data with other (more simple) data models than those used the generate the data sets.

- Explore the accuracy and numerical complexity of other statistical estimators.

- Simulating new instruments:
  The simulation pipeline is constructed on a modular base. Simulating different instruments requires substituting each module with one relevant for the new instrument. The most important module is the primary beam. Different modules for Focal Plane Arrays (i.e APERTIF), phased arrays (i.e. EMBRACE) can be easily incorporated.
6.2 Suggestions for future work

Figure 6.2: In December 2009 a long observing run was carried out on the extragalactic radio source 3C61.1. For this 60 hour observation a total of 20 LOFAR HBA stations were used, consisting of 16 split core stations and 4 remote stations. For comparison, images of 3C61.1 from other radio telescopes are shown. The VLSS 74 MHz image was made with the Very Large Array (VLA), New Mexico (USA). The VLA 1.5 GHz image is from Leahy and Perley (1991, AJ, 102, 537). The WENSS survey was carried out with the Westerbork Synthesis Radio Telescope located not far away from the central core of LOFAR. (Courtesy of Reinout van Weeren, University of Leiden/ASTRON)

- Square Kilometre Array:
  Whereas in this thesis, we concentrated mostly on the case of the LOFAR EoR KSP, SKA is designed as an array that consists of phased arrays, single dishes and Focal Plane Arrays. The lessons learned from the calibration and data inversion of the LOFAR EoR KSP will be an invaluable asset for similar observations with the SKA and we can thus consider them also as progress made on the SKA calibration.
  
  Our simulation framework can be extended to handle SKA data. Regularized maximum-likelihood techniques can be used as a standard method for SKA core imaging. However, the increased size of the SKA data products might lead to new strategies that revolutionize astronomical data handling.

  Hence, whereas in this thesis a first step was made to simulate, calibrate and invert fully realistic LOFAR EoR data sets, but we expect this work to be far from completed. A continuous learning-process is expected in the coming years, where observations will teach us to improve our data-models and the data-models will further enhance our understanding of the data. This interplay will be critical also in future, with even larger, more complex and ambitious observations of the EoR with the SKA. Exciting times lie ahead as we are experiencing a revolution (as contrasted to evolution) in radio astronomy.
Nederlandse samenvatting

Wanneer je een telescoop buiten zet in een donkere, heldere nacht, kun je er bijna zeker van zijn dat het een groep mensen zal aantrekken die door de lens willen kijken. Onmiddellijk na het bekijken van de hemel zullen ze een overvloed aan vragen stellen. Vragen over een object dat zich al hun hele leven boven zich bevond, maar waarvan zijn schitterende en bizarre oorsprong slechts zichtbaar werd door het kijken door de telescoop. Deze vragen zullen uiteindelijk leiden tot meer fundamentele vragen: vragen waarom wij hier zijn en waar we heen gaan. Het wetenschappelijke gebied dat zich bezighoudt met het verleden en de bestemming van het universum is kosmologie.


De publicatie van Einsteins werk over de theorie van generale relativiteit kan beschouwd worden als het begin van de fysische kosmologie. Een aantal mensen hebben geprobeerd om het universum te beschrijven met deze theorieën en markeerden daarmee het ontstaan van fysische kosmologie. Een lijst aan krachtige gereedschappen en technologieën hebben wetenschappers geholpen om metingen van ongeëvenaarde precisie te produceren over de parameters van het universum. Daarmee brachten zij kosmologie op een gelijk niveau met andere wetenschappen zoals natuur- en scheikunde.

De nieuwe metingen en theoretische vooruitgangen (voornamelijk in de theorie van deeltjesfysica) hebben geholpen om een bolstaand paradigma vast te leggen dat gaat over de vorming en evolutie van structuur in het universum, genaamd het koude donkermatteriemodel (cold dark matter model of CDM). Volgens dit model begon het universum in een begintoestand met oneindige dichtheid en temperatuur, genaamd de oerknal. In zijn eerste fase was het universum homogeen en isotroop gevuld met een enorm hoge energiedichtheid, terwijl het zeer snel uittijdt en afkoelt. De zeer complexe kosmos waarin we leven is het resultaat van de ontwikkeling van zeer vroege, minuscule dichtheidsfluctuaties in een verder isotroop en homogeen universum.
Wat is de herionisatieperiode? Een schematische weergave van de kosmologische geschiedenis (bron: NASA/WMAP science team)

Waarschijnlijk het meest aangrijpende verhaal der astrofysica is de vorming van structuur in ons universum: hoe de buitengewoon complexe objecten die ons hedendaags omringen gegroeid zijn uit een opvallend simpel medium dat ontsprong in de oerknal. De afgelopen decennia hebben een enorme vooruitgang gezien in het ontrafelen van de vele draden in dit verhaal. Een basisparadigma voor structuurvormatie is nu gevormd.

Ongeveer 400.000 jaar na de oerknal begon de dichtheid en temperatuur van het universum significant te dalen, waardoor ionen en elektronen de kans kregen om zich te hercombineren tot voornamelijk waterstof- en heliumatomen. De andere elementen zijn
Het pad van het kosmische herionisatiesignaal.

verwaarloosbaar. Onmiddellijk daarna begonnen fotonen zich te ontkoppelen van de baryonen en het universum werd transparant, waardoor het universum transparant werd en straling zo kon ontsnappen. Dit liet de reliëfstraling genaamd de kosmologische achtergrondstraling (cosmic microwave background, CMB) achter. Deze gebeurtenis bracht het universum in een periode van duisternis, bekend als de “dark ages” van het universum. Deze donkere periode eindigde 400 miljoen jaar later, toen de eerste sterren, zwarte gaten, etc., begonnen te vormen en ioniserende straling begonnen af te geven. Toen het aantal ultravioletstralende objecten voldoende groeide, begon de temperatuur en hoeveelheid geioniseerde gas toe te nemen, waardoor uiteindelijk de neutrale waterstof volledig ioniseerde. Deze periode, waarin het gas van bijna volledig neutraal de overgang maakte naar bijna volledig geioniseerd, staat bekend als de periode der herionisatie (Epoch of Reionization, vanaf hier “EoR”).

De EoR splitste de geschiedenis van het universum in tweeën. Voor de EoR domineerde donkere materie (onzichtbare materie waarvan zijn bestaan is gedetecteerd door gravitationele interactie) de vorming en evolutie van structuur, terwijl baryonische materie slechts een marginale rol speelde. Na de EoR begon de rol van gas prominent te worden op de ontwikkeling van de structuur. Op kleine schalen begon het zelfs te domineren.

De studie naar deze periode raakt vele fundamentele vragen in de kosmologie, de ontwikkeling van sterrenstelsels en het vormen van quasars en metaalarme sterren. Veel theoretische aandacht wordt op het moment besteed aan het begrijpen van de natuurkundige processen die de evolutie van deze periode gestart en gecontroleerd heeft, alsmede aan de onderverdeling van verdere structuurvorming.

Ondanks zijn fundamentele aard bestaan er slechts drie sterke observationele limieten aan de EoR. De CMB-temperatuur en polarisatiegegevens die verkregen zijn door de WMAP-satelliet hebben het mogelijk gemaakt om de totale Thomsonverstrooing te kwantificeren van de eerste CMB-fotonen op tussenliggende vrije electronen die gepro-
duceerd zijn door de EoR in zijn kijklijn. Zij tonen dat de CMB-intensiteit slechts voor negen procent geremd is, wat aangeeft dat het universum 400 miljoen jaar voornamelijk neutraal was en toen ioniseerde. Desalniettemin is de Thomsonverstrooiingmeting een integrale limiet die ons weinig verteld over de bronnen der herionisatie, de duur van de periode, noch hoe het zich heeft verspreid over het hele universum. Een andere limiet is te danken aan specifieke kenmerken in de spectra van verre quasars, die bekend staan als het Lyman-alfawoud. Deze kenmerken, waarvan neutrale waterstof de oorzaak is, geven ons twee belangrijke feiten over herionisatie.

