Model reduction of port-Hamiltonian systems
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6. Structure preserving moment matching at an arbitrary point in the complex plane

The goal of this chapter is to show that the rational Arnoldi and Lanczos methods, apart from equalizing a certain amount of moments at an arbitrary point in the complex plane, also preserve the port-Hamiltonian structure, and, as a consequence, passivity.

6.1. Introduction

In this chapter structure preserving model reduction of port-Hamiltonian systems is considered by employing the rational Krylov methods. The rational Arnoldi method is shown to preserve not only a specific number of moments at an arbitrary point in the complex plane but also the port-Hamiltonian structure. Furthermore, it is shown how the rational Lanczos method, applied to a subclass of port-Hamiltonian systems characterized by an algebraic condition (different from that of Chapter 5), preserves the port-Hamiltonian structure. In fact, for the same subclass of port-Hamiltonian systems the rational Arnoldi and the rational Lanczos methods turn out to be equivalent in the sense of producing reduced order port-Hamiltonian models with the same transfer function.

As discussed in Section 5.1, the moment matching methods are an important class of model reduction methods, when a specific number of moments of the full order system at certain points in the complex plane are preserved by the reduced order system.

The goal of this chapter is to show that the rational Arnoldi and Lanczos methods (distinguish from the Arnoldi and Lanczos methods) serve the purpose of preserving the port-Hamiltonian structure, apart from matching a certain amount of moments at an arbitrary point in the complex plane. This chapter is an extension of Chapter 5, [69], where the port-Hamiltonian structure is preserved along with matching the moments at infinity. A similar discussion is presented in [57], where the authors make use of the rational Arnoldi method, which results in a reduced order port-Hamiltonian model which is slightly different from the one obtained in this chapter. In fact, we
show that the reduced order port-Hamiltonian model of this chapter is equivalent to that of [57].

In Section 6.2 we briefly discuss the rational Arnoldi and Lanczos methods as well-known moment matching methods. In Section 6.3 we demonstrate how to preserve the port-Hamiltonian structure using the rational Arnoldi method. In Section 6.4 we exploit the rational Lanczos method for structure preserving model reduction of port-Hamiltonian systems, showing that it can be applied to a subclass of port-Hamiltonian systems, characterized by an algebraic condition. We will prove that the reduced order port-Hamiltonian models obtained by the rational Lanczos method for the given subclass of port-Hamiltonian systems are equivalent to those obtained by the rational Arnoldi method, matching $2r$ moments at an arbitrary point in the complex plane. Finally, in Section 6.5 we present a numerical example of a physical system and apply the rational Arnoldi method to obtain a reduced order port-Hamiltonian model. This example illustrates that, even though we applied the rational Arnoldi method, which in general only preserves $r$ moments, in our case $2r$ moments are preserved, since the considered port-Hamiltonian model belongs to the subclass of port-Hamiltonian systems described above.

This chapter is primarily based on the paper [70].

### 6.2. Moment matching for linear systems at an arbitrary point in the complex plane

In this section we briefly recall the use of the rational Krylov methods (see [2]), in particular, the rational Arnoldi method and the rational Lanczos method in order to obtain reduced order linear systems preserving the first moments of the full order systems at an arbitrary point in the complex plane.

Consider a linear, single-input, single-output, continuous-time system $\Sigma$ described by equations of the form

$$\begin{align*}
\dot{x} &= Ax + bu, \\
y &= cx,
\end{align*}$$

(6.1)

with the state-space vector $x(t) \in \mathbb{R}^n$, input $u(t) \in \mathbb{R}$, output $y(t) \in \mathbb{R}$, and constant matrices $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$, $c \in \mathbb{R}^{1 \times n}$.

