Chapter 4

The $N$-body problem

The $N$-body scattering problem cannot simply be regarded as an extension of the two-body scattering problem. In the two-body case the scattering problem can be cast into a well-defined mathematical form through time-independent scattering theory. The two-body Lippmann–Schwinger (LS) equation is mathematically well behaved provided the interaction satisfies a number of reasonable assumptions (cf. Chapter 3), since the LS kernel is compact under these assumptions.

Formally, there is no problem in the construction of $N$-body equivalents of the Lippmann–Schwinger equation. The solution of such an equation, however, is problematic. The source of the problems is the presence of singularities in the kernel, which destroys (at least) one of the useful properties of LS equations: describing the dynamics plus the boundary conditions in a single integral equation. This can be fixed by including constraints to ensure the correct boundary conditions. However, without a compact kernel, we find ourselves in a mathematical wasteland where little is known about existence or uniqueness of solutions, or even how to obtain such a solution.

The mathematically correct (and really the only practical) solution to this problem lies in a reformulation of the Lippmann–Schwinger equations into a new equation, which has a kernel that becomes compact after a sufficient number of iterations. This is what Faddeev achieved for the three-body problem [Faddeev, 1960; 1965]. The nontrivial task of finding a systematic generalization for any number of bodies was taken up, and brought to a successful end by Yakubovsky seven years later [Yakubovskii, 1967].

The remainder of this chapter is organized as follows. In Sec. 4.1 the $N$-body scattering problem is defined. Notation which is used throughout the rest of this thesis is established, formal $N$-body Lippmann–Schwinger equations are derived, and the problems these equations suffer from are discussed. In Sec. 4.2 the mathematically rigorous solution by formulation of the Faddeev–Yakubovsky equations is discussed. In Sec. 4.3 the Faddeev–Yakubovsky equations are formulated in configuration space. The general structure and the boundary conditions of the Faddeev–Yakubovsky equations and specific formulations are discussed in detail.
4.1 $N$-body scattering

The scattering process for more than two particles is much richer than the two-body scattering process. If the particles are assumed to be structureless, two particles can only scatter elastically, i.e., their energies and momenta may redistributed, but the input (two free particles) equals the output. This situation changes dramatically if a third particle is brought into play. Let us consider the scattering of a particle $a$ on a bound state ($bc$) of two other particles $b$ and $c$. There are several possibilities for the final product. Firstly, the final product may be a free particle $a$ and a pair ($bc$), possibly with redistributed energy and momentum. This process is known as the elastic channel, and is not essentially different from two-body scattering. However, if the center-of-mass energy is sufficiently high, a bound state ($ab$) or ($ac$) may be formed, replacing the incoming bound pair ($bc$). These processes are known as rearrangement. Finally, if there is sufficient energy to break up the pair without formation of a new pair, the final state may include three essentially free particles. This is known as the breakup channel.

Channels can be open (i.e., the particular process may occur) or closed depending on the center-of-mass energy. In the example of neutron-deuteron scattering one rearrangement channel is open (forming a “different” deuteron), the breakup channel opens at the binding energy of the deuteron ($\approx 2.2$ MeV). The energy at which a channel opens is known as the threshold energy. (Note that in this example the second rearrangement channel is inaccessible, since two neutrons cannot form a bound state.)

When more than three particles scatter, the scattering structure becomes even more involved. If the interactions allow the formation of bound states with more than two particles, several types of rearrangement must be distinguished. In fact, the number of channels grows exponentially with the number of particles.

In the following subsections I will discuss the failure of few-body scattering theory before the Faddeev formalism provided the necessary breakthrough. But first it is necessary to establish notation. This is done in Sec. 4.1.1. After that, the development of few-body scattering theory and its failure to describe the physics in a mathematically sound way will be discussed in Sec. 4.1.2. A somewhat more thorough look at the specific problems will be given in Sec. 4.1.3. Faddeev’s solution to these problems will be discussed in Sec. 4.2.

4.1.1 Definitions and notation

Since the scattering structure is closely related to the clustering structure, it is convenient to introduce the notion of partitions and partition chains. (These were first used by Yakubovsky [Yakubovskii, 1967].) An $N$-body system can be broken up or partitioned in fragments. A partition can be described by writing down the particle numbers and collecting and bracketing the fragment members. For example, there are seven two-fragment partitions of four particles: (123)4, (124)3,
(12)(34), (13)(24), and (14)(23). Note that one-particle fragments are not bracketed. The order in which the particles or the fragments appear is irrelevant. Partitions will be denoted by a letter and a subscript to define its number of fragments. The following relation between partitions is defined:

**Definition 4.1** A partition \( a_i \) is said to follow partition \( a_k \), if it can be obtained by breaking up one or more of the fragments of \( a_k \). This relationship is written as \( a_i \subseteq a_k \). As will become clear in the section describing the Faddeev–Yakubovsky equations, another important concept is that of a partition chain. This is an ordered chain of partitions \((a_k, a_{k+1}, \ldots, a_{N-1})\), for which the short-hand notation \( a_k \) will be used. Note that for all pairs of partitions in a chain \( a_k = (a_k, a_{k+1}, \ldots, a_{N-1}) \) we must have \( a_i \subseteq a_n \) if \( i > n \). There are eighteen different partition chains in the four-body system: \((12)(34), (12)(34), (13)(24), (12)(34), (12)(34), (12)(34), \ldots, (34)(1, (34)(12) \) (twelve in total), and \((12)(34), (12)(34), (12)(34), (12)(34), (12)(34), (12)(34), \ldots, (14)(23), (23)(14) \) (six in total). Note the distinction between the first twelve and the last six chains. (These will later be described as being of the \( 3 + 1 \) and the \( 2 + 2 \) type, respectively.) It can easily be shown that an \( n \)-body system has \( 2^{1-n}n!(n-1)! \) different partition chains. The number of chains of an \( n \)-body system, \( N_n \), can be written as

\[
N_n = \frac{1}{2} \sum_{i=1}^{n-1} \binom{n}{i} \binom{n-2}{i-1} N_i N_{n-i}. \tag{4.1}
\]

The sum is over all possible ways of subdividing the system into two pieces. The first factor in the sum counts the number of ways of selecting \( i \) bodies out of \( n \), the last two factors count the number of chains arising from these two subsystems, and the second factor counts the number of possibilities in which the \( n-2 \) remaining subdivision steps can be distributed over the two fragments. The factor \( \frac{1}{2} \) is to avoid double counting. It follows by induction, that

\[
N_n = 2^{1-n}n!(n-1)!. \tag{4.2}
\]

A partition chain can be thought of as the process of breaking up an \( N \)-body cluster into smaller and smaller fragments, or the building up of cluster from smaller fragments. With a partition chain, a natural (Jacobi) set of coordinates can be associated. It is constructed by taking, for every step in the building process, the vector connecting the centers of mass of the two fragments that are being put together. Since there are \( N-1 \) steps in the process (going from \( N \) down to 1 fragment), one more coordinate is needed to fully describe the positions of all the particles. The most natural choice is the center of mass of the total system. (Since the particles are assumed to interact only with each other, the dependence on this variable can, and will, be factored out.)
In Fig. 4.1 two Jacobi “trees” and the partition chains from which they were constructed are shown. As in the two-body case (cf. Chapter 2), the Jacobi vectors will be scaled to remove the masses from the kinetic energy operator:

\[ r_f^g = \left(2m_f^g\right)^{1/2}(r_f - r_g), \]  

(4.3)

where \( f \) and \( g \) are the fragments being coupled, \( r_f \) and \( r_g \) are the positions of their centers of mass, and \( m_f^g \) is their reduced mass \( m_f m_g/(m_f + m_g) \). Center-of-mass positions \( r_{fg} \) of fragments and fragment masses \( m_{fg} \) are defined by

\[ r_{fg} = \frac{m_f r_f + m_g r_g}{m_{fg}}, \]  