Ten eerste is waterstof in het recente universum sterk geioniseerd: slechts een op 10.000 is neutraal. Ten tweede is de neutrale fractie van waterstof in het verre universum op een roodverschuiving van 6,5 plotseling gestegen, dat wil zeggen, ongeveer 900 miljoen jaar na de oerknal, waarmee het einde van de herionisatie is afgebakend. Ondanks heeft de opgewaardeerde Hubble Space Telescope verre stervormende sterrenstelsels geobserveerd welke zich in de EoR bevinden. Ondanks dat deze gegevens sterke limieten leveren aan de geioniseerde toestand van het universum op een roodverschuiving van 6,5, vertellen ze erg weinig over het intrinsieke herionisatieprocess.

Om waterstof te ioniseren zijn fotonen met een energie van 13,6 elektronvolt of meer nodig: de herionisatie van het universum heeft dus ultraviolette fotonen nodig. Een cruciale vraag is welke natuurlijke bronnen de UV-fotonen geleverd hebben die benodigd zijn om het universe van waterstof te ioniseren en in deze toestand vast te houden. Triviale kandidaten zijn de eerste sterren (zogenaamde derdepopulatiesterren) en (mini)quasars – objecten met een enorm zwarte gat. Verschillende artikelen hebben andere bronnen van herionisatie beschouwd, zoals verdwijnende of zelfoplossende donkermateriedeeltjes of verdwijnende kosmische snaren. Echter, de beperkingen op zulke objecten maken het onwaarschijnlijk dat zij alleen het universe hebben kunnen herioniseren.

Het standaardscenario voor de EoR is eenvoudig. De eerste stralende objecten geioniseerden zijn aangrenzende omgeving, waardoor geioniseerde bellen ontstonden die zich uitbreidden totdat de ionisatie alle fotonen geconsumeerd heeft. Naarmate het aantal objecten stijgt zal ook het aantal bellen stijgen en uiteindelijk de volledige ruimte vullen. Er zijn echter veel details die opgehelderd moeten worden. Wat bestuurde de vorming van de eerste objecten en hoeveel ioniserende straling produceren zij? Hoe hebben de bellen zich uitgedijt naar een intergalactisch medium en wat ioniseerden zij het eerst, dichte of minder dichte regio’s? Om het antwoord op zulke vragen te verkrijgen proberen kosmologen de EoR te simuleren door modellen te combineren. Deze modellen volgen de vorming van de donkermateriehalo’s met stralingstransportmechanismen en de evolutie van geioniseerde bellen.

Drie studies van de 21cm-emissielijn van neutrale waterstof zouden onze beste hoop kunnen zijn om de vorming van structuur in de donkere en de herionisatieperioden te onderzoeken. De sterkte van de 21cm-emissie of -absorptie hangt af van de relatieve bezettingsaantallen van de grondtoestand en de aangeslagen toestand. De 21cm-lijn kan in aangeslagen toestand komen ofwel door botsingen, ofwel door Lyman-alfa excitatie – het zogenaamde Wouthuysen-Field-effect.

De 21cm-linemissie van de EoR is roodverschoven door de kosmische uitdijing naar het metergolflengtebereik. Op een roodverschuiving van negen (550 miljoen jaar na de oerknal) heeft de 21cm-lijn een golflengte van 2,1 meter (wat overeenkomt met een frequentie van ongeveer 140 MHz). Deze eigenschap geeft ons de mogelijkheid om het
herionisatieproces te bestuderen en in kaart te brengen, tijdsnede voor tijdsnede. Het observeren van de 21cm-straling van het diffuse intergalactische medium van voor en tijdens de EoR biedt een grote potentie voor het bestuderen van de materiële verdeling tijdens deze vroege stadia. Het zal ook de oorsprong van de eerste bronnen ontsluieren, evenzo de verzadiging, verdeling en voortgang van de EoR.

Het observeren van de 21cm-straling van de EoR en eerder vereist radiotelescopen, welke op dit moment de vereiste gevoeligheid ontbreken. Dit verandert gelukkig in de komende jaren, door het voltooien van moderne radiotelescopen met het specifieke doel om de 21-cmlijnemissie van de EoR te observeren. Tot deze groep behoren LOFAR\(^1\)(de “Low Frequency Array”), die gebouwd wordt in een aantal Europese landen en geleid wordt door Nederland; de Murchison Widefield Array\(^2\)(MWA), op het moment in de constructiefase in het westen van Australië door instituten uit Australië en de VS; en de internationale Square Kilometer Array\(^3\)(SKA), zonder twijfel het meest ambitieuze project wat gebouwd gaat worden in decennia. LOFAR en MWA zullen het verzamelen van gegevens binnen een jaar starten en zijn er op ingesteld om de spatioele spreiding van neutrale waterstof in het universum over honderden vierkante graden van de hemel in kaart te brengen.

Ieder punt in deze kaarten zal geobserveerd worden op vele verschillende frequentie- of tijdsnedes tussen een roodverschuiving van 11,5 en 6,5, waarin de significante evolutie van de fractie van neutrale waterstof wordt verwacht. Deze telescopen zijn interferometrische verzamelingen van antennes met een grote verzameloppervlakte en zullen de gevoeligheid hebben om zeer zwakke signalen te detecteren die zijn verzonden door het vroege universum. Deze metingen zullen niet gemakkelijk zijn door een groot aantal complicerende factoren. Zo zal de roodverschoven 21cm-emissie door overschaduwende galactische en buitengalactische voorgronden sterk aangepast worden. Het radiosignaal wordt verder sterk veranderd door de ionosfeer van de aarde en de instrumentele respons. Al deze effecten moeten op voorzichtigte wijze worden geneutraliseerd door een superieure ijking van de telescoop.

Deze nieuwe telescopen zijn op fase gebrachte verzamelingen van eenvoudige antennes, in plaats van de traditionele parabolische schotelantennes. De ontvangende elementen zijn dipolen met een intrinsiek groot gezichtsveld, waardoor golven van vele richtingen gelijktijdig ontvangen worden. Een signaal vanuit een gegeven richting arriveert bij de verschillende ontvangers met een kleine vertraging. De signalen kunnen met behulp van deze vertraging op een bepaalde manier gecombineerd worden waardoor enkel de signalen van een bepaalde richting maximaal versterkt worden. Verder kunnen signalen van verschillende richtingen tegelijkertijd geobserveerd worden met een hoge richtingsgevoeligheid. Dit proefschrift bestudeert de moeilijkheden in de observatie van het EoR-signaal die optreden door de instrumentele invloed van de telescoop.