**Definition 6.1.** [2] The 0-moment of the system (6.1) at $s_0 \in \mathbb{C}$ is the complex number

$$\eta_0(s_0) = c(s_0I - A)^{-1}b.$$  

The $r$-moment of the system (6.1) at $s_0 \in \mathbb{C}$ is the complex number

$$\eta_r(s_0) = \frac{(-1)^r}{r!} \left[ \frac{d^r}{ds^r} c(sI - A)^{-1}b \right]_{s=s_0} = c(s_0I - A)^{-(r+1)}b.$$
6.2. Moment matching for linear systems at an arbitrary point

6.2.1. The rational Arnoldi method

The idea of the rational Arnoldi method is to construct a reduced order model by applying a Galerkin projection \( V_r V_r^T, V_r \in \mathbb{R}^{n \times r} \), to the full order linear system (6.1). The maps \( V_r, r = 1, \ldots, n, \) satisfy the following properties:

(i) \( V_r^T V_r = I_r \), i.e., the columns of \( V_r \) are orthonormal,

(ii) \( \text{span } \text{col } V_r = \mathcal{K}^{\text{shifted}}_{r, \text{input}}, r = 1, 2, \ldots, n, \)

where

\[
\mathcal{K}^{\text{shifted}}_{r, \text{input}} = \text{span } \text{col } \mathcal{R}_r((A - s_0 I)^{-1}, (A - s_0 I)^{-1} b)
\]

is the so-called shifted input Krylov subspace, and

\[
\mathcal{R}_r(A, b) = [b : A b : \ldots : A^{r-1} b] \in \mathbb{R}^{n \times r}
\]

is the partial reachability matrix of the system (6.1).

**Theorem 6.2.** [29], [36] Let \( V_r \) be a matrix satisfying (6.2). Then the \( r \)th order system \( \hat{\Sigma} \)

\[
\begin{align*}
\dot{\hat{x}} &= \hat{A} \hat{x} + \hat{b} u, \\
\hat{y} &= \hat{c} \hat{x},
\end{align*}
\]

where \( \hat{A} = V_r^T A V_r, \hat{b} = V_r^T b, \hat{c} = c V_r, \) defines a reduced order system with the moments \( \hat{\eta}_i(s_0), i = 0, \ldots, r - 1 \) at \( s_0 \in \mathbb{C} \) equal to the first \( r \) moments \( \eta_i(s_0), i = 0, \ldots, r - 1, \) of the full order system \( \Sigma. \)

**Proof.** The idea of the proof is based on the moment matching around \( s_0 = 0 \) employing the properties of the corresponding input Krylov subspace with the consequent shift to an arbitrary point \( s_0 \) using the shifted input Krylov subspace. Details of the proof can be found in [29], [36]. \( \Box \)

In a similar way we can construct the projection maps \( W_r \in \mathbb{R}^{n \times r}, r = 1, \ldots, n, \) based on the shifted output Krylov subspace:

\[
\mathcal{K}^{\text{shifted}}_{r, \text{output}} = \text{span } \text{col } \mathcal{R}_r((A - s_0 I)^{-T}, (A - s_0 I)^{-T} c^T),
\]

satisfying the following properties:

(i) \( W_r^T W_r = I_r \), i.e., the columns of \( W_r \) are orthonormal,

(ii) \( \text{span } \text{col } W_r = \mathcal{K}^{\text{shifted}}_{r, \text{output}}, r = 1, 2, \ldots, n. \)

**Theorem 6.3.** Let \( W_r \) be satisfying (6.3). Define the reduced order system \( \bar{\Sigma} \)

\[
\begin{align*}
\dot{\bar{x}} &= \bar{A} \bar{x} + \bar{b} u, \\
\bar{y} &= \bar{c} \bar{x},
\end{align*}
\]
where $\bar{A} = W_r^T A W_r$, $\bar{b} = W_r^T b$, $\bar{c} = c W_r$. Then the reduced order moments $\bar{\eta}_i(s_0)$, $i = 0, \ldots, r - 1$ at $s_0 \in \mathbb{C}$ are equal to the first $r$ moments $\eta_i(s_0)$, $i = 0, \ldots, r - 1$, of the full order system $\Sigma$.

**Proof.** The proof is similar to the proof of Theorem 6.2, hence omitted. □

### 6.2.2. The rational Lanczos method

In order to apply the rational Lanczos method one has to construct a reduced order model by applying a Petrov-Galerkin projection $V_r W_r^T$, $V_r, W_r \in \mathbb{R}^{n \times r}$, to a full order linear system (6.1). The maps $V_r, W_r$ satisfy property (ii) of (6.2), (6.3). But in this case $V_r, W_r$ are no longer assumed to be orthonormal but instead biorthogonal: $W_r^T V_r = I_r$.

