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Discrete-time supervisory control of input-constrained neutrally stable linear systems via state-dependent dwell-time switching

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Abstract

This paper presents a discrete-time supervisory control scheme for an input-constrained neutrally stable linear plant in the presence of modelling uncertainties. The small gain control is employed as the multi-controller, and is shown to stabilize the plant in the sense of integral-input-to-state stability. As the switching logic, a state-dependent dwell-time switching is employed. The proposed supervisory control guarantees that all signals in the closed loop are bounded and the state of the uncertain plant converges to the origin.
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Keywords: Switching adaptive control; Input constraint; Dwell time; IISS

1. Introduction

Traditionally, an adaptive controller is obtained by combining a controller with a single estimator. The estimator updates the plant or controller parameters continuously [15,3,23,21]. Recently, switching adaptive control or supervisory control has attracted attention; in such a control system, a bank of controllers and estimators, a monitoring signal generator and a switching logic are employed [12,10,8,25,27]. The monitoring signals indicate similarity between each of the estimators and the uncertain plant. On the basis of the certainty equivalence principle, the switching logic selects one between several controllers, which corresponds to a model resulting in the smallest monitoring signal. Supervisory control of this type can overcome various limitations of conventional adaptive control schemes [11], and can improve control performance [26]. For more details of the supervisory control, see [12,10,8,25,27] and references therein.

Almost all real plants are subject to input saturation, which may result in instability or serious performance degradation when the saturation is ignored at the time of control design. For the last decades, many research results have been presented on the problem of input-constrained control [9]. Among these, small gain control [4,6,28,14,20], anti-windup [33,18], and model predictive control (MPC) [24] are recognized as important solutions in the literature. In this paper, attention is directed to stabilization of input-constrained neutrally stable plants using small gain control. When there are no uncertainties, such plants can be stabilized by linear feedback as discussed in [32,4–6,29,13,16,8,30,20]. However, as is the case with all model-based controllers, instability or performance degradation may occur in the presence of modelling errors.

This paper is aimed at adaptively stabilizing an uncertain discrete-time input-constrained neutrally stable plant using a new supervisory control scheme. In the proposed supervisory method, small gain control is employed as the multi-controller, and the discrete-time state-dependent dwell-time (SDDT) switching logic recently proposed in [19] as the switching logic. The main contributions of the paper are as follows:

- The SDDT switching logic ensures stability provided that the control law employed integral-input-to-state stabilizes...
the estimated model; designing the multi-controller boils down to finding an integral-input-to-state stabilizing controller for the estimated model. Hence, we prove in Section 3.2 that small gain control integral-input-to-state stabilizes the input-constrained neutrally stable linear plant in the presence of an external disturbance.

- State-shared multi-estimators for SISO systems as in [25,22,8] are not generally applicable to state-space models. Hence, we propose in Section 3.1 a state-shared multi-estimator for such state equations. A useful feature of the proposed multi-estimator is that the state of the uncertain plant can be expressed in terms of the pth model and the estimation error.

- Using the SDDT switching logic, we prove in Section 3.4 that all signals in the closed-loop system are bounded and the state of the uncertain plant goes to zero, without relying on the assumption that switching stops in a finite time.

The rest of the paper is organized as follows: Section 2 gives the definition of integral-input-to-state stability (iISS) and its Lyapunov function characterization, the class of the system under consideration, and an overview of supervisory control. Section 3 presents the proposed supervisory control, and proves closed-loop stability. Simulation results are then given in Section 4, and conclusions in Section 5.

2. Preliminaries

The concept of iISS [1,31,2] is used to design a stable closed loop in this paper. Hence, the definition of iISS and a theorem regarding the Lyapunov function characterization are introduced first. Then, several properties of an input-constrained neutrally stable linear system are stated together with nominal stability resulting from the small gain control. Finally, a general structure of supervisory control is briefly described.

2.1. Integral-input-to-state stability

Consider the following discrete-time nonlinear system with an external input:

\[ x(k+1) = F(x(k), d(k)), \]

where \( x(k) \) and \( d(k) \) are the state and external disturbance, respectively.