Zeer sterke bronnen aan de hemel kunnen worden gebruikt als startmodel om de geobserveerde gegevens te corrigeren. De gegevens zullen gecorrigeerd moeten worden voor de effecten van de ionosfeer, de richtingsgevoeligheid van de antenne en de effecten van de elektronische ontvangers. Al deze effecten kunnen beschreven worden met een vergelijking genaamd Hamaker-Bregman-Sault (HBS) meetvergelijking (measurement

\(^1\)http://www.lofar.org  
\(^2\)http://www.mwatelescope.org  
\(^3\)http://www.skatelescope.org
equation, ME). Gebruikmakend van verscheidene modellen voor de aanpassingen en de HBS ME simuleren we de gegevens in een gecontroleerde omgeving en zien we hoe zij de observatie beïnvloeden. Verderijken we de gegevens door gebruik te maken van de standaard kalibratieprocedure van LOFAR om te testen of deze naar tevredenheid werkt en testen we tevens de efficiëntie.

Een belangrijke vraag is hoe goed we de gegevens moeten kalibreren om het EoR-signalen te kunnen extraheren zodat we de genoemde fundamentele astrofysische vraagstukken kunnen beantwoorden. Daarvoor berekenen we de statistisch-optimale oplossing voor het maken van afbeeldingen gebruikmakend van de maximale waarschijnlijkheidsmethode. De oplossing op deze methode is eigenlijk de oplossing van een groot systeem van lineaire vergelijkingen. Ondanks dat het probleem wiskundig haalbaar is, zijn de algoritmen numeriek instabiel en is de hoeveelheid gegevens te groot wanneer de verwerkingstijd beschouwd wordt.

Gelukkigerwijs is er in de laatste jaren een grote vooruitgang geboekt in multicore architecturen en dataparallele algoritmen vullen het gat tussen GPUs en CPUs. Wetenschappelijke problemen als het verwerken van de LOFAR-gegevens vereisen dezelfde numerieke operaties op een grote hoeveelheid gegevens. GPUs kunnen zulke problemen significant sneller berekenen vergeleken met traditionele computers. Dit versnelt het verwerken van de gegevens aanzienlijk.

Ondanks de moeilijkheden zal de nabije toekomst geweldig opwindend zijn voor dit wetenschappelijk gebied, aangezien een observationeel succes een compleet nieuwe deur zal openen in de kosmologie, en haar licht zal laten schijnen op de donkere tijden van het universum en de periode der herionisering. Het zal het observationele gat in onze kennis van het universum dichten dat ligt tussen 400.000 jaar na de oerknal toen hercombinatie ontstond en een miljard jaar later toen het universum volledig geioniseerd was. Om dit doel te bereiken moeten radiointerferometers gekalibreerd worden met ongerekende precisie. Deze dissertatie demonstreert de uitdagingen die gesteld worden door de instrumentele en rekenkundige aspecten van de geplande EoR-experimenten. We staan aan de voet van een spannende periode nu radioastronomie zich uitbreidt met ongekende mogelijkheden en de ambitieuze doelen kunnen we slechts dan halen, wanneer we onze metingen begrijpen tot een fundamenteel niveau.
Περίληψη

Εάν κάποιος τοποθετήσει ένα τηλεσκόπιο σε μια σκοτεινή τοποθεσία κατά τη διάρκεια μιας ξάστερης νύχτας, είναι σχεδόν βέβαιο ότι θα προσελκύσει την προσοχή ενός πλήθους ανθρώπων που θα θέσουν πολλά ερωτήματα για το αντικείμενο που παρατηρούσαν από το τηλεσκόπιο. Αυτές οι ερωτήσεις με μαθηματική βεβαιότητα οδηγούν σε πιο βασικά ερωτήματα του τύπου γιατί είμαστε εδώ και προς τα που οδεύουμε. Το επιστημονικό πεδίο που ασχολείται με το παρελθόν και το έργο του Σύμπαντος ονομάζεται κοσμολογία.

Οι νοηματικές διεργασίες των τεσσάρων τελευταίων αιώνων οδήγησαν στη γέννηση του πεδίου της φυσικής κοσμολογίας. Η κοσμολογία είναι το φυσικό συμπλήρωμα των θετικών επιστημών. Ξεκινά εκεί που οι άλλες επιστήμες εγκαταλείπουν και το πεδίο της είναι πραγματικά διακριτό. Ο επιστήμονας προσδιορίζει τις νομοτελείες φαινομένων που παρατηρεί στον φυσικό κόσμο και διατυπώνει νόμους. Οι νόμοι συντίθενται σε γενικές θεωρίες όπως αυτές του φωτός, της θερμότητας και των ηλεκτρομαγνητικών φαινομένων. Από την άλλη πλευρά, ο κοσμολόγος αναζητά τα αίτια, όχι μιας κατηγορίας φυσικών φαινομένων, αλλά των αντικειμένων που είναι παράδειγμα του Σύμπαντος. Η ετυμολογία της λέξης κοσμολογία υποδηλώνει ακριβώς αυτό: κοσμός + λόγος - γνώση αληθής μετά λόγου για το Σύμπαν.

Ως αρχή της φυσικής κοσμολογίας, μπορεί να θεωρηθεί η δημοσίευση των εργασιών του Albert Einstein στη θεωρία της Γενικής Σχετικότητας. Αρκετοί ερευνητές προσπάθησαν να εξηγήσουν το Σύμπαν με βάση αυτή τη θεωρία και να κάνουν προβλέψεις για αυτό. Πανίσχυρα επιστημονικά όργανα και προχωρημένες τεχνολογίες βοήθησαν να μετρήσουν με απαράμιλλη ακρίβεια τις παραμέτρους του Σύμπαντος, και έτσι η κοσμολογία μετετράπει από φιλοσοφική ενασχόληση σε επιστήμη.

Οι νέες μετρήσεις και θεωρητικές ανακαλύψεις (κυρίως στη θεωρία στοιχειωδών σωματιδίων) οδήγησαν σε μια νέα ενασχόληση της κοσμολογίας με την ζωή και τέλος Θεών. Αρκετοί ερευνητές προσπάθησαν να εξηγήσουν το Σύμπαν με βάση γνώση της διάφορων σκηνών στη θεωρία της Μεγάλης Έκρηξης (Cold Dark Matter model or CDM). Σύμφωνα με την θεωρία της Μεγάλης Έκρηξης, το Σύμπαν προκύπτει αρχικά σε μία κατάσταση άγνωστης, πληθώρας πυκνότητας και θερμότητας. Κατά τη διάρκεια της έκρηξης, αυτή η πυκνότητα και θερμότητα έχει περισυλλήψιμος αντικείμενων, και στη συνέχεια έχει σχηματίσει τον κόσμο.
Ο πολύπλοκος κόσμος στον οποίο ζούμε είναι το προϊόν της μεγέθυνσης μικροσκοπικών, πρωταρχικών διαταραχών πυκνότητας σε ένα κατά τα άλλα ομογενές και ισότροπο Σύμπαν.

Η περισσότερη αναπόφευκτη ερώτηση στην αστροφυσική είναι κατά πάσα πιθανότητα η ερώτηση του σχηματισμού δομών στο Σύμπαν: πώς τα αυξανόμενα πολύπλοκα αντικείμενα που μας περιβάλλουν (πλανήτες, αστέρια, γαλαξίες, σμήνη και υπερσμήνη γαλαξιών) σχηματίστηκαν από ένα ισχυρό μικροαστρικό μεσογαλαξιακό αέριο που προέκυψε μετά την Μεγάλη Έκρηξη; τις δύο τελευταίες δεκαετίες σημειώθηκε μεγάλη πρόοδος στο ξετύλιγμα του νήματος αυτής της ιστορίας και το βασικό μοντέλο για το σχηματισμό δομών έχει πλέον γενική απήχηση στην επιστημονική κοινότητα.

