3. Reduced order Dirac structures and model reduction of port-Hamiltonian systems

In this chapter we use reduced order Dirac structures in order to introduce the effort- and flow-constraint reduction methods as structure preserving port-Hamiltonian reduction methods.

3.1. Introduction

The Dirac structure representation of port-Hamiltonian systems motivates a model reduction scheme which involves the construction of a reduced order Dirac structure. This scheme results in a family of structure preserving reduction methods for general port-Hamiltonian systems (possibly also including the algebraic constraints). In this chapter we treat this family of four structure preserving port-Hamiltonian reduction methods and concentrate on two of them, namely, the effort-constraint reduction method (seen already in Chapter 2) and the flow-constraint reduction method. We explain that these two methods are well motivated not only by the power conservation point of view and by the Dirac structure representation of port-Hamiltonian systems, but also by the bond-graph modeling framework. We show how the effort-constraint method in suitable coordinates is related to the projection-based model reduction methods. We suggest these coordinates for the effort-constraint method and balanced coordinates for both the effort- and flow-constraint methods as a possible choice of the coordinate system in order to obtain the reduced order models.

In Section 3.2 we explain the idea behind structure preserving model reduction based on power conservation. Equational representations of the reduced order models are given in Section 3.3. These equational representations give rise to the effort- and flow-constraint reduced models for linear port-Hamiltonian systems in Section 3.4.

The results of this chapter are reported in [68].
3. Model reduction using reduced order Dirac structures

3.2. Structure preserving model reduction based on power conservation

Consider a general port-Hamiltonian system, with state variables \( x \) and total stored energy \( H(x) \). Let us assume that we have been able to find (e.g. by some balancing technique) a splitting of the state-space variables \( x = (x_1^T, x_2^T)^T, x_1 \in \mathbb{R}^r, x_2 \in \mathbb{R}^{n-r} \), having the property that the \( x_2 \) coordinates hardly contribute to the input-output behavior of the system, and thus could be omitted from the state-space description. It is easily seen that the usual truncation method for obtaining a reduced order model in the reduced state \( x_1 \) in general does not preserve the port-Hamiltonian structure. It will also not preserve the passivity property, see e.g. [2], Remark 2.12. The same holds for the so-called singular perturbation reduction method, as was mentioned in Remark 2.14; see also [26], [35].

In which way is it possible to retain the port-Hamiltonian structure in model reduction? Recall that in the definition of a port-Hamiltonian system the vector of flow and effort variables (1.2) is required to be in the Dirac structure

\[
(f_1^x, f_2^x, e_1^x, e_2^x, f_R, e_R, f_P, e_P) \in D,
\]

(3.1)

while the flow and effort variables \( f_x, e_x \) are linked to the constitutive relations of the energy-storage by

\[
\dot{x}_1 = -f_1^x, \quad \frac{\partial H}{\partial x_1}(x_1, x_2) = e_1^x,
\]

\[
\dot{x}_2 = -f_2^x, \quad \frac{\partial H}{\partial x_2}(x_1, x_2) = e_2^x,
\]

which is shown in Fig. 3.1. This figure is a zoomed-in version of Fig. 1.1. The basic idea of structure preserving model reduction for port-Hamiltonian systems is to cut the interconnection

\[
\dot{x}_2 = -f_2, \quad \frac{\partial H}{\partial x_2}(x_1, x_2) = e_2,
\]

(3.2)

between the energy storage corresponding to \( x_2 \) and the Dirac structure, in such a way that no power is transferred. Then the exchange of power between the Dirac structure and the Hamiltonian happens only via the energy storage corresponding to \( x_1 \), with \( x_1 \) being the reduced order state vector.

The interconnection (3.2) can be cut by making both power products

\[
(\frac{\partial H}{\partial x_2})^T \dot{x}_2 \quad \text{and} \quad (e_2)^T f_2
\]

equal to zero.