**Definition 1 (Angeli [1]).** System (1) is iISS if there exist \( \alpha, \gamma \in \mathcal{K}_\infty \) and \( \beta \in \mathcal{KL} \) such that

\[ \alpha(\|x(k)\|) \leq \beta(\|x(0)\|, k) + \sum_{i=0}^{k-1} \gamma(\|d(i)\|). \]

**Theorem 1 (Angeli [1]).** System (1) is iISS if and only if there exists an iISS Lyapunov function \( V : \mathbb{R}^n \rightarrow \mathbb{R}_0^+ \) such that for \( x_1, x_2, \gamma \in \mathcal{K}_\infty \) and a positive definite function \( \rho, V \) satisfies

\[ x_1(\|x\|) \leq V(x) \leq x_2(\|x\|) \]

and

\[ V(F(x, d)) - V(x) \leq -\rho(x) + \gamma(d). \]

If system (1) is iISS, then the system is robust against an external disturbance in the sense that the external input with bounded energy leads to bounded state. It is also implied by iISS that the unforced system is globally asymptotically stable. The converse is also shown to hold in discrete time, i.e. global stability implies iISS as well [1]. The \( \mathcal{K}_\infty \) function \( \gamma(\cdot) \) in (2) is referred to as the gain function, which is importantly used in the performance monitoring signal generator to be designed in Section 3.3. The following fact plays an important role in sections below, which is a discrete-time version of Proposition 6 in [31].

**Lemma 1.** If system (1) is iISS, then, for any initial state and for any \( d(k) \) such that \( \sum_{k=0}^{\infty} \|d(k)\| < \infty \) with \( \gamma(\cdot) \) being a class \( \mathcal{K}_\infty \) function as in (2), it follows that \( x(k) \to 0 \) as \( k \to \infty \).

2.2. Small gain control of an input-constrained neutrally stable linear system

In this subsection, a stability result in [20] is briefly reviewed for the small gain control of a neutrally stable disturbance-free linear plant with input constraints. Consider the following plant with input saturation:

\[ x(k+1) = Ax(k) + Bu(u(k)), \]

where \( x \in \mathbb{R}^n \) and \( u \in \mathbb{R}^m \) are the state and input, \( A \) is neutrally stable, i.e. all its eigenvalues lie within the unit circle with those on the unit circle being simple, and the saturation function \( \text{sat}(\cdot) \) is defined as follows:

\[ \text{sat}(u) = [\text{sat}(u_1) \text{sat}(u_2) \cdots \text{sat}(u_m)]^T, \]

where

\[ \text{sat}(u_i) = \begin{cases} u_{\text{max}}, & u_i > u_{\text{max}}, \\ u_i, & |u_i| \leq u_{\text{max}}, \\ -u_{\text{max}}, & u_i < -u_{\text{max}} \end{cases} \]

with \( u_{\text{max}} \) being a positive constant. Then for any \( L_u \) satisfying \( L_u u_{\text{max}} > 1 \), we have

\[ \|\text{sat}(u) - u\| \leq L_u u^T \text{sat}(u). \]

It also follows from the neutral stability that there exists a positive definite matrix \( M_c \) satisfying

\[ A^T M_c A - M_c \leq 0. \]
Now a globally stabilizing small gain controller is given by
\[ u(k) = -\kappa B^T M_c A x(k), \]  
where \( \kappa > 0 \) satisfies
\[ \kappa B^T M_c B < I. \]  
Note that the control law (8) is derived directly without using any coordinate transformation unlike such methods as in [4,6,30], where a coordinate transformation is used.

It can then be shown that there exists a positive definite matrix \( M_q \) such that
\[ (A - \kappa BB^T M_c A)^T M_q (A - \kappa BB^T M_c A) - M_q = -\sigma I, \]  
where \( \sigma > 0 \) is a design parameter. On the basis of this Lyapunov equation, the global stability of the resulting closed loop is given below.