**Theorem 6.4.** [29], [36] Let $V_r W_r^T$ be a Petrov-Galerkin projection. Define the reduced order system $\tilde{\Sigma}$

\[
\begin{align*}
\dot{\tilde{x}} &= \tilde{A} \tilde{x} + \tilde{b} u, \\
\tilde{y} &= \tilde{c} \tilde{x},
\end{align*}
\]

where $\tilde{A} = W_r^T A V_r$, $\tilde{b} = W_r^T b$, $\tilde{c} = c V_r$. Then the reduced order moments $\tilde{\eta}_i(s_0)$, $i = 0, \ldots, 2r - 1$ at $s_0 \in \mathbb{C}$ are equal to the first $2r$ moments $\eta_i(s_0)$, $i = 0, \ldots, 2r - 1$, of the full order system $\Sigma$.

**Proof.** The proof is similar to the proof of Theorem 6.2 apart from the fact that in this case both the (shifted) input and output Krylov subspaces are used. Details of the proof can be again found in [29], [36]. □

Thus the rational Lanczos method preserves twice as many moments of the full order model at an arbitrary point $s_0$ as the rational Arnoldi method.

### 6.3. Reduction of port-Hamiltonian systems by the rational Arnoldi method

In this section we will apply the rational Arnoldi method to linear port-Hamiltonian systems.

#### 6.3.1. Energy coordinates, transforming $Q$ to the identity matrix

Consider a port-Hamiltonian system (1.16)

\[
\begin{align*}
\dot{x} &= (J - R)Qx + bu, \\
y &= b^T Q x,
\end{align*}
\] (6.4)
6.3. Reduction by the rational Arnoldi method

with \( A = (J - R)Q \in \mathbb{R}^{n \times n}, b \in \mathbb{R}^n, c = b^T Q \in \mathbb{R}^{1 \times n}, \ Q > 0. \) As discussed in Chapter 5, there exists a coordinate transformation \( S, \ x = Sx_I, \) such that in the new coordinates

\[
Q_I = S^T QS = I. \tag{6.5}
\]

This coordinate transformation leads to the transformed port-Hamiltonian system (5.9)

\[
\begin{aligned}
\dot{x}_I &= (J_I - R_I)x_I + b_I u, \\
y &= b_I^T x_I
\end{aligned} \tag{6.6}
\]

with energy \( H(x_I) = \frac{1}{2} \|x_I\|^2. \)

**Theorem 6.5.** Consider a full order port-Hamiltonian system (6.6) and define \( V_r \) satisfying (6.2) using the Arnoldi procedure. Then the \( r \)th order reduced system

\[
\begin{aligned}
\dot{\hat{x}}_I &= (\hat{J}_I - \hat{R}_I)\hat{x}_I + \hat{b}_I u, \\
\hat{y} &= \hat{c}_I \hat{x}_I
\end{aligned} \tag{6.7}
\]

with the interconnection matrices \( \hat{J}_I, \hat{b}_I, \) energy matrix \( \hat{Q}_I, \) dissipation matrix \( \hat{R}_I \) and output matrix \( \hat{c}_I \) given as

\[
\begin{aligned}
\hat{J}_I &= V_r^T J_I V_r, \quad \hat{R}_I = V_r^T R_I V_r, \quad \hat{Q}_I = I, \\
\hat{b}_I &= V_r^T b_I, \quad \hat{c}_I = b_I^T V_r,
\end{aligned} \tag{6.8}
\]

is a port-Hamiltonian system. Furthermore the first \( r \) moments at \( s_0 \in \mathbb{C} \) of the reduced order port-Hamiltonian system (6.7) and the full order port-Hamiltonian system (6.6) are equal:

\[
(\hat{\eta}_I(s_0))_i = (\eta_I(s_0))_i = \eta_i(s_0), \ i = 0, \ldots, r - 1.
\]

**Proof.** Clearly \( \hat{J}_I \) is skew-symmetric and \( \hat{R}_I \) is symmetric and positive semi-definite. Moreover \( \hat{c}_I = \hat{b}_I^T \hat{Q}_I. \) Therefore the reduced order model (6.7) is port-Hamiltonian. The equality of the first \( r \) moments at \( s_0 \in \mathbb{C}, \ (\hat{\eta}_I(s_0))_i = (\eta_I(s_0))_i \) follows directly from Theorem 6.2. The equality \( (\eta_I(s_0))_i = \eta_i(s_0) \) is due to the fact that the moments are invariant under state-space coordinate transformations. \( \square \)