The following main scenarios arise:
3.2. Structure preserving reduction based on power conservation

\[ -\dot{x}_1, \frac{\partial H}{\partial x_1}(x_1, x_2) = 0, \quad e_2 = 0. \quad (3.3) \]

The first equation imposes an algebraic constraint on the space variables \( x = (x_1^T, x_2^T)^T \). Under the general conditions on the Hamiltonian \( H \), this constraint allows one to solve \( x_2 \) as a function of \( x_1 : x_2 = x_2(x_1) \), leading to a reduced Hamiltonian

\[ H_{\text{red}}^{\text{ec}}(x_1) := H(x_1, x_2(x_1)). \]

Furthermore, the second equation defines the reduced Dirac structure\(^1\)

\[ \mathcal{D}_{\text{red}}^{\text{ec}} := \{(f_1^1, e_1^1, f_R, e_R, f_P, e_P) \mid \exists f_2 \text{ such that } (f_1^1, e_1^1, f_2, 0, f_R, e_R, f_P, e_P) \in \mathcal{D}\}, \]

leading to the reduced port-Hamiltonian system

\[ (-\dot{x}_1, \frac{\partial H_{\text{red}}^{\text{ec}}}{\partial x_1}(x_1), -\varphi(e_R), e_R, f_P, e_P) \in \mathcal{D}_{\text{red}}^{\text{ec}}. \]

We will call this reduction method the effort-constraint reduction method, since it constrains the efforts \( e_2 \) and \( \frac{\partial H}{\partial x_2} \) to zero.

\(^1\mathcal{D}_{\text{red}}^{\text{ec}} \) is the composition of the full order Dirac structure \( \mathcal{D} \) with the Dirac structure on the space of flow and effort variables \( f_2, e_2 \) defined by \( e_2 = 0 \). It is proven in [14] that \( \mathcal{D}_{\text{red}}^{\text{ec}} \) is indeed a Dirac structure.
3. Model reduction using reduced order Dirac structures

2. Set
\[ \dot{x}_2 = 0, \quad f_2 = 0. \] (3.4)

The first equation imposes the constraint
\[ x_2 = c, \]
where the constant \( c \) can be taken to be zero, and thus defines the reduced Hamiltonian
\[ H_{\text{fc}}^\text{red}(x_1) := H(x_1, c), \] (3.5)
while the second equation leads to the reduced Dirac structure
\[ D_{\text{fc}}^\text{red} := \{ (f_x^1, e_x^1, f_R, e_R, f_P, e_P) | \exists e_2 \text{ such that } \}
\[ (f_x^1, e_x^1, 0, e_2, f_R, e_R, f_P, e_P) \in D \}, \] (3.6)
and the corresponding reduced port-Hamiltonian system
\[ (-\dot{x}_1, \frac{\partial H_{\text{fc}}^\text{red}}{\partial x_1}(x_1), -\varphi(e_R), e_R, f_P, e_P) \in D_{\text{fc}}^\text{red}. \] (3.7)

We call this approach the \textit{flow-constraint} reduction method, because it constrains the flows \(-\dot{x}_2, f_2\).

3. Set
\[ \dot{x}_2 = 0, \quad e_2 = 0. \] (3.8)

This leads to the reduced order port-Hamiltonian system with the reduced Hamiltonian \( H_{\text{ec}}^\text{red}(x_1) \) and the reduced Dirac structure \( D_{\text{ec}}^\text{red} \).

4. Set
\[ \frac{\partial H}{\partial x_2}(x_1, x_2) = 0, \quad f_2 = 0. \] (3.9)

This leads to the port-Hamiltonian system with the reduced Hamiltonian \( H_{\text{ec}}^\text{red}(x_1) \) and the reduced Dirac structure \( D_{\text{fc}}^\text{red} \).

