4. Positive real balancing for port-Hamiltonian systems

Positive real balancing is used in this chapter for model reduction of port-Hamiltonian systems.

4.1. Introduction

In this chapter we use a well-known passivity preserving balancing technique, positive real balancing, to reduce port-Hamiltonian systems with the preservation of the port-Hamiltonian structure. In Section 4.2 we discuss the theoretical aspects of positive real balancing. A set of positive real characteristic values is associated with port-Hamiltonian systems as dissipative systems. Physical examples of port-Hamiltonian systems, for which the minimal and the maximal storage functions are equal, are presented in Section 4.3. In Section 4.4 we apply positive real balancing to obtain families of the reduced order models which inherit the port-Hamiltonian structure in both energy and co-energy coordinates.

4.2. Positive real balancing

We start with a definition of passivity for a general linear multi-input multi-output dynamical system

\[
\begin{align*}
\dot{x} &= Ax + Bu, \\
y &= Cx + Du.
\end{align*}
\]

(4.1)

Definition 4.1. [93] The linear system in (4.1) is said to be dissipative with respect to the supply rate \(u^T y\), or passive, if there exists a nonnegative storage function \(S(x)\) such that the dissipation inequality

\[
S(x(t_1)) - S(x(t_0)) \leq \int_{t_0}^{t_1} u(t)^T y(t) dt
\]

(4.2)

holds for all \(t_0 \leq t_1\) and for all trajectories \((u, x, y)\) satisfying the system equations (4.1).
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For a passive system \((4.1)\) the continuum of storage functions (see \([93]\)) is bounded by two universal functions

\[
S_a(x) \leq S(x) \leq S_r(x),
\]

namely, the available storage

\[
S_a(x_0) = \sup_u \left\{ -\int_0^\infty u(t)^T y(t) dt, \; x(0) = x_0, \; x(\infty) = 0, \right. \]

\[
\left. \quad (u, x, y) \text{ satisfying (4.1)} \right\}
\]

which is the maximal storage that can be extracted from the system starting at a fixed initial state, and the required supply

\[
S_r(x_0) = \inf_u \left\{ \int_{-\infty}^0 u(t)^T y(t) dt, \; x(-\infty) = 0, \; x(0) = x_0, \right. \]

\[
\left. \quad (u, x, y) \text{ satisfying (4.1)} \right\}
\]

which is the minimal storage required to achieve a fixed initial state.

It is a well-known fact that for a general minimal linear system with quadratic storage functions \(\frac{1}{2}x^T K x, \; K = K^T > 0\), the passivity property is equivalent (see \([93], [87]\)) to the Kalman-Yakubovich-Popov Linear Matrix Inequality (LMI)

\[
\begin{bmatrix}
A^T K + KA & KB - C^T \\
B^T K - C & -D - D^T
\end{bmatrix} \leq 0,
\]

(4.5)

with the minimal solution \(K_a\) and the maximal solution \(K_r\)

\[
0 < K_a \leq K \leq K_r,
\]

(4.6)

which are symmetric matrices satisfying

\[
S_a(x) = \frac{1}{2}x^T K_a x, \quad S_r(x) = \frac{1}{2}x^T K_r x.
\]

(4.7)

The LMI \((4.5)\) specializes in the lossless case, studied in \([87]\), to

\[
A^T K + KA = 0, \quad C = B^T K, \quad D = -D^T,
\]

(4.8)

and in the passive case with \(D = 0\) to the LMI

\[
A^T K + KA \leq 0, \quad C = B^T K,
\]

(4.9)

which is of interest in this chapter.
4.2. Positive real balancing

**Remark 4.2.** The minimal solution $K_a$ and the maximal solution $K_r$ of the LMI (4.5) can be also found by solving the following Algebraic Riccati Equation (ARE) (see [65], [39])

$$A^T K + KA + (KB - C^T)(D + D^T)^{-1}(KB - C^T)^T = 0$$

under the condition that $D + D^T > 0$.

Having found $K_a, K_r$, one can compute the so-called positive real characteristic values $\sigma_i$, as the square roots of the eigenvalues of the product $K_aK_r^{-1}$ which are invariant under state-space coordinate transformations and satisfy

$$1 \geq \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0,$$

as shown in [23], [65], [39].

