2. Balancing for port-Hamiltonian systems

This chapter discusses reduction to minimality and balancing of port-Hamiltonian systems. It is shown that by the use of the Kalman decomposition an uncontrollable and/or unobservable port-Hamiltonian system is reduced to a controllable/observable system that inherits the port-Hamiltonian structure. Reduction to minimality is presented in both the energy and co-energy variable representations for port-Hamiltonian systems. These exact reduction procedures motivate two approximate reduction procedures, namely the effort- and flow-constraint reduction methods, which are structure preserving for general port-Hamiltonian systems and will be further discussed in detail in Chapter 3. In this chapter we employ Lyapunov balancing for model reduction of general port-Hamiltonian systems. A numerical example illustrating model reduction of a port-Hamiltonian ladder network is presented.

2.1. Introduction

In this chapter we will first investigate the exact model reduction of non-minimal port-Hamiltonian systems. In Section 2.2 we demonstrate that in general controllability of port-Hamiltonian systems is not equivalent to observability, even though there are special port-Hamiltonian systems with this property. We will show by applying the Kalman decomposition in Section 2.3 that the reduction of the dynamics of an uncontrollable/unobservable linear port-Hamiltonian system to a dynamics on the reachable/observable subspace preserves the port-Hamiltonian structure. This result holds both for the energy and co-energy variable representations of linear port-Hamiltonian systems. The co-energy variable representation of port-Hamiltonian systems is considered in Section 2.4, where it is shown that the reduced models in co-energy coordinates take a somewhat “dual” form to the reduced models obtained in the standard energy coordinates.

Within the systems and control literature a popular and elegant tool for model reduction is balancing, going back to [63], [62]. One favorable property of model reduction based on balancing, as compared with other techniques such as modal analysis, is that the approximation of the dynamical
2. Balancing for port-Hamiltonian systems

System is explicitly based on its input-output properties. Balancing for port-Hamiltonian systems is considered in Section 2.5, see also [88]. We will apply the effort-constraint method of model reduction in Section 2.6 to a linear port-Hamiltonian system and show that the reduced order model is again port-Hamiltonian. Preliminary results of this chapter are reported in [71] as well as in [74] using scattering coordinates. Similar reduced order port-Hamiltonian models are obtained in [46, 45] employing perturbation analysis. In Section 2.7 we consider numerical simulations of a ladder network, and apply the effort-constraint method and the balanced truncation method in order to obtain reduced order models and compare the results.

2.2. Special cases of linear port-Hamiltonian systems

Consider port-Hamiltonian systems in energy coordinates (1.16)

\[
\begin{align*}
\dot{x} &= (J - R)Qx + Bu, \\
y &= B^T Qx.
\end{align*}
\]  

(2.1)

In the sequel we will often abbreviate \( J - R \) to \( F = J - R \) (see Remark 1.4). Clearly

\[ F + F^T \leq 0. \]  

(2.2)

Conversely, any \( F \) satisfying (2.2) can be written as \( J - R \) as above by decomposing \( F \) into its skew-symmetric and symmetric parts

\[ J = \frac{1}{2}(F - F^T), \quad R = -\frac{1}{2}(F + F^T). \]  

(2.3)

Two special cases of port-Hamiltonian systems correspond to either \( R = 0 \) or \( J = 0 \). In fact, if \( R = 0 \) (no internal energy dissipation) then the dissipation inequality reduces to an equality

\[ \frac{d}{dt} \frac{1}{2} x^T Q x = u^T y. \]  

(2.4)

In this case the transfer matrix \( G(s) = B^T Q(sI - JQ)^{-1} B \) of the system (for invertible \( Q \)) satisfies

\[ G(s) = -G^T(-s). \]  

(2.5)

Conversely, any transfer matrix \( G(s) \) satisfying \( G(s) = -G^T(-s) \) can be shown to have a minimal realization

\[
\begin{align*}
\dot{x} &= JQx + Bu, \\
y &= B^T Qx.
\end{align*}
\]  

(2.6)
2.2. Special cases of linear port-Hamiltonian systems

(with in fact $Q$ being invertible again).

The other special case corresponds to $J = 0$, in which case the system takes the form

$$
\begin{align*}
\dot{x} &= -RQx + Bu, \\
y &= B^T Qx,
\end{align*}
$$

(2.7)

with the transfer matrix $G(s) = B^T Q(sI + RQ)^{-1}B$ satisfying (for invertible $Q$)

$$
G(s) = G^T(s).
$$

(2.8)

Conversely, any transfer matrix $G(s)$ satisfying (2.8) is represented by a minimal state-space representation (2.7) with $Q$ invertible, where, however, $R$ need not necessarily be positive semi-definite.

In these two special cases, either $R = 0$ or $J = 0$, there is a direct relationship between controllability and observability properties of the port-Hamiltonian system.