Despite their common basis the above reduction schemes have different physical interpretations and consequences. To illustrate this in a simple context, consider an electrical circuit where \( x_2 \) corresponds to the charge \( Q \) of a single (linear) capacitor. Application of the effort-constraint method would correspond to removing the capacitor (and setting its charge equal to zero) and short-circuiting the circuit at the location of the capacitor. On the other hand, the flow-constraint method would correspond to open-circuiting the circuit at the location of the capacitor, and keeping the charge of the capacitor constant. The method 3 is in this case very similar to the effort-constraint method, and corresponds to short-circuiting, with the minor difference of setting the charge of the capacitor equal to a constant. Finally, the method 4 corresponds
3.3. Equational representations of the reduced order models

to open-circuiting while setting the charge of the capacitor equal to zero (and thus is similar to the flow-constraint method).

An important open question is how to make the splitting \( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \) in such a way that power transfer between the energy storage corresponding to \( x_2 \) and the Dirac structure is very small. Then the approximations (3.3), (3.4), (3.8) and (3.9) are well justified.

3.3. Equational representations of the reduced order models

We will now provide explicit equational representations of the above four methods for structure preserving model reduction starting from the general representation by DAEs of the full order model as in (1.12):

\[
F_x \dot{x} = E_x \frac{\partial H}{\partial x}(x) - F_R \varphi(e_R) + E_R e_R + F_P f_P + E_P e_P,
\]

(3.10)

where the matrices \( F_x, E_x, F_R, E_R, F_P, E_P \) satisfy (1.11). Corresponding to the splitting of the state vector \( x \) into \( x = (x_1^T, x_2^T)^T, x_1 \in \mathbb{R}^r, x_2 \in \mathbb{R}^{n-r} \), where \( r \) is the dimension chosen for the reduced order model, and the respective splitting of the flow and effort vectors \( f_x, e_x \) into \( f_x^1, f_x^2 \) and \( e_x^1, e_x^2 \), we write

\[
F_x = \begin{bmatrix} F_x^1 & F_x^2 \end{bmatrix}, \quad E_x = \begin{bmatrix} E_x^1 & E_x^2 \end{bmatrix}.
\]

(3.11)

Now the reduced Dirac structure \( \mathcal{D}_{\text{red}}^{\text{ec}} \) corresponding to the effort-constraint \( e_x^2 = 0 \) is given by the explicit equations (see [14])

\[
L^{\text{ec}} F_x^1 f_x^1 + L^{\text{ec}} E_x^1 e_x^1 + L^{\text{ec}} F_R f_R + L^{\text{ec}} E_R e_R + L^{\text{ec}} F_P f_P + L^{\text{ec}} E_P e_P = 0,
\]

(3.12)

where \( L^{\text{ec}} \) is any matrix of maximal rank satisfying

\[
L^{\text{ec}} F_x^2 = 0.
\]

(3.13)

Similarly, the reduced Dirac structure \( \mathcal{D}_{\text{red}}^{\text{fc}} \) corresponding to the flow-constraint \( f_x^2 = 0 \) is given by the equations

\[
L^{\text{fc}} F_x^1 f_x^1 + L^{\text{fc}} E_x^1 e_x^1 + L^{\text{fc}} F_R f_R + L^{\text{fc}} E_R e_R + L^{\text{fc}} F_P f_P + L^{\text{fc}} E_P e_P = 0,
\]

(3.14)

where \( L^{\text{fc}} \) is any matrix of maximal rank satisfying

\[
L^{\text{fc}} E_x^2 = 0.
\]

(3.15)
3. Model reduction using reduced order Dirac structures

It follows that the reduced order model resulting from applying the effort-constraint method is given by

\[
L^{ec} F^1_x \dot{x}_1 = L^{ec} E^1_x \frac{\partial H^{ec}}{\partial x_1}(x_1) - L^{ec} F_R \varphi(e_R) + L^{ec} E_R e_R + L^{ec} F_P f_P + L^{ec} E_P e_P, \tag{3.16}
\]

whereas the reduced order model resulting from applying the flow-constraint method is given by

\[
L^{fc} F^1_x \dot{x}_1 = L^{fc} E^1_x \frac{\partial H^{fc}}{\partial x_1}(x_1) - L^{fc} F_R \varphi(e_R) + L^{fc} E_R e_R + L^{fc} F_P f_P + L^{fc} E_P e_P. \tag{3.17}
\]

Similar expressions follow for the reduced order models arising from applying methods 3 and 4.