(4.4)

\[ m_{fg} = m_f + m_g. \]  

(4.5)

The Jacobi and center-of-mass coordinates have the following property:

\[ (r_f^g)^2 + 2m_{fg}(r_{fg})^2 = 2m_f(r_f)^2 + 2m_g(r_g)^2, \]  

(4.6)

Momenta conjugated to the Jacobi and center-of-mass momenta can be defined:

\[ k_{fg} = k_f + k_g, \]  

(4.7)

\[ k_g^f = \left(2m_g^f\right)^{-1/2}\left(\frac{m_g k_f - m_f k_g}{m_{fg}}\right). \]  

(4.8)

These have the following properties:

\[ k_g^f \cdot r_f^g + k_{fg} \cdot r_{fg} = k_f \cdot r_f + k_g \cdot r_g, \]  

(4.9)

\[ \frac{(k_{fg})^2}{2m_{fg}} + (k_g^f)^2 = \frac{k_f^2}{2m_f} + \frac{k_g^2}{2m_g}. \]  

(4.10)
Using these definitions, the Hamiltonian for a system consisting of \( N \) particles interacting via pair interactions can be written as

\[
H' = \sum_i T_i + \sum_{i,j>i} V'_{ij}, \quad (4.11)
\]

where \( T_i \) is the kinetic energy of particle \( i \) (i.e., \((\hbar^2/m_u)\partial^2/\partial r_i^2\)), and \( V'_{ij} \) is the potential acting between particles \( i \) and \( j \) (i.e., \( V'_{ij}(r_i-r_j) \), if local potentials are considered). Using the scaling of Eq. (4.3) and factoring out \( m_u/\hbar^2 \), where \( m_u \) is the unit mass in which the fragment masses are expressed, the kinetic energy operator can be reduced to

\[
H_0 = \frac{m_u}{\hbar^2} \sum_{n=1}^{N} T_i = \frac{K^2}{2M} + \sum_{i=1}^{N-1} (k_{fi}^{g_i})^2 = -\frac{\nabla^2}{2\tilde{M}} - \sum_{i=1}^{N-1} \nabla^2_{r'_{fi}}, \quad (4.12)
\]

where \( f_i \) and \( g_i \) are the fragments that are coupled in step \( i \), \( K \) is the total momentum, and \( \tilde{M} \) is the total mass of the system. The potentials will be rewritten as

\[
V_{ij}(r'_{ij}) = \frac{m_u}{\hbar^2} V'_{ij}((2m_j)^{-1/2}r'_{ij}). \quad (4.13)
\]

The reduced Hamiltonian in center-of-mass coordinates will be defined as

\[
H = \sum_{i=1}^{N-1} (k_{fi}^{g_i})^2 + \sum_{i,j>i} V_{ij}. \quad (4.14)
\]

Different sets of coordinates can always be transformed into each other by (products of) rotations:

\[
\begin{pmatrix}
  r'_{hf} \\
  r'_{gh}
\end{pmatrix}
= \begin{pmatrix}
  -\cos \mu_h^{fg} & \sin \mu_h^{fg} \\
  -\sin \mu_h^{fg} & -\cos \mu_h^{fg}
\end{pmatrix}
\begin{pmatrix}
  r'^g_h \\
  r'^f_h
\end{pmatrix}, \quad (4.15a)
\]

\[
\begin{pmatrix}
  r'_{hf} \\
  r'_{gh}
\end{pmatrix}
= \begin{pmatrix}
  -\cos \mu_h^{gf} & \sin \mu_h^{gf} \\
  -\sin \mu_h^{gf} & -\cos \mu_h^{gf}
\end{pmatrix}
\begin{pmatrix}
  r'^g_h \\
  r'^f_h
\end{pmatrix}. \quad (4.15b)
\]

(Note that \( r'^{fg}_{gh} \) was written here to denote the relative coordinate of the two fragments \( f \) and \( gh \). The three fragments \( f \), \( g \), and \( h \) may not have any constituents in common.) The transformation angle \( \mu_h^{fg} \) is defined by

\[
\cos \mu_h^{fg} = \sqrt{\frac{m_f m_g}{m_f m_g h}}, \quad (4.16)
\]

\[
\sin \mu_h^{fg} = \varepsilon_{fg h} \sqrt{\frac{m_f m_g h}{m_f m_g h}}, \quad (4.17)
\]
where \( \varepsilon_{P(f,g,h)} \) is +1 (-1) if \( P(f,g,h) \) is an even (odd) permutation of \( \{f,g,h\} \). For \( N \leq 3 \) all Jacobi coordinate systems are “topologically” the same, i.e., they can be obtained from each other by exchanging particle labels. For \( N > 3 \) the different Jacobi coordinate systems may be topologically different (this is due to the fact that a fragment of three particles can be divided into two fragments in just one manner (apart from permutations), whereas for more than three particles there is more than one possibility; the actual number of possibilities is \( \lfloor N/2 \rfloor \), but it is still possible to transform any coordinate system into any other using a product of rotations as shown above.

Let me finish by defining a number of operators for fragmented systems, i.e., systems where the interaction between different fragments is thought to be zero, which play an essential role in the description of the \( N \)-body scattering process. **Definition 4.2** The *intrafragment* interaction of a partition \( a_i \) is described by the potential \( V_{a_i} \), which is obtained by switching off the potential operating between particles from different fragments.

A number of operators can be defined using this definition:

\[
\begin{align*}
H_{a_i} &= H_0 + V_{a_i}, \quad (4.18) \\
G_{a_i}(z) &= (z - H_{a_i})^{-1}, \quad (4.19) \\
T_{a_i}(z) &= V_{a_i} + V_{a_i} G_{a_i}(z) V_{a_i}. \quad (4.20)
\end{align*}
\]

Special cases are:

\[
\begin{align*}
V_{a_1} &= V, \quad (4.21) \\
V_{a_N} &= 0, \quad (4.22) \\
H_{a_1} &= H, \quad (4.23) \\
H_{a_N} &= H_0, \quad (4.24) \\
G_{a_1}(z) &= G(z), \quad (4.25) \\
G_{a_N}(z) &= G_0(z), \quad (4.26) \\
T_{a_1}(z) &= T(z), \quad (4.27) \\
T_{a_N}(z) &= 0, \quad (4.28)
\end{align*}
\]

where \( T(z) \) is the full \( N \)-body \( T \) operator. Note that all \( T \) operators satisfy the following algebraic equations:

\[
\begin{align*}
T_{a_i}(z) &= V_{a_i} + V_{a_i} G_0(z) T_{a_i}(z), \quad (4.29) \\
T_{a_i}(z) &= V_{a_i} + T_{a_i}(z) G_0(z) V_{a_i}. \quad (4.30)
\end{align*}
\]

These equations immediately follow from the definitions. Solving these algebraic equations is equivalent to solving the \( N \)-body problem. The problems encountered when trying to solve these equations are discussed in the next section.
4.1 N-body scattering

4.1.2 Formal scattering theory

Formal N-body scattering theory started to develop in the 1950s. Lippmann and Schwinger [1950] were the first to construct practical mathematical equations by connecting time-dependent scattering theory to the time-independent formalism. This work was extended using a somewhat different point of view by Gell-Mann and Goldberger [1953]. They briefly touch upon the subject of multichannel scattering. A generalization of time-dependent scattering theory to multichannel processes and a re-examination of the relationship between the physical observables and mathematical expressions was taken up by Ekstein [1956]. Lippmann [1956] looked further into these questions and found a result, which was appreciated as being a potential source of problems by Foldy and Tobocman [1957]. Not driven by a desire to circumvent these problems, but to simplify the practical solution of the three-body problem, Eyges [1959] proposed to split the wave function into three components, each associated with just one of the three pair interactions.