**Proposition 2.1.** Consider a port-Hamiltonian system (2.6) or (2.7), and assume that $Q$ is invertible. The system is controllable if and only if it is observable, while the unobservable subspace $\mathcal{N}$ is related to the reachable subspace $\mathcal{R}$ by

$$
\mathcal{N} = \mathcal{R}^\perp
$$

(2.9)

with $\perp$ denoting the orthogonal complement with respect to the (possibly indefinite) inner product defined by $Q$.

**Proof.** For any port-Hamiltonian system (2.1) with $F = J - R$ we have

$$
\begin{bmatrix}
B^T Q \\
B^T QFQ \\
\vdots
\end{bmatrix} =
\begin{bmatrix}
B & F^T QB & F^T QFQ & \ldots
\end{bmatrix}^T Q.
$$

(2.10)

Since the kernel of the matrix on the left-hand side defines the unobservable subspace, while on the right-hand side the image of the matrix preceding $Q$ defines the reachable subspace if $F^T = F$ or $F^T = -F$, the assertion follows.

Nevertheless, in general controllability and observability for port-Hamiltonian systems are not equivalent, as the following example shows.

**Example 2.2.** Consider a port-Hamiltonian system

$$
\begin{align*}
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} &=
\begin{bmatrix}
-1 & 1 \\
-1 & -1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} +
\begin{bmatrix}
1 \\
0
\end{bmatrix}u, \\
y &=
\begin{bmatrix}
1 & 0 \end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix},
\end{align*}
$$

(2.11)
corresponding to \( J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \) and \( R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \). The system is observable but not controllable.

### 2.3. The Kalman decomposition of port-Hamiltonian systems

#### 2.3.1. Reduction to a controllable port-Hamiltonian system

Consider a port-Hamiltonian system on a state space \( \mathcal{X} \) which is not controllable. Take linear coordinates \( x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \) such that vectors of the form \( \begin{bmatrix} x_1 \\ 0 \end{bmatrix} \) span the reachable subspace \( \mathcal{R} \subset \mathcal{X} \):

\[
\begin{align*}
\dot{x}_1 &= (F_{11} Q_{11} + F_{12} Q_{21}) x_1 + B_1 u, \\
\dot{x}_2 &= (F_{21} Q_{11} + F_{22} Q_{22}) x_2 + B_2 u, \\
y &= B_1^T Q_{11} x_1.
\end{align*}
\]  

(2.12)

Invariance of \( \mathcal{R} \) and the fact that \( imB \subset \mathcal{R} \) (see e.g. [67]) imply

\[
F_{21} Q_{11} + F_{22} Q_{21} = 0, \quad B_2 = 0.
\]  

(2.13)

It follows that the dynamics restricted to \( \mathcal{R} \) is given as

\[
\begin{align*}
\dot{x}_1 &= (F_{11} Q_{11} + F_{12} Q_{21}) x_1 + B_1 u, \\
\dot{x}_2 &= B_1^T Q_{11} x_1.
\end{align*}
\]  

(2.14)

Now let us assume that \( F_{22} \) in (2.13) is invertible. Then it follows from (2.13) that \( Q_{21} = -F_{22}^{-1} F_{21} Q_{11} \). Substitution in (2.14) yields

\[
\begin{align*}
\dot{x}_1 &= (F_{11} - F_{12} F_{22}^{-1} F_{21}) Q_{11} x_1 + B_1 u, \\
\dot{x}_2 &= B_1^T Q_{11} x_1.
\end{align*}
\]  

(2.15)

which is again a port-Hamiltonian system. Indeed, \( F + F^T \leq 0 \) implies that the Schur complement \( \tilde{F} = F_{11} - F_{12} F_{22}^{-1} F_{21} \) satisfies \( \tilde{F} + \tilde{F}^T \leq 0 \).

**Remark 2.3.** Note that \( \tilde{F} \) is skew-symmetric if \( F \) is skew-symmetric, and is symmetric if \( F \) is symmetric.

**Remark 2.4.** The Schur complement of a general \( F \) with a singular \( F_{22} \) is not defined. Nevertheless, it is still possible to extend the definition of the Schur complement of \( F \) to the case where \( F_{22} \) is singular if \( F \) is symmetric, which corresponds to the purely damped port-Hamiltonian systems (2.7). For details see Lemma A.1 in Appendix A at the end of the thesis.
2.3. The Kalman decomposition of port-Hamiltonian systems

2.3.2. Reduction to an observable port-Hamiltonian system

Consider again a port-Hamiltonian system (2.1) and suppose the system is not observable. Then there exist coordinates $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ such that the unobservable subspace $\mathcal{N}$ is spanned by vectors of the form $\begin{bmatrix} 0 \\ x_2 \end{bmatrix}$. From invariance of $\mathcal{N}$ and the fact that $\mathcal{N} \subset \ker C$ (see again [67]) it follows that

\begin{align*}
F_{11}Q_{12} + F_{12}Q_{22} &= 0, \\
B_1^T Q_{12} + B_2^T Q_{22} &= 0.
\end{align*}