The steps of model reduction leading to the reduced order models (3.16), (3.17) and those of the methods 3 and 4 are depicted in Fig. 3.2. Firstly, we consider a full order port-Hamiltonian system with the corresponding full order Dirac structure. Secondly, we reduce the full order Dirac structure to obtain the reduced order Dirac structure. Finally, we derive the reduced order model based on the reduced order Dirac structure. At the same time we are approximating the full order Hamiltonian of the full order model in order
3.4. Reduced models for linear port-Hamiltonian systems

to obtain the reduced order Hamiltonian of the reduced order model. Note that the reduced order models obtained in this way are port-Hamiltonian by construction.

3.4. Reduced models for linear input-state-output port-Hamiltonian systems

In the case of linear input-state-output port-Hamiltonian systems

\[
\begin{aligned}
\dot{x} &= (J - R)Qx + Gu,
\quad y = G^T Qx.
\end{aligned}
\]  

(3.18)

the above reduced order models take the following form.

The model (3.18) is obtained after the termination of the resistive port. In order to use the DAE representation of this model (3.10) we rewrite (3.18) in the form (1.15), where the resistive port is not terminated:

\[
\begin{aligned}
\dot{x} &= JQx + G_R f_R + Gu, \\
y &= G^T Qx, \\
e_R &= G^T_{R_1} Qx, \quad f_R = -\bar{Re}_R.
\end{aligned}
\]  

(3.19)

Splitting of the state vector into \(x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\), \(x_1 \in \mathbb{R}^r, x_2 \in \mathbb{R}^{n-r}\), for \(r\) being the dimension of the reduced order model, then leads to the following partitioned system description

\[
\begin{aligned}
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} G_{R_1} \\ G_{R_2} \end{bmatrix} f_R + \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} u, \\
y &= \begin{bmatrix} G_1^T \\ G_2^T \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \\
e_R &= \begin{bmatrix} G_{R_1}^T \\ G_{R_2}^T \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} f_R, \quad f_R = -\bar{Re}_R.
\end{aligned}
\]  

(3.20)

3.4.1. Effort-constraint method

Rewriting these equations as DAEs (3.10), and applying the effort-constraint reduction method as above, yields (assuming that \(Q_{22}\) is invertible) the reduced model

\[
\begin{aligned}
\dot{x}_1 &= F_{11}(Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})x_1 + G_1 u, \\
y_{ec} &= G_1^T (Q_{11} - Q_{12}Q_{22}^{-1}Q_{21}) x_1,
\end{aligned}
\]  

(3.21)
3. Model reduction using reduced order Dirac structures

where \( F_{11} = J_{11} - R_{11} \) (recall that \( F := J - R \) in (1.19)). This model was already shown by direct methods in [71] and Chapter 2, see (2.45); as well as in scattering coordinates in [74].

The way to derive this reduced order port-Hamiltonian model by reducing the full order Dirac structure is shown in Appendix B.

Remark 3.1. Note that the reduced order port-Hamiltonian model (2.45) was obtained starting from the full order model with the terminated resistive port, while in case of the reduced model (3.21) the resistive port was initially open. Nevertheless, the results are the same.