Model reduction using positive real balancing involves transforming the original system (4.1) to positive real balanced coordinates, defined in [23], [65], [39], where

$$K_a = K_r^{-1} = \text{diag} (\sigma_1, \sigma_2, \ldots, \sigma_n) = \begin{bmatrix} K_{11} & 0 \\ 0 & K_{22} \end{bmatrix},$$

$$K_{11} = \text{diag} (\sigma_1, \sigma_2, \ldots, \sigma_r),$$

$$K_{22} = \text{diag} (\sigma_{r+1}, \ldots, \sigma_n),$$

$$1 \geq \sigma_1 \geq \cdots \geq \sigma_r \geq \sigma_{r+1} \geq \cdots \geq \sigma_n > 0,$$

and leaving out certain state components. These state components correspond to the most of the internal energy dissipation and the smallest positive real characteristic values $\sigma_{r+1} \ldots \sigma_n$, for $r$ being the dimension of the reduced order system.

**Remark 4.3.** Positive real characteristic values $\sigma_i$ may have multiplicities $m_i$, $i = 1, \ldots, q$, different from 1, where $q$ is the number of distinct $\sigma_i$. Then the first line in (4.10) reads

$$K_a = K_r^{-1} = \text{diag} (\sigma_1 I_{m_1}, \sigma_2 I_{m_2}, \ldots, \sigma_q I_{m_q}), \quad m_1 + m_2 + \ldots + m_q = n.$$

After splitting the state vector $x^b$ in positive real balanced coordinates the full order system is

$$\begin{cases}
\begin{bmatrix}
\dot{x}_1^b \\
\dot{x}_2^b
\end{bmatrix} =
\begin{bmatrix}
A_{11}^b & A_{12}^b \\
A_{21}^b & A_{22}^b
\end{bmatrix}
\begin{bmatrix}
x_1^b \\
x_2^b
\end{bmatrix} +
\begin{bmatrix}
B_1^b \\
B_2^b
\end{bmatrix} u,
\end{cases}
\quad (4.11)
$$

$$y = \begin{bmatrix} C_1^b & C_2^b \end{bmatrix} \begin{bmatrix} x_1^b \\
x_2^b
\end{bmatrix} + Du.$$
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where $A_{11}^b \in \mathbb{R}^{r \times r}, B_1^b \in \mathbb{R}^{r \times m}, C_1^b \in \mathbb{R}^{m \times r}$ and the rest of the matrices are of corresponding dimensions. As explained in [23], [39], if the original system (4.1) is asymptotically stable, minimal and passive, then the reduced order system

$$\begin{align*}
\dot{x}_1^b &= A_{11}^b x_1^b + B_1^b u, \\
\hat{y} &= C_1^b x_1^b + Du,
\end{align*}$$

(4.12)

obtained after truncating the system (4.11), is asymptotically stable, minimal and passive.

**Remark 4.4.** Error bounds for positive real balanced truncation are derived in [65], [39], [18], [95]. Positive real balancing in a behavioral framework, as well as other model reduction methods, was studied in [61]. In a nonlinear setting positive real balancing was addressed in [48] and the references therein.

We will concentrate on the LMI (4.9), since the port-Hamiltonian systems in energy coordinates (1.16)

$$\begin{align*}
\dot{x} &= (J - R)Qx + Bu, \\
y &= B^T Qx,
\end{align*}$$

(4.13)

and in co-energy coordinates (1.17)

$$\begin{align*}
\dot{\varepsilon} &= Q(J - R)\varepsilon + QBu, \\
y &= B^T \varepsilon,
\end{align*}$$

(4.14)

have no feed-through term $D$.

For port-Hamiltonian systems a natural choice for the storage function is the Hamiltonian or the physical energy of the system

$$\begin{align*}
\frac{1}{2} x^T K_a x &= S_a(x) \leq H(x) = \frac{1}{2} x^T Q x \leq S_r(x) = \frac{1}{2} x^T K_r x.
\end{align*}$$

(4.15)

In fact there are cases when $K_a = Q = K_r$, which demonstrates that we need not have strict inequalities in (4.15), (4.6), even for systems with non-zero internal energy dissipation, as the following examples show.