The dynamics on the quotient space $\mathcal{X}/\mathcal{N}$ is

\begin{align*}
\dot{x}_1 &= (F_{11}Q_{11} + F_{12}Q_{21})x_1 + B_1 u, \\
y &= B_1^T Q_{11} x_1 + B_2^T Q_{21} x_1.
\end{align*}

Assuming invertibility of $Q_{22}$ it follows from (2.16) that $F_{12} = -F_{11}Q_{12}Q_{22}^{-1}$ and $B_2^T = -B_1^T Q_{12}Q_{22}^{-1}$. Substitution in (2.17) yields

\begin{align*}
\dot{x}_1 &= F_{11}(Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})x_1 + B_1 u, \\
y &= B_1^T (Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})x_1,
\end{align*}

which is again a port-Hamiltonian system with Hamiltonian $\bar{H}(x_1) = \frac{1}{2} x_1^T (Q_{11} - Q_{12}Q_{22}^{-1}Q_{21}) x_1$.

**Remark 2.5.** Note that $(Q_{11} - Q_{12}Q_{22}^{-1}Q_{21}) \geq 0$ if $Q \geq 0$.

**Remark 2.6.** Since $Q$ is symmetric the definition of the Schur complement of $Q$ can be extended to the case that $Q_{22}$ is singular. For details see Lemma A.1 in Appendix A at the end of the thesis.

2.3.3. The Kalman decomposition

It is well known that a linear system $\dot{x} = Ax + Bu$, $y = Cx$ can be represented in a suitable basis as (see [67], [99])

\[ A = \begin{bmatrix}
A_{11} & A_{12} & 0 & 0 \\
0 & A_{22} & 0 & 0 \\
A_{31} & A_{32} & A_{33} & A_{34} \\
0 & A_{42} & 0 & A_{44}
\end{bmatrix}, \quad B = \begin{bmatrix}
B_1 \\
0 \\
B_3 \\
0
\end{bmatrix}, \quad C = \begin{bmatrix}
C_1^T \\
C_2^T \\
0 \\
0
\end{bmatrix}^T,
\]

with $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4$, where $\mathcal{X}_1$ is the part of the system that is both controllable and observable, $\mathcal{X}_2$ is uncontrollable but observable, $\mathcal{X}_3$ is
2. **Balancing for port-Hamiltonian systems**

Controllable but unobservable, while \(\mathcal{X}_4\) is uncontrollable and unobservable, that is

\[
\mathcal{N} = \mathcal{X}_3 \oplus \mathcal{X}_4, \\
\mathcal{R} = \mathcal{X}_1 \oplus \mathcal{X}_3.
\]

Writing out

\[
\begin{bmatrix}
  A_{11} & A_{12} & 0 & 0 \\
  0 & A_{22} & 0 & 0 \\
  A_{31} & A_{32} & A_{33} & A_{34} \\
  0 & A_{42} & 0 & A_{44}
\end{bmatrix}
\begin{bmatrix}
  Q_{11} & Q_{12} & Q_{13} & Q_{14} \\
  Q_{21} & Q_{22} & Q_{23} & Q_{24} \\
  Q_{31} & Q_{32} & Q_{33} & Q_{34} \\
  Q_{41} & Q_{42} & Q_{43} & Q_{44}
\end{bmatrix}
\]

implies that the blocks of the \(F\) and \(Q\) matrices satisfy

\[
\begin{align*}
(a) & F_{11}Q_{13} + F_{12}Q_{23} + F_{13}Q_{33} + F_{14}Q_{43} = 0, \\
(b) & F_{11}Q_{14} + F_{12}Q_{24} + F_{13}Q_{34} + F_{14}Q_{44} = 0, \\
(c) & F_{21}Q_{11} + F_{22}Q_{21} + F_{23}Q_{31} + F_{24}Q_{41} = 0, \\
(d) & F_{21}Q_{13} + F_{22}Q_{23} + F_{23}Q_{33} + F_{24}Q_{43} = 0, \\
(e) & F_{21}Q_{14} + F_{22}Q_{24} + F_{23}Q_{34} + F_{24}Q_{44} = 0, \\
(f) & F_{41}Q_{11} + F_{42}Q_{21} + F_{43}Q_{31} + F_{44}Q_{41} = 0, \\
(g) & F_{41}Q_{13} + F_{42}Q_{23} + F_{43}Q_{33} + F_{44}Q_{43} = 0.
\end{align*}
\]

Similarly by writing out

\[
\begin{bmatrix}
  B_1^T \\
  0 \\
  B_3^T \\
  0
\end{bmatrix}
\begin{bmatrix}
  Q_{11} & Q_{12} & Q_{13} & Q_{14} \\
  Q_{21} & Q_{22} & Q_{23} & Q_{24} \\
  Q_{31} & Q_{32} & Q_{33} & Q_{34} \\
  Q_{41} & Q_{42} & Q_{43} & Q_{44}
\end{bmatrix}
= \begin{bmatrix}
  C_1 \\
  C_2 \\
  0 \\
  0
\end{bmatrix},
\]

we obtain

\[
\begin{align*}
B_1^TQ_{13} + B_3^TQ_{33} &= 0, \\
B_1^TQ_{14} + B_3^TQ_{34} &= 0.
\end{align*}
\]