**Theorem 2 (Kim et al. [20])**. For the closed-loop system (5) and (8), there exists a Lyapunov function \( W(\cdot) \) such that
\[ W(x(k)) = x^T(k) M_q x(k) + \lambda (x^T(k) M_c x(k))^{3/2}, \]  
\[ W(x(k+1)) - W(x(k)) \leq -\sigma \| x(k) \|^2, \]  
where \( \lambda \) is
\[ \lambda = \frac{2L_u \kappa \sigma_{\text{max}}(A_c^T M_q B)}{\sqrt{\lambda_{\text{min}}(M_c)}} \]  
with \( \lambda_{\text{min}} \) and \( \sigma_{\text{max}} \) denoting the minimum eigenvalue and the maximum singular value, respectively.

Using a quadratic Lyapunov function as in [6], we cannot obtain a quadratic upper bound for its difference. However, using the non-quadratic Lyapunov function in (11), the upper bound of the difference can be expressed as a negative definite quadratic function of the state as in (12). This quadratic upper bound is used to give an iISS characterization in Section 3.2.

### 2.3. An overview of supervisory control

In this subsection, the supervisory control is briefly described. For more details of the general architecture of supervisory control, see [25,12,22,10,8]. Fig. 1 depicts a block diagram showing the architecture of the supervisory control.

As seen in the figure, the overall structure is similar to that of conventional adaptive control. However, the difference is that the adaptation is carried out via switching in the supervisory control. This is in sharp contrast with conventional adaptive control where adaptation is based on continuous tuning. The block \( P \) in Fig. 1 is the uncertain plant to be controlled. The multi-estimator \( E \) is a bank of estimators, which correspond to different models for the uncertain plant. The output of the monitoring signal generator \( M \) indicates the size of the estimation error. Smallness of a monitoring signal implies that the corresponding model is close to the plant. The multi-controller \( C \) is a bank of controllers; each controller should be devised such that the corresponding model can be integral-input-to-state stabilized when the estimation error is regarded as an external input [10]. The switching logic \( S \) places in the feedback loop the controller designed using the model from an estimator corresponding to the smallest monitoring signal. The switching logic should be designed such that the closed loop is asymptotically stable.

### 3. Supervisory control of input-constrained linear systems

One of the main features of the supervisory control is its modularity; the multi-controller and the multi-estimator are designed almost independently. The switching logic is devised in such a way that guarantees closed-loop stability. In this section, each module in the proposed supervisory control is described for an uncertain input-constrained neutrally stable system. We then prove that the proposed supervisory control leads to the boundedness of all signals in the closed-loop system and asymptotic convergence of the state of the uncertain plant.

#### 3.1. Process (P) and multi-estimator (E)

We consider the discrete-time input-constrained plant (5), which is neutrally stable. It is assumed that this plant is unknown but is a member of a known set of admissible models as follows:
\[ (A, B) = (A_p, B_p) \in \bigcup_{p \in \mathcal{P}} (A_p, B_p). \]  

In this paper, the index set \( \mathcal{P} \) is assumed to be finite for simplicity.

An estimator for the \( p \)th model is expressed as
\[ x_p(k+1) = A_E x_p(k) + B_p \text{sat}(u(k)) + (A_p - A_E) x(k), \]  
where \( A_E \) is a Hurwitz matrix. Note that the state is used as the measurement for the estimator. The uncertain part can be
parametrized in the form

\[ A_p - A_E = A^0 + \sum_{i=1}^{M} a_{pi} A^i, \]  

\[ B_p = B^0 + \sum_{i=1}^{N} b_{pi} B^i, \]  

where the matrices \( A^i \) and \( B^i \) are known, and \( a_{pi} \) and \( b_{pi} \) (\( r = \mathbb{R} \)) are model parameters. Using Eqs. (14)–(16), the estimator equation can also be written as follows:

\[
\begin{bmatrix}
A_E & 0 & \cdots & 0 \\
0 & A_E & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & A_E
\end{bmatrix}
\begin{bmatrix}
x_E(k+1) \\
\vdots \\
x_E(k)
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
\vdots \\
0
\end{bmatrix}
\begin{bmatrix}
x_E(k) \\
x_E(k)
\end{bmatrix}
+ 
\begin{bmatrix}
B^0 \\
B^N \\
0 \\
\vdots \\
0
\end{bmatrix}
\begin{bmatrix}
\text{sat}(u(k)) \\
\vdots \\
\text{sat}(u(k))
\end{bmatrix}
+ 
\begin{bmatrix}
A^0 \\
\vdots \\
A^M
\end{bmatrix}
\begin{bmatrix}
x(k) \\
\vdots \\
x(k)
\end{bmatrix},
\]