Using the projection map \( W_r \) satisfying (6.3) instead of \( V_r \) in Theorem 6.5 we obtain a different, but analogous \( r \)th order reduced port-Hamiltonian system preserving the first \( r \) moments at \( s_0 \in \mathbb{C}: \)

\[
\begin{aligned}
\dot{\bar{x}}_I &= (\bar{J}_I - \bar{R}_I)\bar{x}_I + \bar{b}_I u, \\
\bar{y} &= \bar{c}_I \bar{x}_I,
\end{aligned} \tag{6.9}
\]

with the port-Hamiltonian matrices \( \bar{J}_I, \bar{R}_I, \bar{Q}_I, \bar{b}_I \) given as in (6.8) after substituting \( W_r \) for \( V_r. \)
6. Structure preserving moment matching at an arbitrary point

In general, the reduced order models (6.7) and (6.9) obtained by applying the projection maps \(V_r, W_r\) constructed using the rational Arnoldi method are not equivalent. Nevertheless, under the condition stated in the following theorem we can prove that these reduced order models indeed are equivalent sharing the same transfer function.

**Theorem 6.6.** The reduced order port-Hamiltonian model (6.7) obtained using the projection map \(V_r\) based on the shifted input Krylov subspace \(K_{r,\text{input}}^{\text{shifted}}\) and the reduced order port-Hamiltonian model (6.9) obtained using the projection map \(W_r\) based on the shifted output Krylov subspace \(K_{r,\text{output}}^{\text{shifted}}\) share the same transfer function if the condition

\[
\text{span col } R_r((FQ - s_0I)^{-1}, (FQ - s_0I)^{-1}b) = \text{span col } R_r((F^TQ - s_0I)^{-1}, (F^TQ - s_0I)^{-1}b)
\]

for \(F = J - R\) is satisfied.

**Proof.** The transfer functions of (6.7) and (6.9) are

\[
G_V(s) = b_I^T V_r (sI - V_r^T F_1 V_r)^{-1} V_r^T b_I,
\]

\[
G_W(s) = b_I^T W_r (sI - W_r^T F_1 W_r)^{-1} W_r^T b_I,
\]

correspondingly. Then since

\[
(F_1 Q I - s_0 I)^{-1} = (S^{-1} F Q S - s_0 I)^{-1} = S^{-1} (F Q - s_0 I)^{-1} S,
\]

it follows that

\[
\text{span col } V_r = \text{span col } R_r((F_1 Q I - s_0 I)^{-1}, (F_1 Q I - s_0 I)^{-1} b_I) = S^{-1} \text{span col } R_r((F Q - s_0 I)^{-1}, (F Q - s_0 I)^{-1} b).
\]

(6.11)

Similarly it can be shown that

\[
\text{span col } W_r = S^{-1} \text{span col } R_r((F^T Q - s_0 I)^{-1}, (F^T Q - s_0 I)^{-1} b).
\]

Therefore, the condition (6.10) is equivalent to

\[
\text{span col } W_r = \text{span col } V_r.
\]

Hence, under the condition (6.10)

\[
G_V(s) = G_W(s),
\]

showing the equivalence of the reduced order models (6.7) and (6.9). \(\Box\)
6.3. Reduction by the rational Arnoldi method

A different yet structure preserving approach to model reduction of port-Hamiltonian systems is considered in [57], where the reduced order moment matching port-Hamiltonian model is defined as

\[
\begin{align*}
\dot{x}_m &= (J_m - R_m)Q_m x_m + b_m u, \\
y_m &= b_m^T Q_m x_m,
\end{align*}
\]  

(6.12)

with \( J_m = U^T Q J Q U, \) \( R_m = U^T Q R Q U, \) \( b_m = U^T Q b, \) \( c_m = b_m^T Q_m, \) together with a reduced order energy matrix \( Q_m \) which, in general, is not an identity matrix: \( Q_m = (U^T Q U)^{-1}, \) and reduced order state \( x_m. \) The projection matrix \( U \) from (6.12) used in [57] is different from \( V_r \) used in Theorem 6.5 and is such that

\[
\text{span col } U = \text{span col } R_r ((J - R) - s_0 I)^{-1}, ((J - R) - s_0 I)^{-1} b).
\]  

(6.13)

The advantage of constructing the reduced order port-Hamiltonian model as in (6.12) is that when \( Q \) is not diagonal in most cases it will be more efficient from numerical point of view to compute the inverse of the \( r \times r \) matrix \( (U^T Q U)^{-1}, \) than the coordinate transformation \( S \) transforming the \( n \times n \) matrix \( Q \) to identity.