The resulting dynamics on \(\mathcal{X}_1\) (the part of the system that is both controllable and observable) can be identified in port-Hamiltonian form, by combining the previous two reduction schemes corresponding to controllability and observability. Indeed, application of Section 2.3.2 yields the following observable system on \(\mathcal{X}_1 \oplus \mathcal{X}_2\)

\[
\begin{align*}
\dot{x} &= \begin{bmatrix} F_{11} & F_{12} \\
  F_{21} & F_{22} \end{bmatrix} \dot{Q} \begin{bmatrix} x_1 \\
  x_2 \end{bmatrix} + \begin{bmatrix} B_1 \end{bmatrix} u, \\
y &= \begin{bmatrix} B_1^T \\
  0 \end{bmatrix} \dot{Q} \begin{bmatrix} x_1 \\
  x_2 \end{bmatrix},
\end{align*}
\]

\[\text{(2.22)}\]
2.4. The co-energy variable representation

where

\[ \bar{Q} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} - \begin{bmatrix} Q_{13} & Q_{14} \\ Q_{23} & Q_{24} \end{bmatrix} \begin{bmatrix} Q_{33} & Q_{34} \\ Q_{43} & Q_{44} \end{bmatrix}^{-1} \begin{bmatrix} Q_{31} & Q_{32} \\ Q_{41} & Q_{42} \end{bmatrix}. \] (2.23)

Next, application of Section 2.3.1 to (2.22) yields the following minimal port-Hamiltonian description of the dynamics on \( \mathcal{X}_1 \)

\[
\begin{aligned}
\dot{x}_1 &= (F_{11} - F_{12}F_{22}^{-1}F_{21})\bar{Q}_{11}x_1 + B_1u, \\
y &= B_1^T\bar{Q}_{11}x_1,
\end{aligned}
\] (2.24)

having the same transfer matrix as the original system (2.1).

Further analysis (using the well-known matrix inversion formula) yields

\[
\begin{aligned}
\bar{Q}_{11} &= Q_{11} - Q_{13}(Q_{33} - Q_{34}Q_{44}^{-1}Q_{43})^{-1}Q_{31} + \\
&+ Q_{14}Q_{44}^{-1}Q_{43}(Q_{33} - Q_{34}Q_{44}^{-1}Q_{43})^{-1}Q_{31} + \\
&+ Q_{13}Q_{33}^{-1}Q_{34}(Q_{44} - Q_{43}Q_{33}^{-1}Q_{34})^{-1}Q_{41} - \\
&- Q_{14}(Q_{44} - Q_{43}Q_{33}^{-1}Q_{34})^{-1}Q_{41}.
\end{aligned}
\] (2.25)

**Remark 2.7.** By first applying the procedure of Section 2.3.1 and then applying the procedure of Section 2.3.2 we obtain the same port-Hamiltonian formulation.

2.4. The co-energy variable representation

In this section we assume throughout that the matrix \( Q \) is invertible. This means that

\[ e = Qx \] (2.26)

is a valid coordinate transformation, leading to the port-Hamiltonian system in co-energy coordinates of the form (1.17)

\[
\begin{aligned}
\dot{e} &= QF e + QBu, \quad F = J - R, \\
y &= B^T e.
\end{aligned}
\] (2.27)

A main advantage of the co-energy variable representation of a port-Hamiltonian system is that additional *constraints* on the system are often expressed as constraints on the co-energy variables (see also Section 2.6).

The reduction of the port-Hamiltonian system to its controllable and/or observable part takes the following form in the co-energy variable representation. Interestingly enough, the formulae take a somewhat “dual” form to the formulae obtained in the energy variable representation.
2. Balancing for port-Hamiltonian systems

Consider the system (2.27) in the co-energy variable representation. Take linear coordinates \( e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \) such that the reachable subspace \( \mathcal{R} \) is spanned by vectors of the form \( \begin{bmatrix} e_1 \\ 0 \end{bmatrix} \):

\[
\begin{bmatrix}
\dot{e}_1 \\
\dot{e}_2 \\
y
\end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} + \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u,
\]

(2.28)

Invariance of \( \mathcal{R} \) and the fact that \( imQB \subset \mathcal{R} \) imply

\[
\begin{align*}
Q_{21}F_{11} + Q_{22}F_{21} &= 0, \\
Q_{21}B_1 + Q_{22}B_2 &= 0.
\end{align*}
\]

(2.29)