(17)

and its estimation error as follows:

\[ x_p(k) = C_p x_E(k), \]

(18)

where \( x_E \) is the shared state vector, and

\[ C_p = [I \ b_{p1} I \ \cdots \ b_{pM} I \ a_{p1} I \ \cdots \ a_{pM} I]. \]

The estimation error is defined as

\[ e_p(k) = x_p(k) - x(k) \]

and its successor is then given by

\[ e_p(k+1) = A_E e_p(k) + (B_p - B) \text{sat}(u(k)) + (A_p - A) x(k). \]

(19)

Note that the estimator in (14) is shown to result from (17) and (18) as follows:

\[
x_p(k+1) = C_p x_E(k+1) = A_E x_p(k) + B_p \text{sat}(u(k)) + (A_p - A_E) x(k).
\]

Now, the plant equation is written in terms of the 1st model and its estimation error as follows:

\[
x(k+1) = A_p x(k) + B_p \text{sat}(u(k)) - e_p(k+1) - A_E e_p(k)
\]

\[ := A_p x(k) + B_p \text{sat}(u(k)) + d_p(k). \]

(20)

This description of the plant makes it more convenient to design and analyze the supervisory control system when compared to conventional approaches in [25,10,8]. Note also that \( d_p \in \ell_1 \) if \( e_p \in \ell_1 \).

### 3.2. Multi-controller: small gain controller (C)

As in [10], we use the iISS property of the closed loop with \( d_p \) (and thus \( e_p \)) regarded as the external disturbance. We employ the small gain controller discussed in Section 2.2 as the multi-controller. Consider the following control law on the basis of the 1st model:

\[ u(k) = -\kappa_p B_p^T M_{c,p} A_p x(k), \quad p \in \mathcal{P} \]

(21)

where \( \kappa_p \) and \( M_{c,p} \) satisfy

\[ A_p^T M_{c,p} A_p - M_{c,p} \leq 0 \quad \text{and} \quad \kappa_p B_p^T M_{c,p} B_p < I \]

as in (7) and (9). In the following theorem, it is proved that the closed-loop system in (20) and (21) is iISS with respect to \( d_p(k) \).

Theorem 3. For the closed-loop system (20) and (21), there exists an iISS Lyapunov function \( V_p(\cdot) \) such that

\[ V_p(x(k)) = \ln(1 + W_p(x(k))), \]

(22)

\[ V_p(x(k+1)) - V_p(x(k)) \leq -\rho_p(||x(k)||) + \tilde{\gamma}_p ||d_p(k)||, \]

(23)

where

\[ W_p(x) = x^T M_{q,p} x + \lambda(x^T M_{c,p} x)^{3/2}, \]

(24)

\[ (A_p - \kappa B_p B_p^T M_{c,p} A_p)^T M_{q,p} (A_p - \kappa B_p B_p^T M_{c,p} A_p) - M_{q,p} \]

\[ = -\sigma I, \]

(25)

\( \rho_p(\cdot) \) is a positive definite function, and \( \tilde{\gamma}_p \) is a positive number.

Proof. Rewrite plant (5) as

\[ x(k+1) = A_p x(k) + B_p \text{sat}( -\kappa B_p^T M_{c,p} A_p x(k) ) + d_p(k) 
\]

\[ := f_p(x(k), d_p(k)). \]

Then we have

\[ V_p(x(k+1)) - V_p(x(k)) = \ln(1 + W_p(x(k+1))) - \ln(1 + W_p(x(k))) \]

\[ = \ln(1 + W_p(f_p(x(k), d_p(k)))) - \ln(1 + W_p(x(k))) \]

\[ = \ln(1 + W_p(f_p(x(k), 0))) - \ln(1 + W_p(x(k))) \]

\[ + \ln(1 + W_p(f_p(x(k), d_p(k)))) - \ln(1 + W_p(f_p(x(k), 0))) \]

\[ \leq \ln(1 + W_p(x(k)) - \sigma ||x(k)||^2) - \ln(1 + W_p(x(k))) \]