In fact, the transfer function \( G_m(s) \) of the reduced order model (6.12) from [57] can be shown to be equal to the transfer function \( G_V(s) \) of (6.7).

**Theorem 6.7.** The reduced order port-Hamiltonian model (6.7) with the transfer function \( G_V(s) \) obtained using the projection map \( V_r \) and the reduced order port-Hamiltonian model (6.12) from [57] obtained using the projection map \( U \) as in (6.13) are equivalent in the sense of sharing the same transfer function:

\[
G_V(s) = G_m(s).
\]  

(6.14)

**Proof.** Firstly, let us derive the relation between the projection maps \( V_r \) and \( U. \) Recall from (6.11) that

\[
\text{span col } V_r = S^{-1}\text{span col } R_r ((FQ - s_0 I)^{-1}, (FQ - s_0 I)^{-1} b),
\]

and therefore \( V_r \) and \( U \) are related as

\[
V_r = S^{-1} U.
\]

Two situations may arise. If \( U^T U = I, \) or the columns of \( U \) are orthonormal, then \( V_r^T V_r = U^T S^{-T} S^{-1} U = U^T Q U \neq I \) and the columns of \( V_r \) are not orthonormal. In this case the transfer function \( G_V(s) \) is

\[
\begin{align*}
G_V(s) &= b_I^T V_r (s V_r^T V_r - V_r^T F V_r)^{-1} V_r^T b_I \\
&= b^T S^{-T} S^{-1} U (s U^T S^{-T} S^{-1} U - U^T S^{-T} S^{-1} F S^{-T} S^{-1} U)^{-1} U^T S^{-T} S^{-1} b, \\
&= b^T Q U (s U^T Q U - U^T Q F Q U)^{-1} U^T Q b,
\end{align*}
\]

85
while the transfer function $G_m(s)$ takes the form

$$G_m(s) = b_m^T Q_m(sI - F_m Q_m)^{-1} b_m = b^T QU(U^T QU)^{-1}(sI - U^T QFQU(U^T QU)^{-1})^{-1}U^T Qb.$$ 

Taking $(U^T QU)^{-1}$ inside the bracket yields $G_V(s) = G_m(s)$.

On the other hand, if $U^T QU = I$, which is equivalent to the orthonormality of the columns of $V_r$, then

$$G_m(s) = b^T QU(sI - U^T QFQU)^{-1}U^T Qb = G_V(s),$$

which completes the proof. \qed

\section{Co-energy coordinates, transforming $Q$ to the identity matrix}

In the analogous manner we can scale the port-Hamiltonian system in co-energy coordinates (1.17)

$$\begin{align*}
\dot{e} &= Q(J - R)e + Qbu, \\
y &= b^T e,
\end{align*}$$

transforming the energy matrix $Q$ to the identity matrix using the coordinate transformation $T : e = Te_1$, as in (5.14). Further projection can be done as shown in Theorem 6.5 using both projection maps $V_r$ and $W_r$ based on the shifted input and output Krylov subspaces respectively. Similar arguments as in Theorem 6.6 show that reduced order port-Hamiltonian systems in co-energy coordinates are equivalent as well.