It was at this stage, that Faddeev published his work [Faddeev, 1960], which would transform few-body physics into a mature, separate field. Faddeev not only realized why there had been so many problems solving the three-body problem, he also found an elegant solution, finally putting the theory of three-body scattering on solid mathematical ground. His solution consists of a reformulation of the Lippmann–Schwinger equations by splitting the wave function into three components in a specific manner.

I will continue by formulating formal three-body scattering equations of the Lippmann–Schwinger type. As mentioned before, there are four different final states. With each an asymptotic wave function can be associated. The first case is that of three free particles, which has an asymptotic form of

\[ \Phi_0(r_i^j, r_k^l) = e^{i k_j^i \cdot r_j^i + i k_l^k \cdot r_l^k}. \]  

(4.31)

The corresponding energy is \( E_0 = (k_j^i)^2 + (k_l^k)^2 \). The other three states are each associated with a bound state of one of the three possible pairs, and a free third particle, i.e., a specific two-fragment partition \( a_2 = (ij)k \). The asymptotic wave functions are:

\[ \Phi_{a_2, n}(r_i^j, r_k^l) = \psi_{a_2, n}(r_i^j) e^{i k_l^k \cdot r_l^k}. \]  

(4.32)

The function \( \psi_{a_2, n} \) is a two-particle bound-state wave function, i.e., a solution of the equation

\[ [-\nabla_r^2 + V_{a_2}(r)] \psi_{a_2, n}(r) = \epsilon_{a_2, n} \psi_{a_2, n}(r), \]  

(4.33)

where

\[ \epsilon_{a_2, n} < 0, \quad n = 1, 2, \ldots \]  

(4.34)

\(^1\text{Gerjuoy [1958] formulated multichannel scattering theory by investigating the asymptotic behavior of the wave functions, rather than by projecting on momentum eigenstates.}\)
The energy associated with $\Phi_{a_2,n}$ is

$$E_{a_2,n} = e_{a_2,n} + (k_{ij}^k)^2. \quad (4.35)$$

Now Lippmann–Schwinger equations for three-particle scattering processes can be constructed in a manner similar to that described for two-particle scattering in Chapter 3. Recall that the full scattering wave function $\Psi$ can be obtained from the incoming asymptotic state $\Phi$ using

$$\Psi = \lim_{\varepsilon \to 0} \varepsilon G(E + i\varepsilon)\Phi, \quad (4.36)$$

and that from this definition one can derive the LS equation by applying one of the operator identities for $G$. For the three-body problem there is more than one operator identity, each of which can naively be thought of producing a particular type of outgoing waves.

Let us first try to find an equation which will have solutions for which all three particles are free in the final state, by applying the identity

$$G(z) = G_0(z) + G_0(z)V G(z) \quad (4.37)$$

to Eq. (4.36):

$$\Psi = \lim_{\varepsilon \to 0} \varepsilon G_0(E_{n} + i\varepsilon)\Phi + G_0(E_{n} + i0)V\Psi. \quad (4.38)$$

Using the fact that

$$\lim_{\varepsilon \to 0} \varepsilon G_0(E_{0} + i\varepsilon)\Phi_0 = \Phi_0, \quad (4.39)$$

we obtain our first LS equation for the three-body scattering problem:

$$\Psi_0 = \Phi_0 + G_0(E_{0} + i0)V\Psi_0. \quad (4.40)$$

Unfortunately, this equation is not an adequate mathematical tool for calculating three-body scattering states due to that fact that when $\Phi_{a_2,n}$ is substituted for $\Phi$ in Eq. (4.36), the corresponding $\Psi_{a_2,n}$ satisfies a homogeneous equation. This is easily seen as follows: writing down Eq. (4.38) with $\Phi_{a_2,n}$ as a starting point,

$$\Psi_{a_2,n} = \lim_{\varepsilon \to 0} \varepsilon G_0(E_{a_2,n} + i\varepsilon)\Phi_{a_2,n} + G_0(E_{a_2,n} + i0)V\Psi_{a_2,n}, \quad (4.41)$$

and realizing that

$$(G_0(z)\Phi_{a_2,n})(r^i_j, r^k_{ij}) = -\frac{e^{ik_{ij}^k r^k_{ij}}}{4\pi} \int d\mathbf{r}' \frac{e^{i[z-(k_{ij}^k)^2]^{1/2}|r^i_j-r'|}}{|r^i_j-r'|}\psi_{a_2,n}(\mathbf{r}'), \quad (4.42)$$
with Green's function satisfying does not have a singularity at \( z = E_{a_2,n} \), we find that \( \Psi_{a_2,n} \) satisfies the equation

\[
\Psi_{a_2,n} = G_0(E_{a_2,n} + 10) V \Psi_{a_2,n},
\]

(4.43)

which implies that the asymptotic behavior of \( \Psi \) is not fixed, since any amount of scattering wave function corresponding to the initial state \( \Phi_{a_2,n} \) may be mixed in. Therefore, Eq. (4.40) fails to give unique solutions.

One might still hope that by using operator identities for \( G \) different from Eq. (4.37) equations can be obtained which do uniquely specify the solutions. To see if this is the case, we apply the identity

\[
G(z) = G_{a_2}(z) + G_{a_2}(z)(V - V_{a_2})G(z)
\]

(4.44)

(which can be associated with final states containing a bound pair), to Eq. (4.36). This procedure does lead to inhomogeneous equations for \( \Psi_0 \) and \( \Psi_{a_2,n} \). However, solutions corresponding to the other asymptotic forms \( \Phi_{b_2,n} \), where \( b_2 \) is a two-fragment partition not equal to \( a_2 \), can still be mixed in. This can be seen by writing out \( G_{a_2}(z) \Phi_{b_2,n} \) in configuration space, which after some simple algebra gives the following expression:

\[
(G_{a_2}(z)\Phi_{b_2,n})(r^i, r^k_{ij}) = \frac{1}{(2\pi)^3} \int dk^k_{ij} e^{ik^k_{ij} r^k_{ij}} \int dr^j \psi_{jk,n}(r^j) e^{i(s^{i_{jk}} k^k_{ij} - t^{i_{jk}} k^k_{ij}) r^j} \times \int dr' G_{a_2}(z - (k^k_{ij})^2; r^i, r') e^{i(s^{i_{jk}} k^k_{ij} + t^{i_{jk}} k^k_{ij}) r'},
\]

(4.45)

with \( b_2 = (jk)i, s^{i_{jk}}_{ij} = \csc \mu_{ij}^{k_j}, t^{i_{jk}}_{ij} = \cot \mu_{ij}^{k_j} \), and where \( G_{a_2}(z; r, r') \) is a two-body Green's function, satisfying

\[
[z + \nabla^2_r - V_{a_2}(r)]G_{a_2}(z; r, r') = \delta(r - r'),
\]

(4.46)

with poles at \( z = e_{a_2,n} \). Equation (4.45) defines a smooth function of \( z \) (due to the integration over \( k^k_{ij} \)), such that

\[
\lim_{\varepsilon \to 0} i\varepsilon G_{a_2}(E_{b_2,n} + i\varepsilon) \Phi_{b_2,n} = 0.
\]

(4.47)

(This identity is known as Lippmann's identity [Lippmann, 1956].) Therefore, the idea of substituting Eq. (4.44) instead of Eq. (4.37) into Eq. (4.36) does not help, since it also leads to an equation which does not specify a wave function uniquely.