\[ + \ln(1 + W_p(f_p(x(k), d_p(k)))) - \ln(1 + W_p(f_p(x(k), 0))) \]

\[ - \ln(1 + W_p(f_p(x(k), 0))). \]

\[ \]
This inequality follows from Theorem 2 and the monotonically increasing property of the logarithmic function. An upper bound of the first two terms in (26) is shown to be a negative definite function of $\|x\|$ as follows:

$$\ln(1 + W_p(x(k)) - \|x(k)\|^2) - \ln(1 + W_p(x(k)))$$

$$= \ln \left( \frac{1 + W_p(x(k)) - \|x(k)\|^2}{1 + W_p(x(k))} \right)$$

$$\leq - \frac{\|x(k)\|^2}{1 + W_p(x(k))} := -\rho_p(\|x(k)\|).$$

Note that $\rho_p(\|x(k)\|)$ is not a $\mathcal{H}_\infty$ function but a positive definite function of $\|x(k)\|$. Unlike the continuous-time results on deriving ISS and iISS [16,8], the mean value theorem is used to deal with the disturbance in the upper bound of the difference of $V_p(\cdot)$ in (26). Since the function $V_p(\cdot)$ is continuously differentiable, the mean value theorem can be applied; the last two terms in (26) are shown to satisfy

$$\ln(1 + W_p(f_p(x(k), d_p(k)))) - \ln(1 + W_p(f_p(x(k), 0)))$$

$$= \ln \left( \frac{1 + W_p(f_p(x(k), 0))}{1 + W_p(x(k))} \right) + \left. \left( \frac{\partial}{\partial y} W_p(y) \right) \right|_{y=z} \cdot d_p(k)$$

$$\leq \bar{\gamma}_p \cdot \|d_p(k)\|,$$  (28)

where $z$ is a point on the line segment joining $f_p(x(k), d_p(k))$ and $f_p(x(k), 0)$ and $\bar{\gamma}_p$ a finite upper bound on $\|\left. \left( \frac{\partial}{\partial y} W_p(y) \right) \right|_{y=z}\|$. For all $z$, such an upper bound exists, since $\|\left. \left( \frac{\partial}{\partial y} W_p(y) \right) \right|_{y=z}\|$ is bounded by a second order polynomial in $\|y\|$, whereas a lower bound on $1 + W_p(y)$ can be given by a second or third order polynomial in $\|y\|$ depending on its size. Therefore, (26) leads to the following:

$$V_p(x(k+1)) - V_p(x(k)) \leq -\rho_p(\|x(k)\|) + \bar{\gamma}_p \cdot \|d_p(k)\|.$$  (29)

This implies that the function $V(\cdot)$ is an iISS Lyapunov function, which completes the proof. □

On the basis of the iISS Lyapunov function characterization in (29), we can derive, as in (22), the following inequality:

$$\bar{z}_p(\|x(k)\|) \leq \beta_p(\|x(0)\|, k) + \sum_{i=0}^{k-1} \gamma_p(\|d_p(i)\|),$$  (30)

where the expressions for the $\mathcal{H}$ functions $z_p(\cdot)$, $\beta_p(\cdot)$, and $\gamma_p(\cdot)$ are given in Appendix A. These functions are importantly used when giving the SDDT in the next subsection.

Theorem 3 indicates that the small gain controller can be used as the multi-controller since it integral-input-to-state stabilizes the input-constrained neutrally stable linear plant with an external disturbance [10]. Note that the gain function in this case is

$$\gamma_p(r) = 2\bar{\gamma}_p \cdot r$$  (31)

and the performance monitoring signal generator in Section 3.3 is obtained in view of this gain function.

**Remark 1.** A number of stability results have been presented for small gain control of neutrally stable linear plants with input constraints both in continuous and discrete time; see [32,6] for Lyapunov stability and [5,4,16,20] for ISS. Recently, iISS is proved in [8] for continuous-time systems, and Theorem 3 is a discrete-time counterpart of this result.