Hence the dynamics restricted to \( \mathcal{R} \) equals

\[
\begin{cases}
\dot{e}_1 = (Q_{11}F_{11} + Q_{12}F_{21})e_1 + (Q_{11}B_1 + Q_{12}B_2)u \\
\dot{y} = B_1^T e_1
\end{cases}
\]

(2.30)

which is a port-Hamiltonian system in the co-energy variable representation, with energy matrix \( \bar{Q} = Q_{11} - Q_{12}Q_{22}^{-1}Q_{21}, \) and interconnection/damping matrix \( F_{11}. \) Notice that these formulae are dual to the corresponding formulae (2.15) for the controllable part of the system in the energy variable representation, where the resulting interconnection/damping matrix is a Schur complement, while the resulting energy matrix is \( Q_{11}. \) This duality is associated with the Legendre transform of the Hamiltonian \( H(x). \)

Analogously, take linear coordinates \( e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \) such that the unobservable subspace \( \mathcal{N} \) is spanned by the vectors \( \begin{bmatrix} 0 \\ e_2 \end{bmatrix}. \) Invariance of \( \mathcal{N} \) and the fact that \( \mathcal{N} \subset \ker B^T \) imply

\[
\begin{align*}
Q_{11}F_{12} + Q_{12}F_{22} &= 0, \\
B_2 &= 0,
\end{align*}
\]

(2.31)

leading to the observable reduced dynamics

\[
\begin{cases}
\dot{e}_1 = (Q_{11}F_{11} + Q_{12}F_{21})e_1 + Q_{11}B_1 u \\
\dot{y} = B_1^T e_1
\end{cases}
\]

(2.32)

Combination of the above leads to a similar Kalman decomposition as in the energy variable representation.
Remark 2.8. To extend the definition of the Schur complements of $Q$, $F$ for singular $Q_{22}$, $F_{22}$ see Remarks 2.4, 2.6.

2.5. Balancing for port-Hamiltonian systems

Consider a linear time invariant (LTI) minimal asymptotically stable system

\[
\begin{align*}
\dot{x} &= Ax + Bu, \\
y &=Cx.
\end{align*}
\] (2.33)

Definition 2.9. The controllability and observability functions of the linear system (2.33) are defined as

\[
L_c(x_0) = \min_{u \in L_2(-\infty, 0)} \frac{1}{2} \int_{-\infty}^{0} \|u(t)\|^2 dt, \quad x(-\infty) = 0, \quad x(0) = x_0
\] (2.34)

and

\[
L_o(x_0) = \frac{1}{2} \int_{0}^{\infty} \|y(t)\|^2 dt, \quad u(t) = 0, \quad x(0) = x_0,
\] (2.35)

respectively.

The value of the controllability function at $x_0$ is the minimum amount of input energy required to reach the state $x_0$ from the zero state, and the value of the observability function at $x_0$ is the amount of output energy generated by the state $x_0$.

Theorem 2.10. [62] The controllability and observability functions of the linear system (2.33) satisfy

\[
L_c(x_0) = \frac{1}{2} x_0^T W^{-1} x_0 \quad \text{and} \quad L_o(x_0) = \frac{1}{2} x_0^T M x_0,
\]

where

\[
W = \int_{0}^{\infty} e^{At} BB^T e^{A^T t} dt \quad \text{and} \quad M = \int_{0}^{\infty} e^{A^T t} C^T C e^{At} dt
\]

are the controllability and observability Gramians of the system (2.33) correspondingly. Furthermore, $W$ and $M$ are symmetric and positive definite, and are the unique solutions of the Lyapunov equations

\[
AW + WA^T = -BB^T
\] (2.36)
2. Balancing for port-Hamiltonian systems

and

\[ A^T M + MA = -C^T C \]  \hspace{1cm} (2.37)

respectively.

In the port-Hamiltonian case equations (2.36), (2.37) specialize to

\[ (J - R) QW + WQ (J - R)^T = -BB^T \]  \hspace{1cm} (2.38)

and

\[ Q (J - R)^T M + M (J - R) Q = -QBB^T Q. \]  \hspace{1cm} (2.39)

Now bringing the system (2.1) into a balanced form where

\[ W = M = \text{diag}(\sigma_1, \ldots, \sigma_n) \]

(see [62], [79]), and computing the square roots of the eigenvalues of \( MW \), which are equal to the Hankel singular values \( (\sigma_1, \ldots, \sigma_n) \) (see [32]), provide us the information about the number of state components of the system to be reduced. The state components corresponding to the smallest Hankel singular values \( (\sigma_{r+1}, \ldots, \sigma_n) \) (for \( r \) being the dimension chosen for the reduced system) require large amount of the incoming energy to be reached and give small amount of the outgoing energy to be observed. Therefore they are less important from the energy point of view and can be removed from the system (see also [2]).