From the above reasoning we conclude that no single Lippmann–Schwinger equation of the type shown above has unique solutions. Uniqueness can only be enforced by adding extra constraints, which makes these equations ill suited for practical calculations.
4.1.3 Source of the problems

The discussion in the previous section clearly shows that there are problems with the LS equations for three particles or more. However, the discussion does not give insight into the precise source of the problem, which is a prerequisite if a solution is to be found. Clearly, uniqueness is guaranteed provided the kernel is well behaved and the homogeneous LS equation has no solutions. The fact that the solution is not unique gives rise to two questions: (i) is the kernel well-behaved, and (ii) why are there solutions to the homogeneous equation. Let us first address the second question. A wave function which is a solution to a homogeneous LS equation consists of outgoing waves only, violating conservation of probability unless the LS kernel itself contains some singularity which acts as a probability source. This leads one to suspect that the answer to the first question has to be negative.

The next step is to determine which part of the kernel is the cause of the problems. Let us look carefully at the following three-body LS equation

\[ \psi_{a_2} = \lim_{\varepsilon \downarrow 0} \varepsilon G_{a_2}(E + i\varepsilon)\phi_{b_2} + G_{a_2}(E + i\varepsilon)(V - V_{a_2})\psi_{a_2}. \] (4.48)

As noted by Lippmann [1956], the inhomogeneous term will be zero unless the expansion of \( \phi_{b_2} \) in eigenfunctions of \( H_{a_2} \) contains a delta function in energy, which only happens if \( b_2 = a_2 \). This implies that the solution for \( \psi_{a_2} \) is not unique, since any amount of \( \psi_{b_2} \) (\( b_2 \neq a_2 \)) can be mixed in. This was pointed out by Foldy and Tobocman [1957]. To bring out the part of the kernel responsible for this, replace \( G_{a_2} \) by its expansion \( G_0 + G_0V_{a_2}G_{a_2} \):

\[ \psi_{a_2} = \delta_{b_2}^{a_2}\phi_{b_2} + (G_0 + G_0V_{a_2}G_{a_2})(V - V_{a_2})\psi_{a_2}, \] (4.49)

and observe that since

\[ G_0V_{b_2}\phi_{b_2} = \phi_{b_2}, \] (4.50)

the part \( G_0V_{b_2} \) lets a component \( \phi_{b_2} \) through unharmed, which can act as a "substitute driving term" (i.e., an incoming asymptote) for the equation. Note that the problems only exist when the limit \( \varepsilon \downarrow 0 \) is taken, which means that in principle a solution can be found by taking the limit after calculating the wave function. It is more than likely, however, that the numerical effort to calculate the wave function accurately (i.e., with sufficient suppression of the spurious terms \( \psi_{b_2} \)) increases when \( \varepsilon \) decreases.\(^2\)

Why is the term \( G_0V_{b_2} \) problematic? In momentum space, the potential \( V_{b_2} \) is really of the form

\[ V_{b_2} = \delta(k_{jk}^i - k_{jk}^{i'})V_{j_{jk}}(k_k^i, k_k^{i'}). \] (4.51)

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\(^2\) Assuming that the problems are caused by some unidentified singularity on the real energy axis, this is indeed true: the numerical effort required to solve an integral equation increases as the singularity approaches the path of integration.
4.1 N-body scattering

Note the presence of the delta function, representing the conservation of momentum of the "free" particle with respect to the interacting pair in the permuted system. Note also, that this delta function cannot simply be factored out, as can a delta function representing conservation of total momentum. (Note that in principle, the two-body problem suffers from ambiguities as well. For example, in the case of neutron–proton scattering, solutions with positive energies exist which are not scattering states, namely states in which a deuteron moves with sufficient kinetic energy so that the total energy of the system is still positive. This state satisfies the homogeneous LS equation, and can therefore be mixed into the scattering states. In practice, this is not a problem, because in the two-body problem we may fix the total momentum, so that the delta function describing the conservation of total momentum is factored out, and the deuteron solution is ruled out.) Physically, the delta function represents the one feature which sets multi-body scattering apart from two-body scattering: the possibility that subsets of the total system cluster and are separated (and moving freely) from the rest of the system. Therefore, it should not strike as a surprise that (i) the delta functions are present, and (ii) they are the cause of the trouble with the LS equations. In configuration space the problems can be explained as follows. The configuration-space interaction does not contain delta functions, but is independent of the distance of the third particle to the pair, and therefore not square integrable. The independence of the interaction on this variable implies that there are configurations for which the interaction does not vanish asymptotically, so that the construction of boundary conditions in the manner illustrated in Chapter 3 is no longer possible.

Let me conclude with a final remark, which illustrates the problems from a different point of view: the three-body kernel has a continuous spectrum. This fact is intimately related to the kernel not being square integrable. It can be understood as follows. Suppose $\lambda$ is in the point spectrum of the two-body kernel, i.e.,

$$[G_0(E)V_{a_2}]\psi_{a_2} = \lambda(E)^{-1}\psi_{a_2},$$

(4.52)

where all operators are now two-body operators, and $\psi_{a_2}$ is a normalizable two-body state. Then a three-body state may be constructed which is an approximate eigenvector of the three-body kernel by taking the two-body bound state with an energy of $E - k^2$ (note that a bound state can be shifted to arbitrary low energies; usually by increasing the strength of the potential) and describing the motion of the third particle with respect to the pair as a wave packet of nearly sharp momentum $k$ keeping sufficient separation between the pair and the third particle. By taking an increasingly sharp momentum $k$ (making sure sufficient separation is kept), we can construct states which are arbitrarily close to an eigenstate of the kernel. This eigenstate will have an eigenvalue of $\lambda(E - k^2)$ by construction, and will not be normalizable. Therefore it is not in the point spectrum (or the residual spectrum), implying it must be in the continuous spectrum, implying that the kernel cannot be a completely continuous operator. This is of course just another
way of expressing the same problem, but it is instructive, and it helps understand why the three-body kernel is such a hard nut to crack: any numerical method will rely on some sort of finite rank approximation, something which is totally ineffective if a continuous spectrum exists.

Having the flaw and its source made explicit, we are now ready to look for a way of fixing it. This will be the topic of the next section.

### 4.2 The Faddeev–Yakubovsky equations

A mathematically sound solution to the problems the three-body LS equations suffer from was first given by Faddeev [1960]. He found that the delta functions which lead to unconnected graphs (cf. discussion below) in the Born series for $G$ or $T$ could be removed by reordering the series, and dealing with the unconnected parts as a two-body plus free particle problem analytically. His work was later generalized to the $N$-body problem by Yakubovsky [Yakubovskii, 1967]. Others, such as Weinberg [1964] derived equations which are similar to the Faddeev–Yakubovsky (FY) equations, but do not give a completely systematic description. (Note that singularities due to unconnected diagrams not only occur in few-body scattering, but also in many-body theory [Goldstone, 1957; Hugenholtz, 1957].)

In Sec. 4.2.1 Faddeev’s solution to the three-body problem will be described. The treatment of the $N$-body problem by Yakubovsky is the subject of Sec. 4.2.2.