### 3.3. Performance monitoring signal generator (M) and SDDT switching logic (S)

The performance monitoring signal $\mu_p$ associated with each $p \in \mathcal{P}$ is defined as follows:

$$\mu_p(k) = \eta \mu_p(k-1) + \|e_p(k)\|,$$  (32)

where $\eta \in (\eta^*, 1)$ and $\eta^*$ satisfies

$$\max |\text{eig}(A_E)| \leq \eta^* < 1.$$  (33)

Note that the input to each monitoring signal generator is set to a linear function of $\|e_p(k)\|$ in view of the gain function in (31). For analysis, define the exponentially weighted performance monitoring signal:

$$\tilde{\mu}_p(k) = \eta^{-k} \mu_p(k),$$

arg min $\mu_p(k) = \arg min \tilde{\mu}_p(k).$  (35)

Each of these performance monitoring signals relates to the smallness of the estimation error and thus to the similarity between the model and the plant. Therefore, when the switching logic chooses a model corresponding to the smallest performance monitoring signal and the controller is designed on the basis of the model selected, the controller is expected to result in improved performance. This idea is based on “the certainty equivalence principle”.

Now, we summarize the SDDT switching logic. An essential requirement for the switching logic in the supervisory control is that switching should occur slowly enough to preserve stability of subsystems via switching. Hence, we do not allow switching to take place during a prescribed amount of time, which is referred to as the dwell time. As the switching logic for the proposed supervisory control, the SDDT switching logic recently proposed in [19] is employed. The SDDT switching logic is similar to the dwell-time switching logic used in [25]. The difference is that the dwell-time switching logic in [25] uses a constant dwell time and works for linear time-invariant systems; in contrast, the dwell time in the SDDT switching logic varies according to the size of the state and is devised for nonlinear systems.
The resulting switched closed-loop system is written as

\[ \dot{x}(t) = A_E x(t) + B_E u(t), \]

and that

\[ k_i \]

where \( \zeta > 2 \), and the functions \( z_p() \) and \( \beta_p() \) given in (A.6) satisfy inequality (30).

The SDDT switching logic is described in Fig. 2. After switching occurs, the switching logic waits for the SDDT without switching. After the SDDT elapses, it switches to a controller associated with the smallest monitoring signal and updates the SDDT for the next switching. As shown in Fig. 2, the difference between two consecutive switching times is bounded below as follows:

\[ k_{i+1} - k_i > \tau_D(\|x(k_i)\|), \]

where \( k_i \) and \( k_{i+1} \) are the \( i \)th and \( (i + 1) \)th switching times. The resulting switched closed-loop system is written as

\[ x(k+1) = f_{\sigma(k)}(x(k), d_{\sigma(k)}(k)), \]

where \( \sigma: [0, \infty) \to \mathcal{P} \) is a piecewise constant switching function such that

\[ \sigma(k) = \arg \min_{p \in \mathcal{P}} \mu_p(k), \quad \forall k_i \leq k < k_{i+1}. \]

Note again that the switched system is iISS with respect to the signal \( d_\sigma(k) \) for a fixed value of \( \sigma(k) \). This iISS property enables us to prove the following theorem.

**Theorem 4** (Kim et al. [19]). Consider the switched system in (38). Suppose that \( \tau_D() \) defined above satisfies

\[ \lim_{r \to 0} \tau_D(r) < \infty, \]

and that \( \sigma \) is determined by the SDDT switching logic. If \( d_\sigma() \) satisfies

\[ \sum_{i=0}^{\infty} \|d_\sigma(i)\| < \infty, \]

then we have

\[ \lim_{k \to \infty} \|x(k)\| = 0. \]

Note that the assumption in inequality (40) can be removed under local exponential stability by modifying the dwell time as in [19,7].

### 3.4. Stability of the proposed supervisory control system

Note that the error dynamics associated with the correct model is given by

\[ e_p(k+1) = A_E e_p(k), \]

Therefore, \( e_p(k) \) decays exponentially, i.e.

\[ \|e_p(k)\| \leq C_1 \eta^k, \]

where \( C_1 > 0 \) and \( \eta^* \) is as in (33). In view of this and the definition of \( \tilde{\mu}_p \), we have

\[ \lim_{k \to \infty} \tilde{\mu}_p(k) < \infty. \]

On the basis of this observation, we have the following.