Περίπου 400.000 χρόνια μετά την Μεγάλη Έκρηξη, η πυκνότητα και η θερμοκρασία του Σύμπαν μειώθηκαν αρκετά, ώστε τα ιόντα και τα ηλεκτρόνια να επανασυνδεθούν και να σχηματίσουν άτομα υδρογόνου και ηλίου. Τα υπόλοιπα στοιχεία του περιοδικού πίνακα ήταν σε αμελητέες ποσότητες τότε. Αμέσως μετά, τα φωτόνια διαχωρίστηκαν από τα βαρυόνια και το Σύμπαν γίνεται διαφανές, επιτρέποντας στην ακτινοβολία να διαφύγει. Αυτό το απολίθωμα της ακτινοβολίας της Μεγάλης Έκρηξης που διέφυγε και φτάνει σε εμάς σήμερα, ονομάζεται κοσμική ακτινοβολία υποβάθρου (Cosmic Microwave Background radiation, CMB).

Μετά αυτό το γεγονός το σύμπαν βρέθηκε σε μία κατάσταση σκότους η οποία ονομάζεται κοσμικός Μεσαίωνας. Ο κοσμικός Μεσαίωνας τελείωσε 400 εκατομμύρια χρόνια μετά, όταν σχηματίστηκαν τα πρώτα αστέρια και μελανές οπές, και άρχισαν να εκπέμπουν ιονίζουσα ακτινοβολία. Όταν ο αριθμός αυτών των αντικειμένων έγινε επαρκής, η θερμοκρασία και το ποσοστό ιονισμού του μεσογαλαξιακού αερίου αυξήθηκαν γρήγορα και το μεγαλύτερο μέρος του ουδετέρου υδρογόνου σταδιακά ιονίστηκε. Αυτή η περίοδος κατά την οποία το μεσοαστρικό αέριο μετέβη από την σχεδόν ουδέτερη φάση στη σχεδόν πλήρως ιονισμένη ονομάζεται Περίοδος του Επανιονισμού (Epoch of Reionization, EoR).

Η ΠτΕ ήταν ένα κατακλυσμιαίο γεγονός στην ιστορία του Κόσμου μας. Πριν από αυτήν, η σκοτεινή ύλη κυριαρχούσε στο σχηματισμό και στην εξέλιξη δομών στο Σύμπαν, ενώ η συνήθης ύλη έπαιζε περιθωριακό ρόλο. Μετά την ΠτΕ, ο ρόλος του μεσοαστρικού αερίου στο σχηματισμό δομών έγινε σημαντικός, και σε μικρές κλίμακες κυρίαρχος.

Η μελέτη αυτής της περιόδου άπτεται βασικών ερωτημάτων της κοσμολογίας, του σχηματισμού γαλαξιών, ενεργών γαλαξιακών πυρήνων και φτωχών σε μέταλλα αστέρων. Τα τελευταία χρόνια παρατηρείται μία μεγάλη θεωρητική προσπάθεια για την κατανόηση των φυσικών διεργασιών που προκάλεσαν και οδήγησαν τις της ΠτΕ και των προεκτάσεως τους στο σχηματισμό δομών που ακολούθησε.

Από την άλλη πλευρά υπάρχουν μόνο τρία παρατηρησιακά δεδομένα για την ΠτΕ. Τα δεδομένα θερμοκρασίας και πόλωσης της κοσμικής ακτινοβολίας υποβάθρου (CMB) επιτρέπουν την μέτρηση του ολοκληρωμένου οπτικού βάθους των πρωταρχικών φωτονίων της ΚΑΥ από τα ελεύθερα φωτόνια που προέκυψαν λόγω του ιονισμού του αερίου κατά την ΠτΕ. Αυτά τα δεδομένα υποδηλώνουν ότι η ΚΑΥ έχει αποσβεστεί κατά εννέα τοις εκατό, πράγμα που σημαίνει ότι ο Σύμπαν ήταν 11 χιλιάδες χιλιάδες χρόνια μετά την Μεγάλη Έκρηξη και μετά ιονίστηκε. Παράλληλα αυτά οι μετρήσεις του οπτικού βάθους της σκέδασης των Φωτονίων των Φάσματα Lyman-α δίνουν ενδείξεις για την τελευταία εποχή του κοσμικού Μεσαίωνα.
είναι ιονισμένο σε μεγάλο βαθμό και μόνο ένα μέρος στα 10.000 είναι ουδέτερο. (2) Το ποσοστό ουδέτερου υδρογόνου συζητάται από τις σηματοδοτήσεις, στην ερυθρό μετατόπιση 6,5 (900 εκατομμύρια χρόνια μετά την Μεγάλη Έκρηξη) και αυτό το γεγονός χαράσσει το τέλος της διεργασίας του επανιονισμού. Πρόσφατα, το αναβαθμισμένο διαστημικό τηλεσκόπιο Hubble παρατηρήσει μακρινά γαλαξίες με έντονο σχηματισμό ακτινοβολίας με έντονο σχηματισμό αστέρων και που βρίσκονται στην ΠτΕ. Αυτά τα δεδομένα αποτελούν σημαντικές επιχειρήσεις για τις συχνότητες της διεργασίας του επανιονισμού του ευρύτερου χρόνου. 

Για να ιονίστο το υδρογόνο απαιτούνται ηλεκτρόνια με ενέργεια 13,6 ελεκτρονιοβόλτ ή παραπάνω. Το βασικό ερώτημα είναι ποια αστροφυσικά σώματα παρήγαγαν τα υπεριώδη φωτόνια που ιόνισαν το Σύμπαν και το διατήρησαν σε αυτή την κατάσταση. Οι προφανείς υποψήφιοι είναι τα πρώτα αστέρια (αστέρες πληθυσμού ΙΙΙ) και οι εκπέμπουν ακτινοβολία synchrotron λόγω της μελανής οπής στο κέντρο τους. Μια σειρά από δημοσιεύσεις έχουν υποθέσει πιο εξωτικές πηγές όπως κοσμικές χορδές και αποσυντιθέμενα σωματίδια σκοτεινής ύλης, όμως η μέχρι τώρα γνώση μας υποδηλώνει ότι αυτές οι πηγές δεν θα μπορούσαν να ιονίσουν το Σύμπαν από μόνες τους.

Το βασικό σενάριο για την ΠτΕ είναι απλό. Τα πρώτα αντικείμενα που εκπέμπουν ιονίζουσα ακτινοβολία, ιονίζουν το άμεσο περιβάλλον τους σχηματίζοντας φυσαλίδες ιονισμένου αερίου οι οποίες διαστέλονται και συνενώνονται μέχρι να εξαντληθούν τα διαθέσιμα ουδέτερα άτομα. Καθώς ο αριθμός των πηγών ενισχύεται, τα φυσαλίδια ιονίζονται και συνενώνονται καθώς και το μέγιστο κατάλειμο τους αγνοείται.