#### 4.2.1 The Faddeev equations

A simple way of visualizing the problems generated by the delta functions in the kernel is to draw graphs representing the Born series which arises when calculating the full Green’s operator by iterating the kernel $VG_0$ of the LS equation (4.29) for the $T$ operator. Drawing a straight horizontal line for every freely moving particle, and connecting the lines with a vertical wavy line when a potential term is encountered leads to the following pictorial representation:

$$ T = V_{a_2} + V_{b_2} + V_{c_2} + V_{a_2} G_0 V_{b_2} + V_{a_2} G_0 V_{c_2} + V_{b_2} G_0 V_{a_2} + V_{b_2} G_0 V_{c_2} + \cdots $$

$$ = \overline{\text{---}} + \overline{\text{---}} + \overline{\text{---}} + \overline{\text{---}} + \overline{\text{---}} + \overline{\text{---}} + \cdots $$

Here the full expansion up to second order is shown. Note the presence of terms of the type $\overline{\text{---}}$. In these terms (which are represented by unconnected graphs) one of the three particles moves freely, implying conservation of momentum, and therefore a delta function. There will always be terms of this type, no matter how far the expansion is taken, so there is no hope that the delta singularities disappear by iterating the kernel.
However, it is possible to rewrite the series as follows

\[
T = \frac{1}{\omega} + \frac{1}{\omega} + \cdots + \frac{1}{\omega} + \frac{1}{\omega} + \cdots + \frac{1}{\omega} + \frac{1}{\omega} + \cdots \\
+ \frac{1}{\omega} + \frac{1}{\omega} + \cdots + \frac{1}{\omega} + \frac{1}{\omega} + \cdots + \frac{1}{\omega} + \frac{1}{\omega} + \cdots \\
= \frac{1}{\omega} + \frac{1}{\omega} + \frac{1}{\omega} + \frac{1}{\omega} + \frac{1}{\omega} + \frac{1}{\omega} + \cdots ,
\]

(4.54)

where \(\frac{1}{\omega}\) is \(T_{a_2}\) as defined by Eq. (4.29). Note that \(T_{a_2}\) is a two-body operator (embedded in three-body space) and can be calculated “cheap.” In the preceding all leading sequences \(\frac{1}{\omega} + \frac{1}{\omega} + \cdots\) were collected into \(\frac{1}{\omega}\). This can be done for the rest of the series as well:

\[
T = \frac{1}{\omega} + \frac{1}{\omega} + \frac{1}{\omega} + \frac{1}{\omega} + \frac{1}{\omega} + \frac{1}{\omega} + \cdots \\
= \frac{1}{\omega} + \frac{1}{\omega} + \frac{1}{\omega} + \frac{1}{\omega} + \frac{1}{\omega} + \frac{1}{\omega} + \cdots \\
+ \frac{1}{\omega} + \frac{1}{\omega} + \frac{1}{\omega} + \frac{1}{\omega} + \frac{1}{\omega} + \frac{1}{\omega} + \cdots ,
\]

(4.55)

where two-body \(T\) matrices immediately following each other are taken to describe different pairs, ensuring connectedness of each term (except for the leading ones). The final step is to define operators \(T^{a,b}\) which satisfy

\[
T^{a_2,b_2} = T_{a_2} \delta(a_2, b_2) + T_{a_2} G_0 \sum_{d_2 \neq a_2} T^{d_2,b_2} .
\]

(4.56)

Obviously the original Born series for the \(T\) matrix, Eq. (4.55), is retrieved by adding the nine equations in Eq. (4.56), so that the following identification can be made:

\[
T = \sum_{a_2 b_2} T^{a_2,b_2} .
\]

(4.57)

Note that the \(T\) operators can also be defined by

\[
T^{a_2,b_2} = V_{a_2} \delta(a_2, b_2) + V_{a_2} G V_{b_2} .
\]

(4.58)

Similar equations can be set up for the Green’s operator or the wave functions, by defining objects \(G^{a_2}\):

\[
G^{a_2} = G_0 \sum_{b_2} T^{a_2,b_2} G_0 ,
\]

(4.59)

which satisfy the equation

\[
G^{a_2} = G_{a_2} - G_0 + G_0 T_{a_2} \sum_{b_2 \neq a_2} G^{b_2} .
\]

(4.60)
The full Green’s operator can be constructed from these objects as follows:

\[ G = G_0 + \sum_{\alpha_2} G^{\alpha_2} . \]  

(4.61)

From the definition of the total scattering wave function, Eq. (4.36), and the Faddeev equations for the full Green’s function, Eq. (4.60), the following equations can be obtained:

\[ \psi_{\alpha_2} = \delta_{\alpha_2, c_2} \Phi_{c_2, n} + G_0 T_{a_2} \sum_{b_2 \neq a_2} \psi_{b_2} , \]  

(4.62)

(when \( \Phi_{c_2, n} \) is used as the initial state) and

\[ \psi_{\alpha_2} = \Phi_{\alpha_2}^{a_2} - \Phi_0 + G_0 T_{a_2} \sum_{b_2 \neq a_2} \psi_{b_2} , \]  

(4.63)

(when \( \Phi_0 \) is used as the initial state), where

\[ \Phi_{\alpha_2}^{a_2} = \lim_{\varepsilon \downarrow 0} i \varepsilon G_{a_2} (E_0 + i \varepsilon) \Phi_0 . \]  

(4.64)

Note that the following identification was made:

\[ \psi_{\alpha_2} = \lim_{\varepsilon \downarrow 0} i \varepsilon G_{a_2} (E + i \varepsilon) \Phi . \]  

(4.65)

The functions \( \psi_{\alpha_2} \) are known as Faddeev amplitudes.

For three particles, the above method of rewriting the series gives rise to a connected kernel (after one iteration; cf. Eq. (4.55)), and we can expect to have gotten rid of the difficulties described earlier. As can be seen by inspection, Eqs. (4.62) and (4.63) can be satisfied only in the inhomogeneous form by wave functions having the appropriate asymptotic behavior. A more profound mathematical description of the properties of the Faddeev equations is given in Faddeev [1965]. The generalization of the Faddeev equations to more than three particles will be given in the following subsection.

4.2.2 The Yakubovsky equations

In this section I describe the findings of Yakubovsky, using a slightly different notation. Let me begin by stating two simple properties of partitions.

**Property 4.1** For any two different partitions \( a_k \) and \( b_k \), there is at most one partition \( a_{k+1} \) which follows both.

**Property 4.2** If two different partitions \( a_k \) and \( b_k \) precede the same partition \( a_{k+1} \) then they must follow one definite \( a_{k-1} \).
These two properties are essential in the proofs that will follow. Yakubovsky defines objects which I call chain matrices. These are objects whose components are operators of an \( N \)-body system which has been partitioned in a certain manner \( (i.e., \) some of the interactions have been switched off):

**Definition 4.3** Chain matrices \( A^k_{a_i(b_j)} \) are matrices whose rows and columns are indexed by partition chains \( a_k \) and \( b_k \), with \( k > i \) and \( a_k \subseteq a_i \) and \( b_k \subseteq a_i \), for given definite \( k \) and \( a_i \). The elements of such a matrix will be denoted by \( A^k_{a_i(b_j)} \).

The chain matrix \( A^N_{a_i} \) has only one element, and is identical to the corresponding operator of the partitioned \( N \)-body system, \( e.g., T^N_{a_i} = T_{a_i} \). It is possible to construct chain matrices for a partition \( a_i \) from chain matrices for the partitions \( a_k \) using the following prescription:

\[
A^a_k(b_k)_{a_i(n)} \equiv \delta(a_k \subset a_i)A^{a_{k+n},b_{k+n}}_{a_i} \prod_{l=0}^{n-1} \delta(a_{k+l},b_{k+l}), \tag{4.66}
\]

with

\[
\delta(a_k \subset a_i) \equiv \begin{cases} 
1, & a_k \subset a_i \\
0, & a_k \nsubseteq a_i 
\end{cases}, \tag{4.67}
\]

\[
\delta(a_k, b_k) \equiv \begin{cases} 
1, & a_k = b_k \\
0, & a_k \neq b_k 
\end{cases}, \tag{4.68}
\]

and under the restrictions \( k + n > j > i \) and \( a_j \subset a_i \). If \( j \) equals \( k \), the subscript \( j \) will be dropped. Another key concept is that of chasing chains; I will write \( b_k \preceq a_j (k > j) \), to denote that \( b_k \) chases \( a_j \), which can be expressed by the following set of conditions:

\[
b_k \subset a_{k+1} \wedge b_{k+1} \subset a_k \wedge \cdots \wedge b_{N-1} \subset a_{N-2} \cdot \tag{4.69}
\]