**Lemma 2.** Let \( \{k_0, k_1, \ldots\} \) be the set of switching times. There exist a switching time \( k^* \) and a set \( \mathcal{P}^* \subset \mathcal{P} \) such that

1. For any switching time \( k_j > k^* \), \( \sigma(k_j) \in \mathcal{P}^* \).
2. For any \( p \in \mathcal{P}^* \),

\[ \sum_{k=0}^{\infty} \|e_p(k)\| < \infty. \]

**Proof.** The proof parallels that of the corresponding lemma for continuous-time linear time-invariant systems\(^3\) given in [25] (Lemma 1), and thus is not given here. \(\square\)

We are now in a position to prove that the overall supervisory control system is stable as follows.

**Theorem 5.** Consider the supervisory control system consisting of the uncertain plant in (5), the multi-controller in (21), the multi-estimator in (17) and (18), the monitoring signal in (32), and the SDDT switching logic described in Section 3.3. Then, all signals in the supervisory control system are bounded and the state of the uncertain plant goes to zero as time goes by.

**Proof.** Note that the closed-loop system in (20) and (21) is iISS with respect to \( d_p(k) \) for any fixed value of \( p \). As switching is allowed persistently, it ensues from Lemma 2 that there exists a switching time \( k^* \) such that for all \( k > k^* \)

\[ \sum_{k=0}^{\infty} \|e_{\sigma(k)}(k)\| = \sum_{k=0}^{k^*} \|e_{\sigma(k)}(k)\| + \sum_{k=k^*+1}^{\infty} \|e_{\sigma(k)}(k)\| < \infty. \]

This means

\[ \sum_{k=0}^{\infty} \|d_{\sigma(k)}(k)\| < \infty. \]

\(^3\) Although the plant is nonlinear due to the input constraints, the multi-estimator has linear dynamics with sat(\(u\)) and \( x \) being the inputs.
It then follows from Theorem 4 that the state $x(k)$ of the uncertain plant asymptotically converges to zero.

The state $x(k)$ is defined for all $k$ and is convergent. Further, since the estimator has the stable linear dynamics driven by bounded signals $\text{sat}(u)$ and $x(k)$ as shown in (17), the state $x_E$ and $x_p$ are bounded, and so are the monitoring signals $\mu_p$. Therefore, all the signals in the closed-loop system are bounded. This completes the proof. □

4. Simulation

As is often the case with adaptive or supervisory control, the proposed control scheme designed for an unknown time-invariant plant is applied to a slowly time-varying uncertain process as follows:

$$
\begin{align*}
x(k+1) &= \begin{bmatrix} 1 & a(k) \\ 0 & 0.8 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{sat}(u(k)) \\
& \quad + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w(k),
\end{align*}
$$

where $a(k)$ is an uncertain parameter that varies with time. (In simulations, $a(k)$ is set to vary linearly from $-0.4$ to $-1.4$ with a slope of $-0.01$.) We also consider the disturbance $w(k)$ that tends to zero. (In simulations, $w(k)$ is set to $2(0.95)^k$.)

For this plant, we employ the multi-estimator in (17) and (18) as follows:

$$
\begin{align*}
x_E(k+1) &= \begin{bmatrix} A_E & 0 & 0 \\ 0 & A_E & 0 \\ 0 & 0 & A_E \end{bmatrix} x_E(k) + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{sat}(u(k)) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} x(k),
\end{align*}

$$

$$
\begin{align*}
x_p(k) &= \begin{bmatrix} 1 & 0 & 1 & 0 & a_p & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} x_E(k),
\end{align*}
$$

where $A_E = \text{diag}(0.7, 0.7)$ and $a_p \in \{-0.02, -0.04, -0.06, \ldots, -2\}$. Note that the dimension of the multi-estimator is independent of the number of models considered; should we decide to use (14), then the dimension would be twice the number of models. The initial state and the value of $\eta$ are assumed to be

$$
\begin{align*}
x_0 &= [-4, -8]^T, \quad \eta = 0.8
\end{align*}
$$

and $M_{\epsilon, p}$ is computed using the LMI toolbox. Fig. 3 shows that the supervisory control successfully stabilizes the time-varying uncertain plant. In the right part of Fig. 3, the dashed line denotes $\text{sat}(u)$ and the solid line $u$. Fig. 4 presents the switching...
signal, which indicates that the multi-estimator keeps tracking the time-varying plant in the transient state.