Η γραμμή των 21 εκατοστών του ουδέτερου υδρογόνου αποτελεί ίσως τον καλύτερο τρόπο για να αποκτήσουμε πρόσβαση στην ΠτΕ και τον κοσμικό 'Μεσαίωνα'. Η ισχύς αυτής της γραμμής σε εκπομπή ή απορρόφηση εξαρτάται από τον σχετικό αριθμό κατάληψης της βασικής κατάστασης και της διεγερμένης κατάστασης της υπέρλεπτης υφής του ουδέτερου υδρογόνου. Η γραμμή των 21 εκατοστών πριν και κατά τη διάρκεια της ΠτΕ είναι πολλά υποσχόμενη και θα μας επιτρέψει να μελετήσουμε και να παράγουμε εικόνες της διεργασίας του επανιονισμού σε διάφορες χρονικές στιγμές. Η μέτρηση της ακτινοβολίας 21 εκατοστών εντός της περιοχής του ουδέτερου υδρογόνου αποτελεί πολλά υποσχόμενη και θα μας επιτρέψει να μελετήσουμε και να παράγουμε εικόνες της διεργασίας του επανιονισμού σε διάφορες χρονικές στιγμές.
χρόνια μία νέα γενιά από πρωτοποριακά ραδιοτηλεσκόπια θα έχει θεωρητικά την δυνατότητα να παρατηρήσει την ΠτΕ. Μεταξύ αυτών είναι το LOFAR (the LOw Frequency Array)4, το οποίο έχει σταθμούς σε διάφορες ευρωπαϊκές χώρες και κατασκευάστηκε υπό την καθοδήγηση του Ολλανδικού Ινστιτούτου Ραδιοαστρονομίας, το Murchison Widefield Array (MWA)5 που κατασκευάζεται στην Αυστραλία υπό την αιγίδα αυστραλιανών και αμερικανικών ιδρυμάτων και φυσικά το διεθνές Square Kilometer Array (SKA) το οποίο είναι το πιο φιλόδοξο αστρονομικό πρόγραμμα μέχρι στιγμής και θα κατασκευαστεί μέσα στην επόμενη δεκαετία.

Το LOFAR θα αρχίσει να συλλέγει δεδομένα στο τέλος του έτους και θα μελετήσει την κατανομή του ουδετέρου υδρογόνου σε αρκετές εκατοντάδες τετραγωνικές μοίρες στον ουρανό. Κάθε σημείο θα παρατηρηθεί σε πολλές διαφορετικές συχνότητες (ή χρονικές στιγμές, μεταξύ ερυθρομετατοπίσεων 6,5 και 11,5), οπότε και αναμένεται το μεγαλύτερο μέρος της εξέλιξης του ουδετέρου υδρογόνου. Αυτά τα τηλεσκόπια είναι συμβολομετρικές συστοιχείες με μεγάλες συλλεκτικές επιφάνειες και θα είναι αρκετά ευαίσθητα ώστε να ανιχνεύσουν ακτινοβολία από το αρχικό Σύμπαν.

Παρόλα αυτά, οι παρατηρήσεις δεν θα είναι καθόλου εύκολες. Η ακτινοβολία των 21 εκατοστών ταξιδεύει διαμέσου ισχυρών γαλαξιακών και εξωγαλιαξικών εκπομπών ακτινοβολίας. Το ασθενές κοσμικό σήμα παραμορφώνεται από την γήινη ιονόσφαιρα και τα συστηματικά σφάλματα των οργάνων. Αυτά τα σφάλματα πρέπει να διορθωθούν με κατάλληλη βαθμονόμηση των οργάνων.

Τα νέα τηλεσκόπια είναι συστοιχεία απλών στοιχείων (διπολικές κεραίες) και όχι κατασκευασμένα παραβολικά πιάτα. Τα δίπολα έχουν εκ φύσεως μεγάλα οπτικά πεδία και επομένως λαμβάνουν ηλεκτρομαγνητικά κύματα από διαφορετικές κατευθύνσεις ταυτόχρονα. Ένα σήμα από μια δεδομένη κατεύθυνση φτάνει στους δέκτες σε διαφορετικές χρονικές στιγμές. Χρησιμοποιώντας τις διαφορές φάσης, τα σήματα από διαφορετικούς δέκτες μπορούν να συν-διαμορφωθούν με τέτοιο τρόπο, ώστε ολόκληρη η συστοιχία να έχει μεγάλη ευαισθησία σε μία κατεύθυνση και ελάχιστη ή καθόλου ευαισθησία στις υπόλοιπες κατευθύνσεις. Σήματα από διαφορετικές κατεύθυνσεις μπορούν να παρατηρηθούν ταυτόχρονα με μεγάλη ευαισθησία ανά κατεύθυνση. Αυτή η διατριβή μελετά τις παρατηρησιακές προκλήσεις της μελέτης της ΠτΕ, οι οποίες είναι αποτέλεσμα των παραμορφώσεων του οργάνου και της ατμόσφαιρας. Οι πιο λαμπερές ραδιοπηγές στον ουρανό, μπορούν να χρησιμοποιηθούν ως ένα αρχικό μοντέλο ώστε να διορθωθούν τα δεδομένα. Όλα τα φαινόμενα κατά μήκος της διαδρομής της ακτινοβολίας μπορούν να μοντελοποιηθούν με την εξίσωση μέτρησης των Hamaker, Bregman και Sault. Μέσω αυτής εξίσωσης μπορούμε να μελετήσουμε διαφορετικά μοντέλα για κάθε πηγή σφάλματος σε ένα ελεγχόμενο περιβάλλον στον υπολογιστή μας και να δούμε πως αλληλεπιδρούν και πως επηρεάζουν την παρατήρηση. Επιπλέον, μελετάμε την ακρίβεια των τυπικών αλγορίθμων επεξεργασίας των δεδομένων του LOFAR.

Το βασικό ερώτημα είναι σε πιο βαθιά πρέπει να διορθώσουμε τα σφάλματα μέτρησης πριν μπορέσουμε να εξαγάγουμε το κοσμικό σήμα και απαντήσουμε στα θεμελιώδη αστροφυσικά και κοσμολογικά ερωτήματα που ήδη αναφέραμε. Για αυτό το σκοπό αναζητούμε την στατιστικά βέλτιστη λύση στο πρόβλημα του σχηματισμού εικόνων. Η μέθοδος απαιτεί την επίλυση ενός μεγάλου συστήματος γραμμικών εξισώσεων. Μέσω αυτής, μπορούμε να μελετήσουμε πιθανές ερμηνείες των δεδομένων του LOFAR.

4http://www.lofar.org
5http://www.mwatelescope.org
Ευτυχώς τα δύο τελευταία χρόνια οι εξελίξεις στις πολυπύρηνες αρχιτεκτονικές και στους παράλληλους αλγορίθμους οδήγησαν στο γεφύρωμα του χάσματος μεταξύ κλασσικών επεξεργαστών και επεξεργαστών γραφικών. Πολλά επιστημονικά προβλήματα όπως η ανάλυση των δεδομένων του LOFAR απαιτούν την τέλεση της ιδια μαθηματικής πράξης σε ένα μεγάλο όγκο δεδομένων. Οι επεξεργαστές γραφικών μπορούν να χειριστούν τέτοια προβλήματα με δραματική αύξηση της ταχύτητας σε σχέση με απλούς επεξεργαστές και αυτό επιταχύνει σημαντικά την ανάλυση δεδομένων.