3.4.2. Flow-constraint method

The application of the flow-constraint method is more involved. The flow-constraint method is seen to lead to the reduced port-Hamiltonian model

\[
\begin{align*}
\dot{x}_1 &= \left( (J_s - \beta^T Z_{sk} \beta) - \beta^T Z_{sym} \beta \right) Q_{11} x_1 + \\
&\quad \left[ (-\alpha^T + \beta^T Z_{sk} \gamma^T) - (-\beta^T Z_{sym} \gamma^T) \right] u,
\end{align*}
\]

\[
y_{fc} = \left[ (-\alpha - \gamma Z_{sk} \beta) - \gamma Z_{sym} \beta \right] Q_{11} x_1 + \\
&\quad \left[ (-\eta + \gamma Z_{sk} \gamma^T) + \gamma Z_{sym} \gamma^T \right] u,
\]

(3.22)

where for simplicity we use the notation

\[
\begin{align*}
\alpha := G_{R_2}^T J_{22}^{-1} J_{21} - G_1^T, & \quad \beta := G_{R_2}^T J_{22}^{-1} J_{21} - G_{R_1}^T, \\
\gamma := G_{R_2}^T J_{22}^{-1} G_{R_2}, & \quad \delta := G_{R_2}^T J_{22}^{-1} G_{R_2}, \\
\eta := G_{R_2}^T J_{22}^{-1} G_2, & \quad Z := \bar{R}(I - \delta \bar{R})^{-1}, \\
Z_{sym} := (\bar{R}^{-1} - \bar{R} \delta)^{-1} > 0, & \quad Z_{sk} := (\bar{R}^{-1} \delta^{-1} \bar{R}^{-1} - \delta)^{-1},
\end{align*}
\]

(3.23)

and \( J_s := J_{11} - J_{12} J_{22}^{-1} J_{21} \) is a Schur complement of \( J \), which is a skew-symmetric matrix since the matrix \( J \) is skew-symmetric. We assume that \( J_{22} \) (which is also skew-symmetric) is an invertible matrix. For even dimensions \( J_{22} \) is going to be necessarily invertible.

The matrices \( G_{R_1} \in \mathbb{R}^{r \times l}, G_{R_2} \in \mathbb{R}^{(n-r) \times l} \), where \( l \) is the dimension of the full rank matrix \( \bar{R} \in \mathbb{R}^{l \times l}, l \leq n \), come from the factorization of the damping matrix \( \bar{R} \):

\[
\bar{R} = \left[ \begin{array}{c|c} G_{R_1} & G_{R_1}^T \\ \hline G_{R_2} & G_{R_2}^T \end{array} \right],
\]

(3.24)

see also (1.14).

The matrix \( Z \) in (3.23) for an invertible \( \bar{R} \) can be equivalently written as

\[
Z = (\bar{R}^{-1} - \delta)^{-1},
\]
3.4. Reduced models for linear port-Hamiltonian systems

which can be further decomposed into its symmetric part $Z_{\text{sym}}$ and its skew-symmetric part $Z_{\text{sk}}$:

$$ Z_{\text{sym}} = \frac{1}{2}(Z + Z^T), \quad Z_{\text{sk}} = \frac{1}{2}(Z - Z^T). \quad (3.25) $$

$Z_{\text{sym}}, Z_{\text{sk}}$ can be shown to have the form as in (3.23) (for details see Lemma (B.1)). For $Z = 0$ (this happens when the full order system is lossless) $Z_{\text{sym}} = Z_{\text{sk}} = 0$. In practice, once having $Z$, one would use expressions in (3.25) to compute $Z_{\text{sym}}, Z_{\text{sk}}$ rather than those in (3.23).

The lengthy derivations of the reduced order model (3.22), which involve computing the reduced order Dirac structures, are shown in Appendix B. Lemma (B.1) demonstrates that $Z_{\text{sym}}$ is positive definite.