### 4.3. Physical examples

**Example 4.5.** [87] Consider a single mass-spring-damper system in Fig. 4.1 with the system equations

$$\begin{align*}
\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} &= \begin{bmatrix} 0 & \frac{1}{m} \\ -k & -\frac{1}{m} \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad u = \text{force}, \\
y &= \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} = \text{velocity},
\end{align*}$$

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physical energy \( H(q,p) = \frac{1}{2m}p^2 + \frac{1}{2}kq^2 \), and internal energy dissipation corresponding to the damper with the damping coefficient \( c > 0 \). The port-Hamiltonian representation (4.13) for this system is

\[
\begin{align*}
\dot{q} &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} q - \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix} p + \begin{bmatrix} k & 0 \\ 0 & \frac{1}{m} \end{bmatrix} q + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \\
y &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} k & 0 \\ 0 & \frac{1}{m} \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix}.
\end{align*}
\]

The LMI (4.9) takes the form

\[
\begin{bmatrix} 0 & -k \\ \frac{1}{m} & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{m} \\ -k & -\frac{c}{m} \end{bmatrix} \leq 0
\]

\[
\begin{bmatrix} 0 & 1 \\ k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{m} \end{bmatrix}.
\]

The last equation yields \( k_{12} = 0 \) as well as \( k_{22} = \frac{1}{m} \). Substituting this in the inequality yields a unique solution \( k_{11} = k \), corresponding to the unique storage function \( H(q,p) = \frac{1}{2m}p^2 + \frac{1}{2}kq^2 \), which is equal to \( S_a \) and \( S_r \): \( S_a(x) = H(x) = S_r(x) \).
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![Ladder network diagram](image)

Figure 4.2.: Ladder network

**Example 4.6.** Consider the two-dimensional ladder network in Fig. 4.2, which is similar to the ladder network from Example 1.5, with the system equations

\[
\begin{align*}
\begin{bmatrix}
\dot{q} \\
\dot{\phi}
\end{bmatrix} &= 
\begin{bmatrix}
\frac{1}{CR} & -\frac{1}{L} \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
q \\
\phi
\end{bmatrix} + 
\begin{bmatrix}
\frac{1}{R} \\
0
\end{bmatrix} u, \quad u = \text{voltage} \\
y &= 
\begin{bmatrix}
\frac{1}{CR} & 0
\end{bmatrix}
\begin{bmatrix}
q \\
\phi
\end{bmatrix}.
\end{align*}
\]

\(x = [q \phi]^T\) is the state vector with \(q\) the charge of the capacitor \(C\) and \(\phi\) the flux of the inductor \(L\) respectively. The input \(u\) is the voltage from an external voltage source and the port-Hamiltonian output \(y\) has the physical dimension of a current and given as \(y = U_C/R\), where \(U_C\) is the voltage over the capacitor. The Hamiltonian of this system is \(H(q, \phi) = \frac{1}{2C}q^2 + \frac{1}{2L}\phi^2\), and internal energy dissipation corresponds to the resistor with the resistance \(R > 0\). The port-Hamiltonian representation (4.13) for this system is

\[
\begin{align*}
\begin{bmatrix}
\dot{q} \\
\dot{\phi}
\end{bmatrix} &= 
\begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
q \\
\phi
\end{bmatrix} - 
\begin{bmatrix}
\frac{1}{R} \\
0
\end{bmatrix} u, \\
y &= 
\begin{bmatrix}
\frac{1}{R} & 0
\end{bmatrix}
\begin{bmatrix}
q \\
\phi
\end{bmatrix}.
\end{align*}
\]

Similarly to Example 4.5 the solution to the LMI (4.9) for this system

\[
\begin{bmatrix}
\frac{1}{CR} & \frac{1}{L} \\
-\frac{1}{L} & 0
\end{bmatrix}
\begin{bmatrix}
k_{11} & k_{12} \\
k_{12} & k_{22}
\end{bmatrix} + 
\begin{bmatrix}
k_{11} & k_{12} \\
k_{12} & k_{22}
\end{bmatrix}
\begin{bmatrix}
\frac{1}{CR} & -\frac{1}{L} \\
0 & \frac{1}{L}
\end{bmatrix} \leq 0,
\]

\[
\begin{bmatrix}
\frac{1}{R} & 0 \\
0 & \frac{1}{R}
\end{bmatrix}
\begin{bmatrix}
k_{11} & k_{12} \\
k_{12} & k_{22}
\end{bmatrix} = 
\begin{bmatrix}
\frac{1}{RC} & 0 \\
0 & \frac{1}{RC}
\end{bmatrix},
\]

is a single matrix

\[
K_a = Q = K_r = \begin{bmatrix}
\frac{1}{C} & 0 \\
0 & \frac{1}{L}
\end{bmatrix}
\]

implying the uniqueness of the storage function \(S_a(x) = H(x) = S_r(x)\).