The balancing coordinate transformation \( S, x = Sx_b, \) where \( x_b \) denotes balanced coordinates, clearly preserves the port-Hamiltonian structure of the system (2.1):

\[
\begin{align*}
\dot{x}_b &= (J_b - R_b) Q_b x_b + B_b u, \\
y &= B_b^T Q_b x_b,
\end{align*}
\]  \hspace{1cm} (2.40)

where \( S^{-1} R S^{-T} = R_b = R_b^T \geq 0 \) is the dissipation matrix, \( S^{-1} J S^{-T} = J_b = -J_b^T \) is the structure matrix and \( S^T Q S = Q_b = Q_b^T \geq 0 \) is the energy matrix in balanced coordinates \( x_b \). In this case \( B_b = S^{-1} B \). In fact, any coordinate transformation would preserve the port-Hamiltonian structure.

Similarly, the port-Hamiltonian structure is preserved applying balancing coordinate transformation \( T, e = Te_b \), to the port-Hamiltonian system (2.27) in co-energy coordinates.

Sometimes it is useful to proceed using the so-called scattering representation for port-Hamiltonian systems (motivated by the electrical domain), where controllability and observability Gramians are related to the energy matrix \( Q \) as \( M \leq Q \leq W^{-1} \), and the Hankel singular values \( \sigma_i \) are all less than one: \( \sigma_i \leq 1, i = 1, \ldots, n \) (see [88, 87, 74]).

Nevertheless in this chapter we proceed without using scattering coordinates.
2.6. Reduction of port-Hamiltonian systems in the general case

For a general port-Hamiltonian system in energy (2.1) or co-energy (2.27) coordinates with no uncontrollable/unobservable but with "hardly" controllable/observable states we may apply balancing, as explained in the previous section, and use one of the following structure preserving reduction techniques. Since the considered techniques are applied to the port-Hamiltonian systems in balanced coordinates, for the sake of simplicity in this section we skip the subscript "b", writing $x$, $e$, $J$, $R$, $Q$, $B$ instead of $x_b$, $e_b$, $J_b$, $R_b$, $Q_b$, $B_b$.

2.6.1. Effort-constraint reduction

Consider a full order port-Hamiltonian system (2.1). We balance the system (2.1), but now in co-energy coordinates (and thus with another change of coordinates (2.26)), obtaining the following balanced representation of our system

$$
\begin{align*}
\dot{e} &= Q(J - R)e + QBu, \\
y &= B^T e,
\end{align*}
$$

(2.41)

where the lower part of the state vector $e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$ is the most difficult to reach and to observe.

Consider the system (2.1) again, but now in the coordinates where the system (2.41) is balanced

$$
\begin{align*}
\dot{x} &= (J - R)e + Bu, \\
y &= B^T e.
\end{align*}
$$

(2.42)

A natural choice for the reduced model would be a model which contains only the $e_1$ dynamics since the lower part of the state vector $e_2$ with the imposed constraint

$$e_2 = Q_{21}x_1 + Q_{22}x_2 \approx 0.$$

(2.43)

is much less relevant from the energy point of view. Therefore the reduced system takes the following form

$$
\begin{align*}
\dot{x}_1 &= (J_{11} - R_{11})e_1 + B_1 u \\
&= (J_{11} - R_{11})(Q_{11}x_1 + Q_{12}x_2) + B_1 u, \\
y &= B_1^T e_1 = B_1^T (Q_{11}x_1 + Q_{12}x_2).
\end{align*}
$$

(2.44)
After substituting $x_2 \approx -Q_{22}^{-1}Q_{21}x_1$ from (2.43) into (2.44), assuming that $Q_{22}^{-1}$ exists, the reduced system will take the final form in energy coordinates

\[
\begin{align*}
\dot{x}_1 &= (J_{11} - R_{11})(Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})x_1 + B_1 u, \\
\bar{y} &= B_1^T(Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})x_1,
\end{align*}
\]

which is again a port-Hamiltonian system produced by the effort-constraint method. For the details on the effort-constraint method see Chapter 3.

**Remark 2.11.** The reduced order port-Hamiltonian system (2.45) is automatically passive since the preservation of the port-Hamiltonian structure implies the preservation of the passivity property (see [86]).

**Remark 2.12.** The effort-constraint method with the reduced model (2.45) may seem to be similar to the well-known balanced truncation (see e.g. [2] and the references therein) which gives the reduced order model of the form

\[
\begin{align*}
\dot{x}_1 &= A_{11}x_1 + B_1 u, \\
\bar{y} &= C_1x_1.
\end{align*}
\]

The balanced truncation method does not preserve the port-Hamiltonian structure. Indeed, writing out $A_{11}, B_1, C_1$ in (2.46) yields

\[
\begin{align*}
\dot{x}_1 &= (F_{11}Q_{11} + F_{12}Q_{21})x_1 + B_1 u, \\
\bar{y} &= (B_1^TQ_{11} + B_2^TQ_{21})x_1,
\end{align*}
\]

which is clearly not port-Hamiltonian, since in general in balanced coordinates $Q_{21} \neq 0$.