Παρόλες τις δυσκολίες, το εγγύς μέλλον προβλέπεται λαμπρό για αυτό το πεδίο καθώς η επιτυχής παρατήρηση της ΠτΕ θα ανοίξει ένα νέο παράθυρο στο Σύμπαν και θα ρίξει φως στον κοσμικό ‘Μεσσαίωνα’ και την ΠτΕ. Έτσι θα μπορέσουμε να γεφύρωσουμε το χάσμα μεταξύ των πρώτων 400.000 χιλιάδων ετών μετά την Μεγαλή Εκρηξή, όταν έγινε η επανασύνδεση ιόντων-ηλεκτρονίων, και ένα δισεκατομμύριο έτη μετά, όταν το Σύμπαν ξανάγινε πλήρως ιονισμένο. Αλλά ο μόνος τρόπος για να επιτυχήσεται αυτό είναι η βαθμονόμηση σε εξαιρετικό βαθμό των νέων ραδιο-συστοιχιών. Αυτή η διατριβή αναδεικνύει τις προκλήσεις οι οποίες προκύπτουν από τα όργανα και τους υπολογιστικούς αλγορίθμους που θα χρησιμοποιηθούν στις προσεχείς παρατηρήσεις της ΠτΕ. Βρισκόμαστε μπροστά σε συναρπαστικές εξελίξεις στην ραδιο-αστρονομία και την κοσμολογία, καθώς το πεδίο μεταμορφώνεται μέσω των νέων παρατηρησιακών δυνατοτήτων, αλλά για να επιτύχουμε το επιθυμητό επιστημονικό αποτέλεσμα πρέπει να κατανοήσουμε τα δεδομένα σε θεμελιακή βάση.
Τι είναι η Περίοδος τους Επανιονισμού· Μια σχηματική απεικόνιση της κοσμολογικής ιστορίας (πηγή: επιστημονική ομάδα NASA/WMAP)
Η διάδρομη του κosμολογικού σήματος από την Περίοδο του Επανιονισμού.
Appendix A

Polarimetric issues

In this appendix we introduce several fundamental concepts in the description of polarized radiation. We feel that they are important to mention and will be used in forthcoming publications to guide us to a better understanding of complex calibration processes in radio interferometry.

A.1 The Electric-Field Vector and Coherency Matrix

The effects of linear passive media on the propagated photons can be represented by linear transformations of the electric field variables. The nature of those effects, the spectral profile of the light and the chromatic and polarizing properties of the medium through which light passes, all affect the degree of mutual coherence. In general coherent interactions can be represented by the Jones calculus (Jones, 1948, 1942), while incoherent interactions of polychromatic light require the Mueller calculus (Barakat, 1963), since the loss of coherence needs more parameters to be described. The two components of the electric field (e.g. those received at two dipoles) can be arranged as the components a $2 \times 1$ complex vector:

$$\mathbf{e}(t) = \begin{pmatrix} E_x(t) \\ E_y(t) e^{i\delta(t)} \end{pmatrix}$$  \hspace{1cm} (A.1)

where $\delta(t)$ is the relative phase. This vector includes all information about the temporal evolution of the electric field. When the parameters have no time dependence this is called the Jones vector. Moreover, the coherency (or polarization or density) matrix of a light beam contains all the information about its polarization state. This Hermitian $2 \times 2$ matrix is defined as

$$\mathbf{C} \equiv \left\langle \mathbf{e}(t) \otimes \mathbf{e}^\dagger(t) \right\rangle = \begin{pmatrix} \left\langle e_1(t) e_1^*(t) \right\rangle & \left\langle e_1(t) e_2^*(t) \right\rangle \\ \left\langle e_2(t) e_1^*(t) \right\rangle & \left\langle e_2(t) e_2^*(t) \right\rangle \end{pmatrix}$$  \hspace{1cm} (A.2)

This is the coherency matrix of the perpendicular dipoles of a single LOFAR HBA antenna. $\otimes$ stands for the Kronecker product and the brackets indicate averaging over time (Boonstra, 2005). The coherency matrix is a correlation matrix whose elements are the
second moments of the signal. Using the ergodic hypothesis the brackets can be considered as ensemble averaging. Due to its statistical nature its eigenvalues ought to be non-negative. The normalized version of this matrix \( \hat{C} = \frac{C}{\text{tr}(C)} \) contains information about the population and coherencies of the polarization states (Fano, 1957). This object is the equivalent of the single brightness point in the scalar version of the theory.

The measurable quantities Stokes \( I, Q, U \) and \( V \) arise as the coefficients of the projection of the coherency matrix onto a set of Hermitian trace-orthogonal matrices, the generators of the unitary SU(2) group plus the identity matrix. Parameters with direct physical meaning can be derived from the corresponding measurable quantities. The Stokes parameters are usually arranged as a \( 4 \times 1 \) vector,

\[
s = \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix}.
\]

An alternative notation is the \( 2 \times 2 \) Stokes matrix:

\[
S = \frac{1}{2} \begin{pmatrix} I + Q & U - iV \\ U + iV & I - Q \end{pmatrix} \equiv C,
\]

which relates the measured coherency matrix quantities to the Stokes parameters (Born & Wolf, 1999).

### A.2 The Jones Formalism

An adequate method to describe a non-depolarizing system is the Jones formalism. It represents the effects on the polarization properties of an EM wave after the interaction with such a system. For passive, pure systems, the electric field components of the light interacting with them is given by the corresponding Jones matrix \( J \),

\[
e' = Je.
\]

As both the initial and final fields can fluctuate, it is useful to describe the properties of partially polarized light with the coherency matrix. Thus,

\[
C' = \langle e' \otimes e'^\dagger \rangle = \langle (Je) \otimes (Je)^\dagger \rangle \\
= \langle Je \otimes e'^\dagger J^\dagger \rangle = J \langle e \otimes e^\dagger \rangle J^\dagger = JCJ^\dagger
\]

As we are dealing with interferometry, the two \( J \) matrices can come from two different telescopes. The effects on the electric field vectors in the coherency matrix \( C' \) can be written as an operation of these Jones matrices on the original unaffected coherency matrix \( C \). As we already mentioned, the coherency matrix can also be written as a four vector with \( c = (\langle e_1 e_1^\dagger \rangle, \langle e_1 e_2^* \rangle, \langle e_2 e_1^* \rangle, \langle e_2 e_2^* \rangle) \). This vector is related to the Stokes vector via

\[
s = Lc \quad \text{with} \quad L = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & -i & i & 0 \end{pmatrix}.
\]
The matrix has the following property \( L^{-1} = \frac{1}{2}L^\dagger \). Using the properties of the Kronecker product we then find, in terms of Stokes parameters, that

\[
 s' = L \left(Je \otimes (Je)^\dagger\right) = Ns
\]

\[
 N = L (J \otimes J) L^{-1}
\]

with

\[
 N_{kl} = \frac{1}{2} \text{tr} \left( \sigma_k J \sigma_l J^\dagger \right).
\]

where \( \sigma_i \) are the Pauli matrices. A Jones matrix can represent a physically realizable state as long as the transmittance condition (gain or intensity transmittance) holds; that is, the ratio of the initial and final intensities must be \( 0 \leq g \leq 1 \). The reciprocity condition describes the effect when the output signal follows the path in the inverse order. For every proper Jones matrix \( e' = J^\dagger e \). This result does not hold when magneto-optic effects are present. In this case the Mueller–Jones matrices have to be used. If a Jones matrix represents a physically realizable state the reciprocal matrix also represents a physical effect.