Even though we started with the full order port-Hamiltonian system (3.18) without the feed-through term, the flow-constraint method, in contrast to the effort-constraint method, results in the reduced order model (3.22) of more general form (1.13) with the feed-through term:

$$ \begin{align*}
\dot{x}_1 &= (J_r - R_r)Q_r x_1 + (G_r - P_r)u, \\
y_{\text{fc}} &= (G_r^T + P_r^T)Q_r x_1 + (M_r + S_r)u,
\end{align*} \quad (3.26) $$

where the reduced order matrices are

$$ J_r = J_s - \beta^T Z_{\text{sk}} \beta, \quad R_r = \beta^T Z_{\text{sym}} \beta, $$

$$ Q_r = Q_{11}, \quad G_r = -\alpha^T + \beta^T Z_{\text{sk}} \gamma^T, $$

$$ P_r = -\beta^T Z_{\text{sym}} \gamma^T, \quad M_r = -\eta + \gamma Z_{\text{sk}} \gamma^T, $$

$$ S_r = \gamma Z_{\text{sym}} \gamma^T. $$

One can easily verify that indeed for this reduced order port-Hamiltonian model $J_r, M_r$ are skew-symmetric, $R_r, S_r$ are positive semi-definite symmetric, $Q_r$ is positive definite symmetric, and $R_r, P_r$ and $S_r$ satisfy

$$ \begin{bmatrix} R_r & P_r \\ P_r^T & S_r \end{bmatrix} \succeq 0. $$

**Remark 3.2.** In case of a lossless full order port-Hamiltonian system (3.18), when $R = 0$ and $\bar{R} = 0$, the reduced order port-Hamiltonian system (3.22) is also lossless and is of the form

$$ \begin{align*}
\dot{x}_1 &= J_s Q_{11} x_1 + (G_1 - J_{12} J_{22}^{-1} G_2)u, \\
y_{\text{fc}} &= (G_1^T - G_2^T J_{22}^{-1} J_{21}) Q_{11} x_1 - G_2^T J_{22}^{-1} G_2 u.
\end{align*} \quad (3.26) $$

If the coordinates chosen for model reduction are such that $G_2 = 0$, then the reduced order port-Hamiltonian system (3.22) specializes to

$$ \begin{align*}
\dot{x}_1 &= [J_s - (G_{R_1} - J_{12} J_{22}^{-1} G_{R_2})] Z(G_{R_1}^T - G_{R_2}^T J_{22}^{-1} J_{21})] Q_{11} x_1 + G_1 u, \\
y_{\text{fc}} &= G_1^T Q_{11} x_1.
\end{align*} \quad (3.27) $$
3. Model reduction using reduced order Dirac structures

In this case the reduced order system by the flow-constraint method (3.27) does not have a feed-through term.

**Remark 3.3.** Note that the approximation given in (2.50), which is structure preserving only if \( G_2 = 0 \) and in general not structure preserving, mimics the flow-constraint method (3.22), but is not the same. This can be explained by the fact that the approximation (2.50) is obtained starting from the full order model (2.1) (or (3.18)), where the resistive port is terminated, which makes the model lose the properties of the underlying Dirac structure. While for the flow-constraint method (3.22) the starting point is the model (3.19) with the corresponding underlying Dirac structure and an open resistive port.

### 3.4.3. Effort- and flow-constraint methods in the bond-graph modeling framework

Effort- and flow-constraint methods have a direct interpretation from the bond-graph modeling point of view. Constraining the efforts

\[
e^2_x = \frac{\partial H}{\partial x_2}(x_1, x_2) = 0, \quad e_2 = 0,
\]

in the lower part of Fig. 3.1, which results in the effort-constraint method, corresponds to the so-called 0-junction, shown in Fig. 3.3 (without directions). While constraining the flows

\[
f^2_x = -\dot{x}_2 = 0, \quad f_2 = 0,
\]

in order to obtain the flow-constraint method corresponds to the 1-junction in Fig. 3.3. The 0- and 1-junctions represent generalized, i.e. domain independent, Kirchhoff current and voltage laws respectively. For details see [19].