**Remark 4.7.** In Example 4.6 there is an obvious misuse of notation of \(R, C\).
In Examples 4.5 and 4.6 $K_a = K_r$ both in energy (4.13) and co-energy (4.14) coordinates. In general, the question (as posed in [93]) under what conditions $K_a = K_r$ for a finite dimensional linear dynamical system with non-zero internal energy dissipation is still open to the author’s knowledge.

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In the port-Hamiltonian case the LMI (4.9) becomes

$$Q(J^T - R)K^x + K^x(J - R)Q \preceq 0, \quad B^T Q = B^T K^x$$

(4.16)

in energy coordinates (4.13) and

$$(J^T - R)QK^e + K^e Q(J - R) \preceq 0, \quad B^T = B^T QK^e$$

(4.17)

in co-energy coordinates (4.14).

The next result shows the relation between solutions of the LMI’s (4.16) and (4.17)

**Proposition 4.8.** For any solution $K^e$ of the LMI (4.16) in energy coordinates, $K^e$, given by

$$K^e = Q^{-1}K^x Q^{-1},$$

(4.18)

is a solution of the LMI (4.17) in co-energy coordinates.

**Proof.** Premultiplying and postmultiplying the LMI (4.16) with $Q^{-1}$ yields

$$(J^T - R)Q Q^{-1}K^x Q^{-1} + Q^{-1}K^x Q^{-1}Q(J - R) \preceq 0, \quad B^T = B^T Q Q^{-1}K^e Q^{-1},$$

which is exactly the LMI (4.17) with solution $K^e$.

**Example 4.9.** One can easily compute solution to the LMI (4.17) for the electrical network from Example 4.6 in co-energy coordinates:

$$K^e = \begin{bmatrix} C & 0 \\ 0 & L \end{bmatrix},$$

which, since $Q = \begin{bmatrix} \frac{1}{C} & 0 \\ 0 & \frac{1}{L} \end{bmatrix}$, demonstrates relation (4.18).

After finding $K_a \leq K_r$ from the LMI (4.16) we can apply a balancing coordinate transformation $S$, $x = Sx^b$, to the system (4.13) such that in positive real balanced coordinates $x^b$ the port-Hamiltonian system (4.13) (with no feed-through) takes the form
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\[
\begin{align*}
\begin{bmatrix}
\dot{x}_1^b \\
\dot{x}_2^b
\end{bmatrix} &= \begin{bmatrix}
A_{11}^b & A_{12}^b \\
A_{21}^b & A_{22}^b
\end{bmatrix}
\begin{bmatrix}
x_1^b \\
x_2^b
\end{bmatrix} + \begin{bmatrix}
B_1^b \\
B_2^b
\end{bmatrix} u, \\
y &= \begin{bmatrix}
C_1^b & C_2^b
\end{bmatrix}
\begin{bmatrix}
x_1^b \\
x_2^b
\end{bmatrix},
\end{align*}
\]

(4.19)

where \(A_{11}^b \in \mathbb{R}^{r \times r}, B_1^b \in \mathbb{R}^{r \times m}, C_1^b \in \mathbb{R}^{m \times r}\) and the rest of the matrices are of corresponding dimensions.

**Remark 4.10.** Even though \(K_a, K_r\) are diagonal in positive real balanced coordinates, the non-unique energy matrix \(Q^b\) which satisfies \(Q^b = S^T QS, K_a \leq Q_b \leq K_r\), need not be diagonal itself.

Truncating the system (4.19) leads to the following result

**Theorem 4.11.** The reduced order system

\[
\begin{align*}
\dot{\hat{x}} &= \hat{A}\hat{x} + \hat{B}u, \\
\hat{y} &= \hat{C}\hat{x},
\end{align*}
\]

(4.20)

of the system (4.13) with

\[
\hat{A} = A_{11}^b = (\hat{J} - \hat{R})\hat{Q}, \\
\hat{Q} = \begin{cases} 
K_{11}, \\
K_{11}^{-1}, \\
\text{any diagonal } \hat{K} \text{ satisfying } 0 < K_{11} \leq \hat{K} \leq K_{11}^{-1},
\end{cases} \\
\hat{J} = \frac{1}{2}(\hat{A}\hat{Q}^{-1} - \hat{Q}^{-1}\hat{A}^T), \\
\hat{R} = -\frac{1}{2}(\hat{A}\hat{Q}^{-1} + \hat{Q}^{-1}\hat{A}^T), \\
\hat{B} = B_1^b, \\
\hat{C} = C_1^b,
\]

with \(K_{11}\) defined as in (4.10), is port-Hamiltonian, minimal and passive. Moreover \(K_{11}\) and \(K_{11}^{-1}\) are the available storage and the required supply for the system (4.20) respectively.