### A.3 Pauli Matrices

Pauli matrices are well known and have been used for the analysis of partially polarized light \cite{Fano1954}. Their major advantage is that they satisfy a set of properties that significantly reduce the complexity of calculations associated with the intensity. The identity plus the Pauli matrices in two dimensions are defined as

\[
\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}
\]

(A.7)

This set of \( 2 \times 2 \) linearly independent matrices constitute a basis for the vector space of \( 2 \times 2 \) Hermitian matrices over the complex numbers. Summarizing their properties, they are Hermitian and they follow the commutation relations \( [\sigma_i, \sigma_j] = \sigma_i \sigma_j - \sigma_j \sigma_i = i2 \epsilon_{ijk} \sigma_k \) where \( \epsilon_{ijk} \) is the Levi-Civita permutation symbol. These matrices are unitary and traceless except for the identity matrix. The linear expansion of the coherency matrix in this basis is

\[
 C = \frac{1}{2} \sum \text{tr} (C \sigma_i) \sigma_i
\]

(A.8)

with \( s_i = \text{tr} (C \sigma_i) \) being the four Stokes parameters: \( i = 0, 1, 2, 3 \) corresponding to the Stokes \( I, Q, U, V \) and \( V \), respectively.
A.4 The Stokes Vector and Matrix

In the literature the Stokes parameters are usually arranged as a $4 \times 1$ vector, $s = (I, Q, U, V)^T$ (Gil, 2007). An alternative notation would be to introduce a $2 \times 2$ Stokes matrix.

$$s = \frac{1}{2} \begin{pmatrix} I + Q & U - iV \\ U + iV & I - Q \end{pmatrix} \tag{A.9}$$

In this case for pure states we have $\|C\|^2 = \frac{1}{2} \|S\|^2$ and in the case of unpolarized light $\|C\|^2 = \frac{1}{4} \|S\|^2$. This is because for unpolarized light $Q, U$ and $V$ are zero and the matrix becomes $I$ multiplied by the identity matrix. Since there is a factor of $\frac{1}{2}$, the Frobenius norm of the matrix has this factor squared. Even if the source is not polarized one should observe the same signal in both orthogonal polarizations, as in this case both polarization states are equiprobable. This is an important fact for calibration. The X and Y dipoles of the LOFAR HBAs should measure the same signal, for unpolarized sources, and any fluctuations should be due to polarization calibration errors.

A.5 Polarization Level

In addition, the degree of polarization is defined as $P \equiv \sqrt{U^2 + V^2 + Q^2} / I$. We can also define the vector (Gil, 2007)

$$p^T = \frac{1}{P I} \begin{pmatrix} Q \\ U \\ V \end{pmatrix}$$

The Stokes vector can be decomposed using those parameters in several ways. A trivial decomposition is between a polarized and unpolarized state. As radio receivers are intrinsically polarized, and in the case of LOFAR orthogonal, a spectral decomposition can express the Stokes vector as a convex linear sum of two orthogonal states. This makes the relationship between the measured signals and the parameters describing the system more clear, since we have to deal with orthogonal states. The spectral decomposition of the coherency matrix is using its eigenvalue structure to decompose it into pure states, i.e. eigenvectors. The spectral decomposition (diagonalization) of the coherency matrix is then equivalent to

$$s = I \times \left( \frac{1 + P}{2} \begin{pmatrix} 1 \\ p^T \end{pmatrix} + \frac{1 - P}{2} \begin{pmatrix} 1 \\ -p^T \end{pmatrix} \right)$$

Of course, for a mixed state there are infinite combinations of independent states into which it can be decomposed, but it would be useful if the selection is such that it matches the characteristics of the system. The parameters $I$ and $P$ are invariant under unitary transformations (changes of coordinate systems). They are directly related to the eigenvalues of $C$. Pure states are related to rank-1 polarization matrices, and mixed states to rank-2. Wolf showed that there always exist two orthogonal reference directions, such that the degree of coherence is maximized and coincides with $P$. This is important because various random distributions correspond to unpolarized light. The measurement of the correlations of the Stokes parameters allows us to distinguish between those different types of non-polarized light.
A.6 Entropy

Concluding with the quality criteria, the von Neumann entropy can be applied to electromagnetic waves (Fano, 1957). In terms of the coherency matrix it is defined as $S = -\text{tr} (\hat{C} \ln \hat{C})$. It is a measure of the difference in the amount of information between pure and mixed states, both with same intensity. It can be expressed as a function of the eigenvalues of $\hat{C}$ as

$$S = -\frac{1}{2} \left[ (1 + P) \ln (\sqrt{1 + P}) + (1 - P) \ln (\sqrt{1 - P}) \right].$$

It is a decreasing monotonic, bounded function of $P$ (Gil, 2007). It attains its maximum value, $S = \ln(2)$, for non polarized light and its minimum, $S = 0$, for totally polarized light. Using this entropy one can define the polarization temperature. Depolarizing effects, such as the ionospheric Faraday rotation, can be studied this way in order to detect spatial heterogeneity. For example, ionospheric TIDs should lead to an increase in the polarization entropy. This is a good scheme of ranking observations: if the polarization entropy differs between two maps, residual Faraday rotation and leakage can be present. Finally, for non-Gaussian distributions of polarization states, higher order moments are needed and the Stokes system is no longer adequate to describe the polarization states.

The standard (self)-calibration procedure for interferometric observation of polarized light aims at recovering the coherency matrix of the polarized radiation. Given the statistical nature of the coherency matrix, we must emphasize the importance of parameters that give a measurement of their polarimetric purity. Polarization entropy is a concept related to the impurities of the media through which radiation propagates, as it was defined above. It is useful for many purposes, especially when depolarization is a relevant subject. An increase in the polarization entropy signifies a decrease in the polarization purity. It is a direct way to access the quality of the data. Ionospheric Faraday rotation causes a depolarization with a certain frequency behavior. We would expect that the entropy would follow this behavior. In Figure A.1 we show the polarization entropy for several lines of sight as a function of frequency in a simulated map. Attention should be brought to the fact that is really hard to distinguish between Faraday rotation and other depolarization effects.

Figure A.1: Left: the polarization entropy of a simulated Galactic diffuse synchrotron emission map. Right: the same plot after applying a depolarizing effect (Faraday rotation) along the horizontal axis.
A.7 Lorentz Transformations

The transformations of the Stokes vector lead to a system of differential equations. From the statistical interpretation of the coherency matrix we derive that the Stokes $I$ parameter has to be positive and that $s_0 = I \geq s_1^2 + s_2^2 + s_3^2 = Q^2 + U^2 + V^2$, which means that there can be no more polarized light than the total light. This is a key concept in physics, namely the conservation of energy. Extending this remark, we can define a Minkowski space (in the mathematical sense) for the Stokes 4-vectors. The norm in this space would be $\|s\| = I^2 - Q^2 + U^2 + V^2$. Positive values of the norm correspond to the time-like vectors of the special theory of relativity. Light-like Stokes vectors correspond to totally polarized states. The Poincaré sphere defines the light cone with the exception that the symmetric part of the cone does not have any special meaning. The Jones vector transforms as usual under the Lorentz group. The coherency matrix is defined through the Kronecker product of two Jones vectors and is a $2\times2$ spinor of rank two. The Stokes vector and the coherency matrix must be realizations of the same irreducible representation of the Lorentz group, as they both have 4 independent components. The Lorentz transformations form a 6 parameter group. The six generators are the 3 spatial rotation generators of $O(3)$ $R$ and the Lorentz boosts $B$. The infinitesimal transformation of the Stokes vector is

$$s' = s - \sum_i (r_i \mathbf{R}_i + b_i \mathbf{B}_i) s d l = s - K s d l$$

where $K$ is a matrix that resembles the absorption matrix. This equation is the radiative transfer equation along the signal path. If one choses to exclude effects that intensify the radiation, this is the complete mathematical description of the effects in the signal path. The Lorentz boosts describe polarizing effects, while the spatial rotations describe the Faraday rotation. The relation between the initial and final Stokes vector is a finite Lorentz transformation. While it sounds straightforward it is not an easy task. The problem arises from the non commutativity of the generators. Magnus has proposed a solution to this problem. We will discuss this soon.