### 2.6.2. Another approximation

Another way of model reduction for port-Hamiltonian systems is the following. We balance the system (2.1) and approximate the lower part of the state vector, but now in energy coordinates, plus its dynamics. Using the notation $F := J - R$ we obtain

\[
\begin{align*}
x_2 &\approx 0, \\
\dot{x}_2 &= (F_{21}Q_{11} + F_{22}Q_{21})x_1 + B_2 u \approx 0,
\end{align*}
\]

resulting in the reduced port-Hamiltonian system of the form

\[
\begin{align*}
\dot{x}_1 &= (F_{11}Q_{11} + F_{12}Q_{21})x_1 + B_1 u, \\
y &= (B_1^TQ_{11} + B_2^TQ_{21})x_1.
\end{align*}
\]
From (2.48) it immediately follows that \( Q_{21}x_1 \approx -F_{22}^{-1}F_{21}Q_{11}x_1 - F_{22}^{-1}B_2u \), assuming that \( F_{22}^{-1} \) exists. Substituting in (2.49) yields
\[
\begin{align*}
\dot{x}_1 &= (F_{11} - F_{12}F_{22}^{-1}F_{21})Q_{11}x_1 + (B_1 - F_{12}F_{22}^{-1}B_2)u, \\
\dot{y} &= (B_1^T - B_2^TF_{22}^{-1}F_{21})Q_{11}x_1 - (B_2^T F_{22}^{-1}B_2)u,
\end{align*}
\]
(2.50)
which is, if \((F_{12}F_{22}^{-1})^T = F_{22}^{-1}F_{21}\), again a reduced system in the port-Hamiltonian form.

**Remark 2.13.** We underline that the approximation (2.50) is not the flow-constraint approximation (even though we called (2.50) as the flow-constraint approximation in [71], [74]), since in general it is not structure preserving: in general the condition \((F_{12}F_{22}^{-1})^T = F_{22}^{-1}F_{21}\) is not satisfied. For the details on the structure preserving flow-constraint method see Chapter 3.

**Remark 2.14.** The approximation (2.50) (as well as the flow-constraint method of Chapter 3) is different from the less well-known singular perturbation method (see [26], [35]) with the reduced order model
\[
\begin{align*}
\dot{x}_1 &= (A_{11} - A_{12}A_{22}^{-1}A_{21})x_1 + (B_1 - A_{12}A_{22}^{-1}B_2)u, \\
y_{sp} &= (C_1 - C_2A_{22}^{-1}A_{21})x_1 - C_2A_{22}^{-1}B_2u.
\end{align*}
\]
(2.51)
A similar argument as in Remark 2.12 shows that the singular perturbation method (2.51) does not preserve the port-Hamiltonian structure.

### 2.7. Example

We consider a ladder network, similar to that of [81]. In our case we take the current \( i \) as the input and the voltage of the first capacitor \( U_{c_1} \) as the port-Hamiltonian output. The state variables are as follows: \( x_1 \) is the charge \( q_1 \) over \( C_1 \), \( x_2 \) is the flux \( \phi_1 \) over \( L_1 \), \( x_3 \) is the charge \( q_2 \) over \( C_2 \), \( x_4 \) is the flux \( \phi_2 \) over \( L_2 \), etc.

In our case, as in [81], the resulting Hankel singular values obtained after balancing are not distinct enough. In order to overcome this difficulty we inject additional dissipative elements, in this case resistors \( R_1, \ldots, R_n \) into the model as shown in Fig. 2.1.

We take unit values of the capacitances \( C_i \) and inductances \( L_i \), while \( R_i = 0.2, \ i = 1, \ldots, n, \ R_{n+1} = 0.4 \). A minimal realization of this port-Hamiltonian ladder network for the order \( n = 6 \) is
\[
A = \begin{bmatrix}
0 & -\frac{1}{C_1} & 0 & 0 & 0 & 0 \\
\frac{1}{C_1} & -\frac{1}{L_1} & -\frac{1}{C_2} & 0 & 0 & 0 \\
0 & -\frac{1}{L_1} & 0 & -\frac{1}{C_2} & 0 & 0 \\
0 & 0 & -\frac{1}{C_2} & -\frac{1}{L_2} & -\frac{1}{C_3} & 0 \\
0 & 0 & 0 & -\frac{1}{L_2} & 0 & -\frac{1}{R_4 + R_5} \\
0 & 0 & 0 & 0 & -\frac{1}{C_3} & -\frac{1}{R_4 + R_5}
\end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} \frac{1}{C_1} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T.
\]
2. Balancing for port-Hamiltonian systems

\[ u = I \]

\[ y = U C_1 \]

\[ C_1, q_1 \]

\[ C_2, q_2 \]

\[ R_{(n/2)} \]

\[ L_{(n/2)}, \phi_{(n/2)} \]

\[ C_{(n/2)} \]

\[ q_{(n/2)} \]

\[ R_{(n/2+1)} \]

\[ \ldots \]

\[ \frac{1}{C_1} \]

\[ \frac{1}{L_1} \]

\[ \frac{1}{C_2} \]

\[ \frac{1}{L_2} \]

\[ \frac{1}{C_3} \]

\[ \frac{1}{L_3} \]