### 3.4.4. The other two approximation methods

For simplicity of exposition we will only consider the case \( G_2 = 0 \). It is seen that the third reduction method yields in this case

\[
\begin{aligned}
\dot{x}_1 &= F_{11}Q_{11}x_1 + G_1u, \\
y &= G^T_1Q_{11}x_1,
\end{aligned}
\]  

(3.28)

while the fourth method (again assuming \( G_2 = 0 \) and invertibility of \( Q_{22} \) and \( F_{22} \)) yields

\[
\begin{aligned}
\dot{x}_1 &= (F_{11} - F_{12}F_{22}^{-1}F_{21})(Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})x_1 + G_1u, \\
y &= G^T_1(Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})x_1.
\end{aligned}
\]  

(3.29)
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![0-junction and 1-junction](image)

Figure 3.3.: 0-junction (left) and 1-junction (right)

Even though the last two methods are structure preserving as well, they do not have a direct interpretation from the bond-graph modeling point of view. Therefore we consider the effort-constraint and the flow-constraint methods as the main model reduction methods of this chapter.

3.4.5. Effort-constraint method and moment matching

Consider a single-input single-output (SISO) port-Hamiltonian system (3.18)

\[
\begin{align*}
\dot{x} &= (J - R)Qx + gu, \\
y &= g^T Qx,
\end{align*}
\]

with an input matrix \( g \in \mathbb{R}^{n \times 1} \). The effort-constraint method, which leads in this case to the following reduced order model

\[
\begin{align*}
\dot{x}_1 &= F_{11}(Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})x_1 + g_1 u, \\
y_{ec} &= g_1^T (Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})x_1,
\end{align*}
\]

turns out to have a relation to the projection based methods matching moments of the full order system at certain points in the complex plane. The moment-matching approach, discussed in Chapters 5, 6 (see also [2]), involves computing (e.g. by the Arnoldi method) a projection map \( V_r \in \mathbb{R}^{n \times r} \), \( x = V_r x_r \), with \( x_r \in \mathbb{R}^r \) being the reduced order state vector, in such a way that \( r \) moments of the full order system (3.30) and the projected reduced order system match at \( s_0 \in \mathbb{C} \) or at infinity.

**Theorem 3.4.** Consider a full order SISO port-Hamiltonian system (3.30) in new energy coordinates \( z =Tx \) (for some coordinate transformation \( T \))

\[
\begin{align*}
\dot{z} &= (J - R)Qz + gu, \\
y &= g^T Qz.
\end{align*}
\]
3. Model reduction using reduced order Dirac structures

Suppose that energy coordinates $z$ are such that the projection map $V_{ez} \in \mathbb{R}^{n \times r}, e_z = V_{ez} e_r$, which matches the first $r$ moments at $s_0 \in \mathbb{C}$ or at infinity of the co-energy variable representation of (3.32) (with the usual coordinate transformation $e_z = Qz$)

$$
\begin{align*}
\dot{e}_z &= Q(J - R)e_z + Qu, \\
y &= g^T e_z,
\end{align*}
$$

(3.33)

has the following splitting

$$
V_{ez} = \begin{bmatrix} V_1 \\ 0 \end{bmatrix}, V_1 \in \mathbb{R}^{r \times r}, V_1 - \text{invertible}.
$$

Then the reduced order port-Hamiltonian model

$$
\begin{align*}
\dot{z}_1 &= F_{11}(Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})z_1 + g_1 u, \\
y_{ec} &= g_1^T (Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})z_1,
\end{align*}
$$

(3.34)

matches the first $r$ moments of the full order system (3.32) at $s_0 \in \mathbb{C}$ or at infinity.