A.8 Clifford Algebra

After having introduced the coherency matrix and the Jones formalism we proceed a step further in the mathematical abstraction in order to unify all possible cases. We note that every system is equivalent to a parallel combination of pure systems, and any pure system to a combination of retarders and de-attenuators. In Hamaker (2000) the author proposes the use of the quaternion algebra to describe the Jones matrices, or their $SO^+(1, 3)$ covering group. Quaternions, as any other space isomorphic to $\mathbb{C}$ constitute a sub-algebra of a Clifford algebra. In particular, in three dimensions the Clifford product is equivalent to spatial rotations. Pure systems can still be described as Lorentz transformations in this algebra. Since algebraic manipulations become more clear, the computational requirements of problems involving partial polarization can be reduced. The Clifford algebra of three-dimensional space $\text{Cl}_3$ represents the four dimensional Minkowski space time. Clifford algebras have been used extensively to describe polarization mode dispersion (PMD) in optical fibers (Reimer et al., 2008). In that field they have to deal
A.9 Magnus Expansion

In the case of polarization mode dispersion, which can occur due to ionospheric Faraday rotation, transversal of the polarized source signal through a set of phase screens etc, one is interested in recovering the original polarization state of the radiation. Two possibly quasi-orthogonal modes, represented by the input frequency-independent and the output Jones vectors, are related through a complex $2 \times 2$ Jones matrix. Despite the fact that we might not have any prior knowledge about the structure of the medium through which radiation propagates, the frequency dependence of that effect contains useful information about that medium. For example the $\sim \lambda^2$ Faraday rotation should leave a distinct imprint on the signal, which can help us distinguish it from other depolarizing effects. In the general case, the coherency matrix is transformed as $C' = JCJ^\dagger$. An arbitrary Jones matrix with determinant 1, can be also expressed in terms of two vectors $a$ and $b$ according to

$$j = \exp\left[-\frac{i}{2}(b + ia) \cdot \sigma\right], \quad (A.10)$$

where $\sigma$ is the Pauli spin vector. This is a Jones matrix representation of the Lorentz transformation in a Clifford algebra. In the case of a frequency dependent effect, $J(\omega)$, the Jones space operator can be decomposed into real and imaginary components as

$$J(\omega) = \exp\left[-\frac{i}{2}(b + ia) \cdot \sigma\right], \quad \tilde{J}J = -\frac{i}{2} \left[\Omega + i\Lambda\right], \quad (A.11)$$

where the subscript $\omega$ represents differentiation with respect to the frequency. A solution for $J$ can be obtained through the Magnus expansion, which specifies $J(\omega) = \exp \left( \sum_{n=0}^{\infty} B_n(\omega) \right) J(\omega_0)$, where the first two coefficients are given by

$$B_1(\omega) = \int_{\omega_0}^{\omega} d\omega_1 J_\omega (\omega_1),$$
$$B_2(\omega) = \int_{\omega_0}^{\omega} \int_{\omega_0}^{\omega_1} d\omega_2 d\omega_1 \left( J_\omega (\omega_1) J_\omega (\omega_2) - J_\omega (\omega_2) J_\omega (\omega_1) \right).$$

The coefficients of the Magnus expansion for $n > 2$, are related to those of lower order through recursion. Taylor expanding the frequency derivatives of the Jones matrix to third order gives us a way to directly evaluate the Magnus coefficients. To the extent of our knowledge, the methods mentioned above have not been applied in the field of radio interferometry.
Appendix B

Mathematical definitions

B.1 Imaging with non-coplanar arrays

The measurement equation for a non-coplanar array can be written as:

\[ R = A C A^H, \]

where

\[ R = R(t) \]
\[ A = A(t) = [a(l_1, m_1), \ldots, a(l_n, m_n)] \]
\[ a(l_n, m_n) = \begin{bmatrix} e^{-2\pi i(u_{10}l_1 + v_{10}m_1 + w_{10}n)} \\ \vdots \\ e^{-2\pi i(u_{1k}l_1 + v_{1k}m_1 + w_{1k}n)} \end{bmatrix}, \]
\[ C = \text{diag} \left[ \frac{1}{\sqrt{1-l_1^2-m_1^2}}, \ldots, \frac{1}{\sqrt{1-l_n^2-m_n^2}} \right]. \]

Then the following equation has to be inverted to solve the imaging problem:

\[ R = G A(l, m) C A^H(l, m) G^H + N, \]

where \( N \) is the covariance matrix of additive white noise.

B.2 Polarimetric imaging

LOFAR antennas receive two orthogonal polarization (along the X and Y axis of the antennas). The array response vector is now replaced by two orthogonal vectors describing the two linear polarization states so that \( a(l, m)_X \perp a(l, m)_Y \). The polarization state of a monochromatic source is described by the coherence matrix.
Then the matrix form of the calibrated ME becomes:

\[
R = GA(l, m)CA^H(l, m)G^H + N
\]

\[
A = A(t) = [a(l_1, m_1)_X, a(l_1, m_1)_Y, \ldots, a(l_n, m_n)_X, a(l_n, m_n)_Y]
\]

\[
C = \begin{bmatrix}
C_{11} & C_{12} \\
C_{21} & C_{22} \\
\vdots & \ddots
\end{bmatrix}
\]

\[
G = \text{diag}(G_1, \ldots, G_p)
\]

\[
G_i = \begin{bmatrix}
\xi_{11, i} & \xi_{12, i} \\
\xi_{21, i} & \xi_{22, i}
\end{bmatrix}
\]
Anzengruber S. W., Ramlau R., 2010, Inverse Problems, 26, 025001


Barkana R., Loeb A., 2001, PhysRep, 349, 125


Box G. E. P., Jenkins G. M., eds., 1976, Time series analysis. Forecasting and control
Bregman J., Brentjens M., 2008, uv coverage, LOFAR Internal Report


Ciardi B., Ferrara A., 2005, Space Science Reviews, 116, 625


Fano U., 1954, Physical Review, 93, 121
—, 1957, Reviews of Modern Physics, 29, 74
Furlanetto S. R., Oh S. P., Briggs F. H., 2006a, PhysRep, 433, 181
—, 2006b, PhysRep, 433, 181
—, 2006c, PhysRep, 433, 181
Harris F., Dick C., Rice M., 2003, IEEE Transactions on Microwave Theory and Techniques, 51, 1395


Jones R. C., 1942, Journal of the Optical Society of America (1917-1983), 32, 486
—, 1948, Journal of the Optical Society of America (1917-1983), 38, 671


Plackett R. L., 1950, Biometrika, 37, 149


Reich P., Reich W., 1988, A&AS, 74, 7


Rudin L. I., Osher S., Fatemi E., 1992, Physica D: Nonlinear Phenomena, 60, 259


—, 1996a, Journal of Atmospheric and Terrestrial Physics, 59, 1432

—, 1996b, Journal of Atmospheric and Terrestrial Physics, 58, 1229


BIBLIOGRAPHY


van Velthoven P. F. J., Spoelstra T. A. T., 1992, Advances in Space Research, 12, 211


Voronkov M. A., Wieringa M. H., 2004, Experimental Astronomy, 18, 13


Yatawatta S., 2007, LOFAR dipole and station beam patterns, LOFAR Memo 255