Another way to reduce port-Hamiltonian systems in co-energy coordinates is shown in Chapter 5 and [69], [73], where instead of transforming the energy matrix $Q$ we take it on the left-hand side of the differential equation

$$\begin{align*}
Q^{-1}\dot{e} &= (J - R)e + bu, \\
y &= b^T e,
\end{align*}$$

which allows for further projection with the projection map $V_r$ such that $V_r^T Q^{-1} V_r = I_r$. Or, in more general case, $V_r^T Q^{-1} V_r = \hat{Q}_r^{-1}$. The reduced order models obtained in this way will be equivalent to the previously obtained reduced order port-Hamiltonian model in co-energy coordinates as well as to (6.7). In fact, under the algebraic condition (6.10) all the reduced order port-Hamiltonian models obtained using the projection maps $V_r, W_r$ based on the shifted input and output Krylov subspaces both in energy and co-energy coordinates will be equivalent sharing the same transfer function. The proof of this fact is similar to the proofs of Theorems 5.13 and 6.6.
6.4. Reduction of port-Hamiltonian systems by the rational Lanczos method

According to Theorem 6.4 the rational Lanczos method preserves 2r moments of the full order system at \( s_0 \in \mathbb{C} \). The problem in the port-Hamiltonian case is that the projection of the full order system to the lower dimensional subspace is two-sided, and, therefore in general the reduced order system is not port-Hamiltonian anymore. Indeed, a reduced order model would take the following form

\[
\tilde{A} = W_r^T(J - R)QV_r, \quad \tilde{b} = W_r^Tb, \quad \tilde{c} = b^TQV_r,
\]

which is in general not port-Hamiltonian.

In this section we show how to overcome this difficulty imposing a condition which characterizes a class of port-Hamiltonian systems for which the rational Lanczos algorithm not only preserves 2r moments at \( s_0 \in \mathbb{C} \) but also produces a port-Hamiltonian reduced order model. In fact, this condition is precisely the algebraic condition given in (6.10).

**Theorem 6.8.** Consider a full order port-Hamiltonian system (6.4) and define \( V_r \) satisfying property (ii) of (6.2) such that \( V_r^TQV_r = I_r \). Then the \( r^{th} \) order reduced system

\[
\begin{cases}
\dot{x} = (\tilde{J} - \tilde{R})x + bu, \\
\tilde{y} = \tilde{c}^T\tilde{x}
\end{cases}
\]

(6.16)

with the interconnection matrices \( \tilde{J}_I, \tilde{b}_I, \) energy matrix \( \tilde{Q}_I, \) dissipation matrix \( \tilde{R}_I \) and output matrix \( \tilde{c}_I \) given as

\[
\begin{align*}
\tilde{J} &= V_r^TQJQV_r, \\
\tilde{R} &= V_r^TQRQV_r, \\
\tilde{Q} &= I, \\
\tilde{b} &= V_r^TQb, \\
\tilde{c} &= b^TQV_r,
\end{align*}
\]

is a port-Hamiltonian system reduced by the rational Lanczos method with \( W_r = QV_r \) if the condition (6.10) holds true. Furthermore the first 2r moments at \( s_0 \in \mathbb{C} \) of the reduced order port-Hamiltonian system (6.16) and the full order port-Hamiltonian system (6.4) are equal:

\[
\tilde{\eta}_i(s_0) = \eta_i(s_0), \quad i = 0, \ldots, 2r - 1.
\]

**Proof.** In this case the subspaces spanned by the columns of \( V_r, W_r \) can be represented as follows:

\[
\begin{align*}
\text{span } \text{col } V_r &= \text{span } \text{col } \mathcal{R}_r((FQ - s_0I)^{-1}, (FQ - s_0I)^{-1}b); \\
\text{span } \text{col } W_r &= \text{span } \text{col } \mathcal{R}_r((QF^T - s_0I)^{-1}, (QF^T - s_0I)^{-1}Qb) \\
&= \text{span } \text{col } \mathcal{R}_r(Q(F^TQ - s_0I)^{-1}Q^{-1}, Q(F^TQ - s_0I)^{-1}b) \\
&= Q \text{span } \text{col } \mathcal{R}_r((F^TQ - s_0I)^{-1}, (F^TQ - s_0I)^{-1}b).
\end{align*}
\]
6. Structure preserving moment matching at an arbitrary point

Then the condition (6.10) implies that

$$\text{span } \text{col } W_r = \text{span } \text{col } Q V_r.$$ 

Therefore one can choose any $W_r$ such that its columns span the same subspace as the columns of $Q V_r$. In particular, taking $W_r$ as $Q V_r$ preserves the port-Hamiltonian structure for the reduced order model. Preservation of $2r$ moments at $s_0$ follows directly from Theorem 6.4 completing the proof. \(\square\)

This scheme of model reduction using the rational Lanczos method works as well in co-energy coordinates resulting in the reduced order port-Hamiltonian model which is equivalent to (6.16).