**Proof.** In positive real balanced coordinates \(x^b\) the LMI (4.16) reads

\[
\begin{bmatrix}
A_{11}^b & A_{12}^b \\
A_{21}^b & A_{22}^b
\end{bmatrix}^T K + K \begin{bmatrix}
A_{11}^b & A_{12}^b \\
A_{21}^b & A_{22}^b
\end{bmatrix} \leq 0, \quad \begin{bmatrix}
C_1^b & C_2^b
\end{bmatrix} = \begin{bmatrix}
B_1^b \\
B_2^b
\end{bmatrix}^T K. 
\]

(4.22)

Since \(K_a, K_r\) from (4.10) satisfy LMI (4.22) it follows that

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\[
\begin{bmatrix}
A_{b11} & A_{b12} \\
A_{b21} & A_{b22}
\end{bmatrix}^T
\begin{bmatrix}
K_{11} & 0 \\
0 & K_{22}
\end{bmatrix}
+ \begin{bmatrix}
K_{11} & 0 \\
0 & K_{22}
\end{bmatrix}
\begin{bmatrix}
A_{b11} & A_{b12} \\
A_{b21} & A_{b22}
\end{bmatrix} \leq 0,
\]

\[
\begin{bmatrix}
C_{1}^b & C_{2}^b
\end{bmatrix}
= \begin{bmatrix}
B_{b1}^b & B_{b2}^b
\end{bmatrix}^T
\begin{bmatrix}
K_{11} & 0 \\
0 & K_{22}
\end{bmatrix},
\]

and

\[
\begin{bmatrix}
A_{b11} & A_{b12} \\
A_{b21} & A_{b22}
\end{bmatrix}^T
\begin{bmatrix}
K_{11}^{-1} & 0 \\
0 & K_{22}^{-1}
\end{bmatrix}
+ \begin{bmatrix}
K_{11}^{-1} & 0 \\
0 & K_{22}^{-1}
\end{bmatrix}
\begin{bmatrix}
A_{b11} & A_{b12} \\
A_{b21} & A_{b22}
\end{bmatrix} \leq 0,
\]

\[
\begin{bmatrix}
C_{1}^b & C_{2}^b
\end{bmatrix}
= \begin{bmatrix}
B_{b1}^b & B_{b2}^b
\end{bmatrix}^T
\begin{bmatrix}
K_{11}^{-1} & 0 \\
0 & K_{22}^{-1}
\end{bmatrix},
\]

which yields two LMIs for the reduced order system (4.20)

\[
(A_{11}^b)^T K_{11} + K_{11} A_{11}^b \leq 0, \quad C_{1}^b = (B_{11}^b)^T K_{11},
\]

and

\[
(A_{11}^b)^T K_{11}^{-1} + K_{11}^{-1} A_{11}^b \leq 0, \quad C_{1}^b = (B_{11}^b)^T K_{11}^{-1},
\]

showing that the reduced order system (4.20) is passive, \( \hat{A} = A_{11}^b \) is a stable matrix and \( K_{11} \) and \( K_{11}^{-1} \) are the available storage and the required supply for the system (4.20) respectively.

Since \( \hat{Q} \) is symmetric positive definite \( 0 < K_a \leq K \leq K_r \) implies \( 0 < K_{11} \leq \hat{Q} \leq K_{11}^{-1} \), and, therefore, the reduced order system (4.20) is minimal, \( \hat{J} \) is skew-symmetric:

\[
\hat{J} + \hat{J}^T = \frac{1}{2}(\hat{A}\hat{Q}^{-1} - \hat{Q}^{-1}\hat{A}^T + \hat{Q}^{-1}\hat{A}^T - \hat{A}\hat{Q}^{-1}) = 0,
\]

\( \hat{R} \) is symmetric and positive semi-definite:

\[
\hat{A}^T\hat{Q} + \hat{Q}\hat{A} \leq 0 \implies - (\hat{A}\hat{Q}^{-1} + \hat{Q}^{-1}\hat{A}^T) \geq 0,
\]

and since from (4.23), (4.24) \( \hat{C} = \hat{B}^T\hat{Q} \), system (4.20) is a port-Hamiltonian system.