5. Conclusion

This paper has presented a discrete-time supervisory control of an uncertain input-constrained neutrally stable linear plant on the basis of the state-dependent dwell-time (SDDT) switching logic. The small gain control is employed as the multi-controller, and is shown to integral-input-to-state stabilize the plant. The multi-estimator is designed such that the plant can be expressed in terms of the multi-estimator and the estimation error. For switching logic, the SDDT logic recently proposed in [19] is employed, which is a discrete-time version of the dwell-time switching in [7]. The resulting control scheme is shown to guarantee the boundedness of all signals in the closed-loop system and the asymptotic convergence of the plant state.

Since switching is not allowed during the dwell time even when there is a better model than the current model, modifying the switching logic is necessary for faster switching without destroying stability properties. Also, other control laws dealing with input constraints such as MPC may be considered for the multi-controller in the supervisory control. These constitute important areas for future research.

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Appendix A. Finding $x_p$, $β_p$, and $γ_p$ in (30)

Using the definitions of the functions $W_p(·)$ and $V_p(·)$ in (22) and (24), we have

\[ a_{wp}\|x\|^2 \leq W_p(x) \leq b_{wp}\|x\|^2 + c_{wp}\|x\|^3, \]

\[ x_{p1}(\|x\|) \leq V_p(x) \leq x_{p2}(\|x\|), \]

where $x_{p1}(\|x\|) = \ln(1 + a_{wp}\|x\|^2)$ and $x_{p2}(\|x\|) = \ln(1 + b_{wp}\|x\|^2) + c_{wp}\|x\|^3$ with $a_{wp}$, $b_{wp}$, and $c_{wp}$ being positive constants. Note that $x_{p1}$, $x_{p2} \in \mathcal{K}_\infty$.

Let $ρ: \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ be a continuous positive definite function. It then follows from [1] that

\[ ρ(r) \geq ρ_1(r)ρ_2(r), \]

where $ρ_1 \in \mathcal{K}_\infty$ and $ρ_2 \in \mathcal{L}$ (i.e. $ρ_2$ is continuous, is decreasing, and converges to 0 as its argument tends to $\infty$).

Using this and the inequality in (A.2), the inequality in (29) can be written as

\[ V_p(x(k + 1)) - V_p(x(k)) \leq -ρ_p(\|x(k)\|) + γ_p\|dp(k)\| \]

\[ \leq -ρ_p(\|x(k)\|)ρ_{p2}(\|x(k)\|) \]

\[ + γ_p\|dp(k)\| \]

\[ \leq -ρ_p(z_{p1}^{-1}(V_p(x(k))))ρ_{p2} \]

\[ \times (z_{p1}^{-1}(V_p(x(k)))) + γ_p\|dp(k)\| \]

\[ := -\bar{ρ}_p(V_p(x(k))) \]

\[ + γ_p\|dp(k)\|, \]

where $ρ_p$ is a $\mathcal{K}_\infty$ function and $ρ_{p2}$ is a $\mathcal{L}$ function. Using Lemma 4.3 in [17] and Theorem 2 in [1], Eq. (A.3) is shown to result in the following:

\[ V_p(x(k)) \leq β_{v_p}(x_p2(k(0))), k - k_0 + \sum_{i=k_0}^{k-1} 2\gamma_i\|dp(i)\|, \]

where

\[ β_{v_p}(r, k) = ρ_{KLp}(r), \]

\[ ρ_{KLp}(r) = \frac{2r - \bar{ρ}_p(r)}{2} \]

and

\[ ρ_{KLp}^0(r) = (ρ_{KLp} \circ ρ_{KLp})(r) \]

with $ρ_{KLp}^0(·)$ being the identity function. It thus ensues from (31), (A.2) and (A.4) that the inequality in (30) holds with

\[ x_p(x) = \ln(1 + a_{wp}\|x\|^2), \]

\[ β_p(r, k) = ρ_{KL}(x_p2(r)) \]

and

\[ γ_p(r) = 2\gamma_i \cdot r. \]

References