The next result establishes a relation between the reduced order port-Hamiltonian models obtained by both the rational Arnoldi and the rational Lanczos methods.

**Theorem 6.9.** The reduced order port-Hamiltonian model (6.7) in energy coordinates obtained by the rational Arnoldi method and the reduced order port-Hamiltonian model (6.16) in energy coordinates obtained by the rational Lanczos method share the same transfer function if the condition (6.10) is satisfied.

**Proof.** The proof is similar to the proof of Theorem 6.6, hence omitted. \(\square\)

**Corollary 6.10.** A natural conclusion of Theorem 6.9 is that a reduced order port-Hamiltonian model (6.7) obtained by the rational Arnoldi method matches $r$, and provided (6.10) is satisfied, $2r$ moments of the original system (6.4) at $s_0 \in \mathbb{C}$.

Note that for an important point $s_0 = 0$ the condition (6.10) specializes to

$$\text{span } \text{col } R_r(F^{-1}Q^{-1}, F^{-1}b) = \text{span } \text{col } R_r(F^{-T}Q^{-1}, F^{-T}b). \quad (6.17)$$

### 6.5. Numerical example

Consider an $n$-dimensional mass-spring-damper system as shown in Fig. 6.1 with masses $m_i$ and spring constants $k_i$, for $i = 1, \ldots, n/2$. This system is an extension of the mass-spring-damper system given in Example 1.7 with a subtle difference that a damper with a damping constant $c_d \geq 0$ is attached
only to the first mass $m_1$. $p_i$ and $q_i$ are the momentum and displacement of the mass $m_i$, respectively. The external force acting on the first mass, $m_1$, is the input $u$, while its velocity is the output, $y$. State variables are defined in the following way: for $i = 1, \ldots, n/2$, $x_{2i-1} = q_i$ and $x_{2i} = p_i$.

A minimal realization of this system for order $n = 6$ (corresponding to three masses with one damper and three springs) is

$$
\begin{align*}
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -1 & 0
\end{bmatrix}
\end{align*}

$$

$$
\begin{align*}
b = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}^T, \\
c = \begin{bmatrix} 0 & \frac{1}{m_1} & 0 & 0 & 0 \end{bmatrix}, \\
R = diag(0, c_d, 0, 0, 0, 0),
\end{align*}

$$

$$
J = \begin{bmatrix}
k_1 & 0 & -k_1 & 0 & 0 & 0 \\
0 & \frac{1}{m_1} & 0 & 0 & 0 & 0 \\
-k_1 & 0 & k_1 + k_2 & 0 & -k_2 & 0 \\
0 & 0 & 0 & \frac{1}{m_2} & 0 & 0 \\
0 & 0 & -k_2 & 0 & k_2 + k_3 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{m_3}
\end{bmatrix}

$$

$$
Q = \begin{bmatrix}
0 & \frac{1}{m_1} & 0 & 0 & 0 & 0 \\
-k_1 & -\frac{1}{m_1} & k_1 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{m_2} & 0 & 0 \\
k_1 & 0 & -k_1 - k_2 & 0 & k_2 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{m_3} \\
0 & 0 & k_2 & 0 & -k_2 - k_3 & 0
\end{bmatrix}

$$

Then the $A$ matrix of this model is given as

$$
A = (J - R)Q = \begin{bmatrix}
0 & \frac{1}{m_1} & 0 & 0 & 0 & 0 \\
-k_1 & -\frac{1}{m_1} & k_1 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{m_2} & 0 & 0 \\
k_1 & 0 & -k_1 - k_2 & 0 & k_2 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{m_3} \\
0 & 0 & k_2 & 0 & -k_2 - k_3 & 0
\end{bmatrix}

$$

Adding another mass with a spring would increase the dimension of the system by two. In that case the main diagonal of the matrix $A$ will obtain zeros in the $(n-1, n-1)$ and $(n, n)$ positions. The superdiagonal of $A$ will have $k_{n/2-1}$ in the $(n-2, n-1)$ position and $1/m_{n/2}$ in the $(n-1, n)$ position. The subdiagonal of $A$ will obtain 0 in the $(n-1, n-2)$ position and $-k_{n/2-1} - k_{n/2}$ in the $(n, n-1)$ position. Additionally $A$ will have $k_{n/2-1}$ in the $(n, n-3)$ position.