This can be generalized to chains of equal length:

\[
b_k \preceq a_k = b_{k+1} \preceq a_k \cdot \tag{4.70}
\]

Finally, I will call two chasing chains \( a_j \) and \( b_k \) \((k \geq j)\) disjunct if \( b_k \preceq a_j \) and none of the partitions in \( b_k \) coincides with any of the partitions in \( a_j \):

\[
b_k \prec a_j = b_k \preceq a_j \wedge b_k \neq a_k \wedge b_{k+1} \neq a_{k+1} \wedge \cdots \wedge b_{N-1} \neq a_{N-1} \cdot \tag{4.71}
\]

The disjunct chasing condition can be cast in matrix form as well; an off-diagonal matrix \( X^a_k(b_k)_{a_i} \) will be defined by:

\[
X^a_k(b_k)_{a_i} \equiv \delta(b_k \prec a_k), \tag{4.72}
\]
where
\[ \delta(b_k \preceq a_k) \equiv \begin{cases} 1, & b_k \preceq a_k \\ 0, & b_k \not\preceq a_k \end{cases}. \] (4.73)

I will now proceed by defining a set of chain matrices, which will take the role of the \( T \) operator in the Yakubovsky formalism.

**Definition 4.4** The chain matrices \( T_{a_i}^k \) contain the Yakubovsky components of the operator \( T_{a_i} \), defined as follows:

\[
T_{a_i}^{a_N, b_N} \equiv T_{a_i}, \tag{4.74a}
\]

\[
T_{a_i}^{a_{N-1}, b_{N-1}} \equiv V_{a_{N-1}} \delta(a_{N-1}, b_{N-1}) + V_{a_{N-1}} G_{a_i}(z) V_{b_{N-1}}, \tag{4.74b}
\]

\[
T_{a_i}^{k-1} \equiv T_{a_i(2)}^{k-1} \left[ I + G_0 X_{a_i(1)}^{k-1} T_{a_i}^k \right], \tag{4.74c}
\]

where in Eq. (4.74c) the matrix \( T_{a_i}^k \) is embedded in the space of matrices \( A_{a_i}^{k-1} \) as follows:

\[
[T_{a_i}^{k}]^{{a_{k-1}}^{a_k}, {b_{k-1}}^{b_k}} = T_{a_i}^{a_k, b_k}. \tag{4.75}
\]

With these definitions in mind, I will now describe and derive the Yakubovsky equations, starting with

**Lemma 4.1** The following algebraic equation is satisfied by \( T_{a_i}^k \) for any \( a_i \), and any \( k \leq N \):

\[
T_{a_i}^k = T_{a_i(1)}^k \left[ I + G_0 X_{a_i(1)}^k T_{a_i}^k \right]. \tag{4.76}
\]

**Proof:** For \( k = N - 1 \) this equation follows from Eq. (4.74b) and the LS equations for \( G_{a_i} \) and \( T_{a_{N-1}} \). We will show that if Eq. (4.76) holds for \( k \), it also holds for \( k - 1 \). In order to do so, another lemma is needed:

**Lemma 4.2** For all \( k \leq N \) the following sum rule holds:

\[
T_{a_i}^{a_k, b_k} = \sum_{a_{k-1}} T_{a_i}^{a_{k-1}, a_k, b_{k-1}, b_k}. \tag{4.77}
\]

**Proof:** This follows immediately from Eq. (4.74c) and the induction assumption Eq. (4.76):

\[
\sum_{a_{k-1}} T_{a_i}^{a_{k-1}, a_k, b_{k-1}, b_k} = [T_{a_i(1)}^k \left[ I + G_0 X_{a_i(1)}^k T_{a_i}^k \right]]^{a_k, b_k} = T_{a_i}^{a_k, b_k}. \tag{4.78}
\]

Note that the following property (which follows from Property 4.2 of partitions) was used to write a sum over matrix elements of \( X_{a_i(1)}^{k-1} \) as matrix elements of \( X_{a_i}^{k} \):

\[
\sum_{a_{k-1}} \sum_{d_k \not\subset a_{k-1}} = \sum_{d_k \not\subset a_k}, \quad \text{provided } a_k \supset d_{k+1}. \tag{4.79}
\]

This proves Lemma 4.2. □
Using Eq. (4.77) we can derive

\[ X_{a_i(1)}^{k-1} T_{a_i}^k = X_{a_i}^k T_{a_i}^{k-1}, \tag{4.80} \]

where embedding of \( X_{a_i}^k \) in the larger space is implied. Substituting this identity into Eq. (4.74c) leads to

\[ T_{a_i}^{k-1} = T_{a_i(2)}^{k-1} [\mathbb{1} + G_0 X_{a_i}^k T_{a_i}^{k-1}] = T_{a_i(2)}^{k-1} [\mathbb{1} + G_0 (X_{a_i}^{k-1} + X_{a_i(1)}^{k-1}) T_{a_i}^{k-1}], \tag{4.81} \]

Transferring the last term on the right-hand side to the left-hand side yields:

\[ [\mathbb{1} - T_{a_i(2)}^{k-1} G_0 X_{a_i(1)}^{k-1}] T_{a_i}^{k-1} = T_{a_i(2)}^{k-1} [\mathbb{1} + G_0 X_{a_i}^{k-1} T_{a_i}^{k-1}]. \tag{4.82} \]

Furthermore, it is easy to show that

\[ \mathbb{1} + T_{a_i(1)}^{k-1} G_0 X_{a_i(1)}^{k-1} = [\mathbb{1} - T_{a_i(2)}^{k-1} G_0 X_{a_i(1)}^{k-1}]^{-1}, \tag{4.83} \]

so that Eq. (4.82) becomes

\[ T_{a_i}^{k-1} = [\mathbb{1} + T_{a_i(1)}^{k-1} G_0 X_{a_i(1)}^{k-1}] T_{a_i(2)}^{k-1} [\mathbb{1} + G_0 X_{a_i}^{k-1} T_{a_i}^{k-1}] = T_{a_i(1)}^{k-1} [\mathbb{1} + G_0 X_{a_i}^{k-1} T_{a_i}^{k-1}], \tag{4.84} \]

where the final step follows from Eq. (4.83). The final equation is just Eq. (4.76) with \( k \) replaced by \( k - 1 \). This completes the proof of Eq. (4.76) for all \( k \). □

Now that Eq. (4.76) has been proven, we take the special case \( i = 1 \) and \( k = 2 \) to arrive at the Yakubovsky equations for the \( T \) matrices:

\[ T^{a_s, b_2} = T^{a_s, b_2}_{a_2} \delta(a_2, b_2) + \sum_{c_3} \sum_{d_2 < (a_2c_3)} T^{a_s, c_3}_{a_2} G_0 T^{d_2, b_2}. \tag{4.85} \]

This is an equation expressing \( T^{a_s, b_2} \) in terms of \( T \) matrices of two-fragment partitions, which are defined by Eq. (4.74), and can be determined by Eq. (4.76), which can be solved using \( T \) matrices of three-fragment partitions, and so on.

The physical properties of the system are contained in the \( T \) matrices, which may be constructed from the \( T \) matrices as follows: from Eq. (4.74b) we find

\[ T_{a_i} = \sum_{a_{N-1} b_{N-1}} T^{a_{N-1}, b_{N-1}}_{a_i}, \tag{4.86} \]

which can be rewritten into

\[ T_{a_i} = \sum_{a_k \ldots a_{N-1} b_{N-1}} T^{a_k \ldots a_{N-1}, b_k \ldots b_{N-1}}_{a_i}. \tag{4.87} \]
by repeated application of Eq. (4.77).