Figure 2.1.: Ladder network

where \( A = (J - R)Q \) with

\[
J = \begin{bmatrix}
0 & -1 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix},
\]

\[
R = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & R_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & R_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & R_3 + R_4
\end{bmatrix},
\]

\[
Q = \begin{bmatrix}
\frac{1}{C_1} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{L_1} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{C_2} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{L_2} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{C_3} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{L_3}
\end{bmatrix}.
\]

Adding another \( LC \) pair to the network (with appropriate resistors), which would correspond to an increase of the dimension of the model by two, will modify the \( ABC \)-model in the following way. The subdiagonal of the matrix \( A \) will contain additionally \( L_{n/2-1}^{-1}, C_{n/2}^{-1} \). The superdiagonal of \( A \) will contain \( -C_{n/2}^{-1}, -L_{n/2}^{-1} \). Furthermore, the main diagonal of \( A \) will have \( -\frac{R_n}{L_{n/2-1}} \) in the \((n-2, n-2)\) position, zero in the \((n-1, n-1)\) position and \( -\frac{R_n}{L_{n/2}} + \frac{R_n}{L_{n/2+1}} \) in the \((n, n)\) position.

We considered the twelve-dimensional full order minimal port-Hamiltonian ladder network and reduced it to a five-dimensional one by the effort-constraint method from the previous section and the usual balanced truncation method. The non-minimal system can be first reduced to a minimal one as shown in Section 2.3. The Hankel singular values (HSV) of the full order system in decreasing order are shown in Table 2.1.
2.7. Example

![Singular Values Diagram](image)

**Figure 2.2.: Frequency response**

It is a well-known fact that the transfer functions of reduced order models obtained by the balanced truncation method approximate the full order transfer functions well in the high-frequency region and not that well in the low-frequency one (of course, depending on the application considered). Since the effort-constraint method is similar to the balanced truncation method (with the above explained modification in order to preserve the port-Hamiltonian structure and passivity) we expected approximations of similar nature. In Fig. 2.2, 2.3 the frequency response of the full order model is shown vs. the frequency responses of the reduced order models, obtained by the balanced truncation method and the effort-constraint method respectively. These fig-

<table>
<thead>
<tr>
<th>Index</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>HSVs</td>
<td>0.8019</td>
<td>0.3746</td>
<td>0.2766</td>
<td>0.2172</td>
<td>0.2150</td>
<td>0.1537</td>
</tr>
<tr>
<td>Index</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
</tr>
<tr>
<td>HSVs</td>
<td>0.1497</td>
<td>0.0966</td>
<td>0.0727</td>
<td>0.0715</td>
<td>0.0125</td>
<td>0.0123</td>
</tr>
</tbody>
</table>
2. Balancing for port-Hamiltonian systems

Figure 2.3.: Frequency response

ures show that the reduced order transfer functions indeed behave quite similarly.

In Table 2.2 the $H_\infty$- and $H_2$-norms are shown for the error systems obtained after the balanced truncation reduction and the effort-constraint reduction. It follows that the error-norms for the effort-constraint method are larger than those for the balanced truncation method. This may imply that the usual balanced coordinates are not the best choice for the effort-constraint method.

Important questions concerning general error bounds for the structure preserving port-Hamiltonian model reduction methods and the physical realization of the obtained port-Hamiltonian reduced order models are proposed for

<table>
<thead>
<tr>
<th>Reduced order system by</th>
<th>$H_\infty$-norm</th>
<th>$H_2$-norm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Balanced truncation method</td>
<td>0.3411</td>
<td>0.1541</td>
</tr>
<tr>
<td>Effort-constraint method</td>
<td>1.1999</td>
<td>0.4491</td>
</tr>
</tbody>
</table>
future investigation.

2.8. Conclusions

We have shown in Section 2.3 that a full order uncontrollable/unobservable port-Hamiltonian system can be reduced to a controllable/observable system, which is again port-Hamiltonian, by exploiting the invariance of the reachable/unobservable subspaces of the original systems. We considered reduction to minimality in both the energy and co-energy variable representations of port-Hamiltonian systems.

Balancing for port-Hamiltonian systems is discussed in Section 2.5. The effort-constraint reduction method is introduced in Section 2.6 along with another approximation method, which is in general not structure preserving. We applied the effort-constraint method to a general port-Hamiltonian full order system showing that the method preserves the port-Hamiltonian structure for the reduced order system, as well as the passivity property. In Section 2.7 we considered a full order ladder network and applied the balanced truncation method and the effort-constraint method in balanced coordinates in order to obtain reduced order models and compared the results.

The effort-constraint method motivates to investigate further important issues about the general error bounds and the physical realization of the reduced order systems, e.g. as an electrical circuit.