**Proof.** The moment matching projection of the rewritten port-Hamiltonian system (3.33)

$$
\begin{align*}
Q^{-1}\dot{e}_z &= (J - R)e_z + Qu, \\
y &= g^T e_z,
\end{align*}
$$

reads

$$
\begin{align*}
V_{ez}^T Q^{-1} V_{ez} \dot{e}_r &= V_{ez}^T (J - R)V_{ez} e_r + V_{ez}^T gu, \\
\hat{y} &= g^T V_{ez} e_r.
\end{align*}
$$

(3.35)

Using the well-known analytic inversion formula we get

$$
V_{ez}^T Q^{-1} V_{ez} = \begin{bmatrix} V_1^T & 0 \end{bmatrix} \begin{bmatrix} Q_s^{-1} & * \\ * & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ 0 \end{bmatrix} = V_1^T Q_s^{-1} V_1,
$$

where $Q_s = Q_{11} - Q_{12}Q_{22}^{-1}Q_{21}$ is the Schur complement of $Q$. Therefore the reduced order system becomes

$$
\begin{align*}
V_1^T Q_s^{-1} V_1 \dot{e}_r &= V_1^T (J_{11} - R_{11})V_1 e_r + V_1^T g_1 u, \\
\hat{y} &= g_1^T V_1 e_r.
\end{align*}
$$

Since $e_z = V_{ez} e_r$ implies that $e_z^1 = V_1 e_r$ and since $V_1^T$ is invertible the reduced order model transforms to

$$
\begin{align*}
Q_s^{-1} \dot{e}_z^1 &= (J_{11} - R_{11})e_z^1 + g_1 u, \\
\hat{y} &= g_1^T e_z^1,
\end{align*}
$$

(3.35)
3.5. Conclusions

which is, after the transformation from co-energy to energy coordinates $e^1_2 = Q_s z_1$, nothing but the reduced order system (3.34) obtained by the effort-constraint method

$$\begin{align*}
\dot{z}_1 &= F_{11}(Q_{11} - Q_{12} Q_{22}^{-1} Q_{21}) z_1 + g_1 u, \\
y_{ec} &= g_1^T (Q_{11} - Q_{12} Q_{22}^{-1} Q_{21}) z_1.
\end{align*}$$

(3.36)

Since there are only linear coordinate transformations involved, the moments of (3.36) and (3.35) are the same which claims the proof. □

One of the questions for future research is how to find such coordinates $z$, as in (3.32), in a numerically efficient way.

3.4.6. The choice of the coordinate system for model reduction

Apparently the full order port-Hamiltonian system (3.18) can be reduced by the effort-constraint method (3.21) and the flow-constraint method (3.22) in any coordinate system in order to preserve the port-Hamiltonian structure. The question is which coordinate system to chose in order to obtain the most accurate approximation from the input-output point of view.

One possible choice of coordinates is balanced coordinates using Lyapunov balancing (see Chapter 2), positive real (Chapter 4) or some other type of balancing. Another choice for the flow-constraint method would be to choose the coordinates where $G_2 = 0$, which would significantly simplify the expression of the reduced order model (3.22), see (3.27). The effort-constraint method for the SISO port-Hamiltonian systems naturally suggests coordinates $z$ as in (3.32) in order to match moments at specific points in the complex plane, which would pose a question of how to find such coordinates in a numerically efficient way.

For the time being it is not yet fully understood how to choose the best coordinates in a systematic way. This is a question for future work.

3.5. Conclusions

In this chapter we considered a family of four port-Hamiltonian structure preserving model reduction methods, arising from the Dirac structure representation of port-Hamiltonian systems. We put the emphasis on two of them: the effort-constraint method and the flow-constraint method, which is motivated by the bond-graph modeling framework. We showed that the effort-constraint method, applied in particular coordinates, matches first moments of the SISO full order port-Hamiltonian system at specific points in
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the complex plane. We suggested this choice of coordinates for the effort-constraint method, as well as balanced coordinates for the effort- and flow-constraint methods. A systematic way of choosing the coordinates for the full order port-Hamiltonian system in order to obtain the most accurate approximation from the input-output point of view, and the error bounds for the effort-constraint and flow-constraint methods, are questions for future research.