The bound-state Yakubovsky equations for wave functions can be constructed by simply replacing $T_{a_i}^{k-1}$ in the homogeneous version of Eq. (4.84) by $\psi_{a_i}^{k-1}$:

$$\psi_{a_i}^{k-1} = G_0 T_{a_i}^{k-1} X_{a_i}^{k-1} \psi_{a_i}^{k-1}. \quad (4.88)$$

By defining

$$\sum_{a_{k-1}} \psi_{a_{k-1}}^{a_k} a_k \equiv \psi_{a_i}^{a_k}, \quad (4.89)$$

it becomes clear that the induction step from $k - 1$ to $k$ can be made, so that Eq. (4.88) for any value of $k$ is equivalent to the same equations for any other value of $k$, specifically so for $k = N$ and $i = 1$, which is just the equation

$$\psi_{a_{N-1}}^{a_N-1} = G_0 T_{a_{N-1}} X_{a_{N-1}} \psi_{a_{N-1}}^{a_{N-1}}$$

$$= G_{a_{N-1}} V_{a_{N-1}} \sum_{b_{N-1} \neq a_{N-1}} \psi_{a_{N-1}}^{b_{N-1}} \quad (4.90)$$

which is the $N$-body analogue of the Faddeev equations, and which can easily be seen to be equivalent to the Schrödinger equation for $\Psi = \sum_{a_{N-1}} \psi_{a_{N-1}}^{a_{N-1}}$.

Note that the kernel of Eq. (4.85) itself is not compact, but all singularities disappear after $N - 2$ iterations, proving that these equations do indeed have the sound mathematical properties we were looking for.

4.3 Configuration-space Faddeev–Yakubovsky equations

The Faddeev equations in configuration space were first formulated by Noyes [1969] to study the three-body scattering problem. (The differential equations were formulated in [Noyes and Fiedeldey, 1968], but not studied in detail.) The configuration-space Faddeev equations have a strikingly simple form. However, careful examination shows that there are mathematical difficulties, which show up when trying to formulate the boundary conditions. In this section, I will formulate the configuration-space Faddeev equations and the boundary conditions for three spinless particles.

4.3.1 Configuration-space Faddeev equations

In this subsection, I will use a somewhat simplified notation for two-fragment partitions appearing as subscripts. For example, I will write $\psi_k$ instead of $\psi_{(ij)k}$, since the spectator particle $k$ uniquely defines the partition $(ij)k$. In configuration-space, the three-body Schrödinger equation can be written as

$$(E - H)\Psi = (E - \nabla^2_{r_{ij}} - \nabla^2_{r_{k,ij}} - V_{12} - V_{23} - V_{31})\Psi = 0. \quad (4.91)$$
4.3 Configuration-space Faddeev–Yakubovsky equations

The Faddeev decomposition, which can be written as \( \Psi = \Phi_0 + \psi_1 + \psi_2 + \psi_3 \) where \( \Phi_0 \) is a wave function describing three free particles and the Faddeev amplitudes, \( \psi_i \), are defined by \( \psi_i = G_i \Psi \), leads to the following set of coupled partial-differential equations:

\[
\begin{align}
(E - \nabla^2_{r_{11}^i} - \nabla^2_{r_{23}^i} - V_1)\psi_1 &= V_1(\psi_2 + \psi_3), \\
(E - \nabla^2_{r_{11}^i} - \nabla^2_{r_{31}^i} - V_2)\psi_2 &= V_2(\psi_3 + \psi_1), \\
(E - \nabla^2_{r_{11}^i} - \nabla^2_{r_{12}^i} - V_3)\psi_3 &= V_3(\psi_1 + \psi_2).
\end{align}
\]  

(4.92a, 4.92b, 4.92c)

(The term \( \Phi_0 \) can be dropped since only scattering of a particle off a bound state of two other particles will be considered here.) These equations are clearly the configuration-space versions of the Faddeev equations Eq. (4.62) (note that \( G_0 T_{a_2} = G_{a_2} V_{a_2} \)). The Schrödinger equation is retrieved by summing the three equations. The equations thus formulated have a very elegant and simple form. Outside the range of the interaction the equations decouple, and we are left with three “two-body plus spectator” (i.e., a third particle which does not interact with the other two) problems.

The partial-differential equations (4.92) are elliptic and must be augmented with boundary conditions on the entire contour bounding the domain to uniquely specify the solution. For \( r_{kj}^i = 0 \) or \( r_{jk}^i = 0 \) it is sufficient to demand that

\[ r_{kj}^i r_{jk}^i \psi_i(r_{k}^j, r_{jk}) \]

goes to zero when either \( r_{kj}^i \) or \( r_{jk}^i \) goes to zero. The boundary conditions at infinity are more involved. Before discussing this subject in Sec. 4.3.2 it is useful to look at how the Faddeev equations can be simplified through angular-momentum analysis.

Angular-momentum analysis

Three particles can be thought of as forming a triangle in three-dimensional space. The position and orientation of the triangle can be separated out by exploiting conservation of total momentum and total angular momentum, respectively. The size and shape of the triangle is described by three independent variables. Several representations are used in the literature. A very symmetric representation is obtained by using the set of coordinates \( \{r_{23}^i, r_{31}^i, r_{12}^i\} \). In the Faddeev framework, it is more convenient to work with the two Jacobi coordinates \( \{r_{kj}^i, r_{jk}^i\} \) augmented with a third, such as \( \nu_{jk}^i \), a quantity which will be defined as

\[ \nu_{jk}^i = \cos \Phi_{jk}^i = \hat{r}_{k}^j \cdot \hat{r}_{jk}^i. \]  

(4.94)

As in the two-body case, the fact that the Faddeev equations are rotationally invariant (as is the Hamiltonian) can be exploited by writing the wave function as a sum of terms, each having a definite total angular momentum and each
being described by an uncoupled “reduced” partial-differential equation. Total-angular-momentum eigenstates can be constructed in various ways, two of which I will describe: the “total-angular-momentum expansion” and the bipolar-harmonic expansion.

**Total-angular-momentum representation**

The notion of three particles forming a triangle in three-dimensional space suggests the following approach: Choose the vector \( r^j_k \) to be along the \( \hat{e}_z \) axis and \( r^i_{jk} \) to be in the \( \hat{e}_x - \hat{e}_z \) plane at an angle of \( \vartheta_k \) to \( r^j_k \). Then apply a general rotation, described by three Euler angles \( \alpha_k, \beta_k, \) and \( \gamma_k \), to rotate the entire triangle to the desired orientation, i.e., take

\[
\begin{align*}
\mathbf{r}^j_k &= x_i D(\alpha_i, \beta_i, \gamma_i) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \\
\mathbf{r}^i_{jk} &= y_i D(\alpha_i, \beta_i, \gamma_i) \begin{pmatrix} \sin \vartheta_i \\ 0 \\ \cos \vartheta_i \end{pmatrix},
\end{align*}
\]

where \( x_i, y_i, \) and \( \vartheta_i \) are introduced as a short-hand notation for \( r^j_k, r^i_{jk}, \) and \( \vartheta_k \), respectively, and \( D(\alpha, \beta, \gamma) \) is as defined in [Edmonds, 1960] (cf. Appendix C). This formulation was proposed by Kostrykin et al. [1989], who refer to it as the total-angular-momentum representation. (This name is in my opinion somewhat awkward, since it does not describe the essential point, which is that this representation treats only the overall orientation of the three-body system using angular-momentum techniques, instead of most others, which also treat some of the variables describing the size of the triangle.)