We considered a 100-dimensional mass-spring-damper system with $m_i = 2$, $k_i = 1$, and $c_d = 1$. We applied the rational Arnoldi method as shown in
6. Structure preserving moment matching at an arbitrary point

Theorem 6.5 with the approximation point $s_0 = 0$. The reduced order systems are constructed for the order from $r = 2$ to $r = 30$ with increments of 2. Evolution of the relative $\mathcal{H}_2$- and $\mathcal{H}_\infty$-norms is shown in Fig. 6.2. The $\mathcal{H}_2$ relative norm decays as the dimension of the reduced order system $r$ increases whereas the $\mathcal{H}_\infty$-norm is almost constant. This implies that the reduced order system is closer to the full order one in the $\mathcal{H}_2$-norm as $r$ growths. Reduced order systems inherit the port-Hamiltonian structure, are asymptotically stable and passive.

The magnitude Bode plots of the full, reduced order for $r = 10$, and error systems are shown in Fig. 6.3. The figure exhibits that the approximation is very accurate for small frequencies which is to be expected since the moments are matched at $s_0 = 0$. The magnitude plot of the reduced order system captures first peaks and zeros of that of the full order system. The error plot demonstrates that the error is accumulated for high frequencies. This is to be predicted since the model reduction scheme used preserves at least the first $r$ moments of the full order transfer function at zero.

In fact, for the mass-spring-damper system considered here the condition (6.17) is satisfied. Therefore even though the reduced order port-Hamiltonian model is obtained using the rational Arnoldi method as shown in Theorem
6.6. Conclusions

In this chapter we applied the rational Krylov methods in order to reduce a full order port-Hamiltonian system to a reduced order system which inherits the port-Hamiltonian structure and preserves a specific number of moments at an arbitrary point in the complex plane. In particular, we showed how the rational Arnoldi method, which preserves $r$ moments, can be employed for this purpose in energy and co-energy coordinates using the projection maps constructed both on the shifted input and output Krylov subspaces. We proved that for a particular class of port-Hamiltonian systems characterized by an algebraic condition all reduced order port-Hamiltonian models obtained by the rational Arnoldi method share the same transfer function.

Remark 6.11. Note that an $n$-dimensional mass-spring-damper system with dampers attached to each mass, as shown in Example 1.7, would not belong to the subclass of port-Hamiltonian systems (6.17).

Figure 6.3.: Amplitude Bode plots for $r = 10$

6.5, it is equivalent to that of the rational Lanczos method as Theorem 6.8 explains. Moreover, due to Corollary 6.10 the obtained reduced order port-Hamiltonian models preserve $2r$ moments at zero, which can be readily checked for the case when $r$, for instance, is equal to 2:

$$\begin{pmatrix} \eta_1(0) & \ldots & \eta_{2r}(0) \end{pmatrix} = \begin{pmatrix} 0 & -50 & 2500 & -39150 \end{pmatrix} = \begin{pmatrix} \tilde{\eta}_1(0) & \ldots & \tilde{\eta}_{2r}(0) \end{pmatrix}.$$
6. Structure preserving moment matching at an arbitrary point

We exploited the rational Lanczos method, which preserves $2r$ moments at an arbitrary point in the complex plane, for structure preserving model reduction of port-Hamiltonian systems. The rational Lanczos method can be applied in a structure preserving way only to a subclass of port-Hamiltonian systems characterized by an algebraic condition. For this subclass of systems the rational Lanczos method is proven to produce a reduced order port-Hamiltonian model, which is equivalent to that of the rational Arnoldi method. Therefore the rational Arnoldi method, applied to a port-Hamiltonian system from the subclass, preserves twice as many moments at an arbitrary point in the complex plane as it does for a general linear system.

Both considered methods preserve the port-Hamiltonian structure, implying, among others, the passivity property, and, therefore, stability.

The results obtained in this chapter seem to allow for an extension to the MIMO case, while the considered algebraic condition requires a description from physical or/and system-theoretic point of view. These questions as well as the numerical efficiency of the considered methods are recommended for future work.