\[\square\]

The reduced order system (4.20) represents, in fact, a family of reduced order port-Hamiltonian systems with the family of Hamiltonians

\[
H_a(\hat{x}) = \frac{1}{2} \hat{x}^T K_{11} \hat{x} \leq H(\hat{x}) = \frac{1}{2} \hat{x}^T \hat{Q} \hat{x} \leq \frac{1}{2} \hat{x}^T K_{11}^{-1} \hat{x} = H_r(\hat{x}).
\]

An analogous result can be established in co-energy coordinates. Consider a port-Hamiltonian system in co-energy coordinates (4.14). The partitioning
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of this system after positive real balancing reads

\[
\begin{bmatrix}
\dot{e}_1^b \\
\dot{e}_2^b
\end{bmatrix} =
\begin{bmatrix}
A_{11}^e & A_{12}^e \\
A_{21}^e & A_{22}^e
\end{bmatrix}
\begin{bmatrix}
e_1^b \\
e_2^b
\end{bmatrix} +
\begin{bmatrix}
B_1^e \\
B_2^e
\end{bmatrix} u,
\]

(4.25)

where \(A_{11}^e \in \mathbb{R}^{r \times r}\), \(B_1^e \in \mathbb{R}^{r \times m}\), \(C_1^e \in \mathbb{R}^{m \times r}\) and the rest of the matrices are of corresponding dimensions.

**Theorem 4.12.** The reduced order system

\[
\begin{bmatrix}
\dot{\hat{e}} \\
\hat{y}^e
\end{bmatrix} =
\begin{bmatrix}
\hat{A}^e & \hat{B}^e \\
\hat{C}^e & \hat{C}_2^e
\end{bmatrix}
\begin{bmatrix}
\hat{e}_1^b \\
\hat{e}_2^b
\end{bmatrix},
\]

(4.26)

of the system (4.14) with

\[
\begin{align*}
\hat{A}^e &= A_{11}^e = \hat{Q}^e (\hat{J} - \hat{R})^e, \\
\hat{Q}^e &= \begin{cases}
K_{11}^e, \\
(K_{11}^e)^{-1}, \\
\text{any diagonal } \hat{K}^e \text{ satisfying } 0 < K_{11}^e \leq \hat{K}^e \leq (K_{11}^e)^{-1},
\end{cases} \\
\hat{J} &= \frac{1}{2} ((\hat{Q}^e)^{-1} \hat{A}^e - (\hat{A}^e)^T (\hat{Q}^e)^{-1}), \\
\hat{R} &= -\frac{1}{2} ((\hat{Q}^e)^{-1} \hat{A}^e + (\hat{A}^e)^T (\hat{Q}^e)^{-1}), \\
\hat{B}^e &= B_1^e, \\
\hat{C}^e &= C_1^e.
\end{align*}
\]

(4.27)

with \(K_{11}^e\) defined as in (4.10) for the system (4.14), is port-Hamiltonian, minimal and passive. Moreover \(K_{11}^e\) and \((K_{11}^e)^{-1}\) are the available storage and the required supply for the system (4.26) respectively.

**Proof.** Proof is similar to proof of Theorem 4.11; hence omitted. \(\square\)

In the family of reduced order models (4.20) at least two energy matrices \(\hat{Q}\) are known: \(K_{11}^e\) and \((K_{11}^e)^{-1}\) leading to at least two known reduced order models. A similar argument is true for the family of reduced order models (4.26). The question of how to systematically characterize all reduced order models in the families (4.20), (4.26) is a subject for future research.

**Remark 4.13.** Note that the usual balanced truncation, as opposed to positive real balanced truncation used in this chapter, does not preserve the port-Hamiltonian structure. For details see Remark 2.12.
4.5. Conclusions

In this chapter we considered positive real balancing, which is a well-known passivity preserving balancing method. We showed that positive real balancing can be used in both energy and co-energy coordinates to reduce port-Hamiltonian systems with the preservation of the port-Hamiltonian structure and passivity. We obtained families of the port-Hamiltonian reduced order models in both energy and co-energy coordinates with at least two known reduced order models in each family. The questions of systematic characterization of the families of the reduced order models, the choice of the best reduced order model from the families to minimize a certain error measure, as well as the relation between the families of the reduced order models in energy and co-energy coordinates, are suggested for future research.