In this representation, the Laplacian takes the following form:

\[
\Delta_i = \frac{\partial^2}{\partial x_i^2} + \frac{2}{x_i} \frac{\partial}{\partial x_i} + \frac{\partial^2}{\partial y_i^2} + \frac{2}{y_i} \frac{\partial}{\partial y_i} + \left( \frac{1}{x_i^2} + \frac{1}{y_i^2} \right) \left( \frac{\partial^2}{\partial \vartheta_i^2} + \cot \vartheta_i \frac{\partial}{\partial \vartheta_i} + \csc^2 \vartheta_i \frac{\partial^2}{\partial \alpha_i^2} \right) - \frac{L^2 + K}{x_i^2},
\]

where \( L^2 \) is given by Eq. (C.21),

\[
K = e^{i\alpha_i} \left( \frac{\partial}{\partial \vartheta_i} + i \cot \vartheta_i \frac{\partial}{\partial \alpha_i} \right) L(-)
\]

\[
+ e^{-i\alpha_i} \left( - \frac{\partial}{\partial \vartheta_i} + i \cot \vartheta_i \frac{\partial}{\partial \alpha_i} \right) L(+) + 2 \frac{\partial^2}{\partial \alpha_i^2},
\]

and

\[
L^{(\pm)} = \pm \frac{\partial}{\partial \beta_i} + i \csc \beta_i \frac{\partial}{\partial \gamma_i} - i \cot \beta_i \frac{\partial}{\partial \alpha_i} = -e^{\pm i\alpha_i} L_{\mp},
\]
4.3 Configuration-space Faddeev–Yukubovskiy equations

and where $L_{\pi}$ are defined by Eq. (C.20). The operator $K$ describes the Coriolis interaction caused by rotation of the entire system. Now the Wigner $D$ functions, which are just the matrix elements

$$D_{m'm}^{(\ell)}(\alpha, \beta, \gamma) = \langle \ell m' | D(\alpha, \beta, \gamma) | \ell m \rangle,$$

(4.100)

(cf. Appendix C), can be used to expand the Faddeev components

$$\psi_i(\mathbf{r}_k^j, \mathbf{r}_j^{ik}) = \sum_{L=0}^{\infty} \sum_{M_k}^{L} \sum_{M}^{L} \psi_{iLM}^{LM}(x_i, y_i, \vartheta_i) D_{LM}^{(L)}(\alpha_i, \beta_i, \gamma_i).$$

(4.101)

The three different Jacobi coordinate systems are related by rotations of the form Eq. (4.15). The Euler rotations can be written as follows:

$$D(\alpha_i, \beta_i, \gamma_i) = D(\alpha_j, \beta_j, \gamma_j)D(0, \omega_{ij}, 0),$$

(4.102)

where

$$\cot \omega_{ij} = \cot \vartheta_j + \frac{\cos \mu_k^{ij} x_j}{\sin \mu_k^{ij} y_j \sin \vartheta_j}. $$

(4.103)

By substituting Eq. (4.101) into the Faddeev equations, we obtain

$$\sum_{M'=-M}^{M+1} \{ \delta_{MM'} E - (H_{0i})_{MM'}^{L} - \delta_{MM'} V_i(x_i) \} \psi_{iLM}^{LM}(x_i, y_i, \vartheta_i)$$

$$= V_i(x_i) \sum_{j \neq k}^{L} \sum_{M'=-L}^{L} D_{LM}^{L}(0, \omega_{jk}, 0) \psi_{jLM'}^{LM}(x_j, y_j, \vartheta_j),$$

(4.104)

where

$$(H_{0i})_{MM'}^{L} = - \langle L M | \Delta_i D(\alpha_i, \beta_i, \gamma_i) | L M' \rangle,$$

(4.105)

and where $\Delta_i$ is given by Eq. (4.97). These matrix elements can be easily calculated using Eqs. (C.23) through (C.26). Note that in the most general case there are $3 \times (2L + 1)$ coupled equations, for a given $\{L, M_L\}$. This number can be somewhat reduced by exploiting the conservation of parity of the system [Kostrykin et al., 1989].

**Bipolar harmonics**

Another useful choice of coordinates is a double set of spherical coordinates, as follows:

$$\mathbf{r}_{k}^{j} = x_i \begin{pmatrix} \sin \vartheta_i \cos \varphi_i \\ \sin \vartheta_i \sin \varphi_i \\ \cos \vartheta_i \end{pmatrix},$$

(4.106)
\[ \mathbf{r}_{jk}^i = y_i \begin{pmatrix} \sin \vartheta_i' \cos \varphi_i' \\ \sin \vartheta_i' \sin \varphi_i' \\ \cos \vartheta_i' \end{pmatrix} \]  \hspace{1cm} (4.107)

The Faddeev components may be expanded on a basis of products of spherical harmonics:

\[
\psi_i(\mathbf{r}_k^j, \mathbf{r}_j^i) = \sum_{\ell m} \sum_{\ell' m'} \frac{\psi_{ij}^{\ell m, \ell' m'}(x_i, y_i) \cdot x_i y_i}{x_i y_i} Y_{\ell m}(\vartheta_i, \varphi_i) Y_{\ell' m'}(\vartheta_i', \varphi_i'). \]  \hspace{1cm} (4.108)

This expression involves a double infinite sum, which may be rewritten by taking specific linear combinations of products of spherical harmonics, which are total-angular-momentum eigenstates:

\[
[Y_{\ell}(\hat{x}) \otimes Y_{\ell'}(\hat{y})]_{LM} = \sum_{m m'} \langle \ell m \ell' m' | L M \rangle Y_{\ell m}(\hat{x}) Y_{\ell' m'}(\hat{y}),
\]  \hspace{1cm} (4.109)

where \( \langle \ell m \ell' m' | L M \rangle \) is a Clebsch–Gordan coefficient (cf. Appendix C). These functions are known as bispherical or bipolar harmonics. The expansion then becomes

\[
\psi_i(\mathbf{r}_k^j, \mathbf{r}_j^i) = \sum_{LM} \sum_{\ell \ell'} \frac{\psi_{ij}^{\ell \ell' LM}(x_i, y_i)}{x_i y_i} Y_{\ell \ell'}(\vartheta_i, \varphi_i) Y_{\ell' m'}(\vartheta_i', \varphi_i'). \]  \hspace{1cm} (4.110)

which involves just a single infinite sum if the system is in a definite total-angular-momentum state. (Note that \( \ell, \ell', \) and \( L \) must satisfy the triangular condition.)

Using the expansion Eq. (4.110), the following form for the Faddeev equations can be obtained:

\[
[E - H_{0i} - V_i(x_i)] \psi_{i}^{\alpha}(x_i, y_i) = V_i(x_i) \sum_{j \neq i} \sum_{\beta} \int_{-1}^{1} \text{d} \nu_i K_{ij}^{\alpha \beta}(x_i, y_i, \nu_i) \psi_{j}^{\beta}(x_j, y_j),
\]  \hspace{1cm} (4.111)

where \( \nu_i = \hat{x}_i \cdot \hat{y}_i, \)

\[
K_{ij}^{\alpha \beta}(x_i, y_i, \nu_i) = 8\pi^2 \frac{x_i y_i}{x_j y_j} [Y_{\ell_n}(\vartheta_i, \varphi_i) \otimes Y_{\ell_n'}(\vartheta_i', \varphi_i')]_{L_n M_n}^* \times [Y_{\ell_{1\beta}}(\vartheta_j, \varphi_j) \otimes Y_{\ell_{1\beta}'}(\vartheta_j', \varphi_j')]_{L_{1\beta} M_{1\beta}},
\]  \hspace{1cm} (4.112)

and

\[
H_{0i}^{\alpha} = - \frac{\partial^2}{\partial x_i^2} + \frac{\ell_{\alpha}(\ell_{\alpha} + 1)}{x_i^2} - \frac{\partial^2}{\partial y_i^2} + \frac{\ell_{\alpha}'(\ell_{\alpha}' + 1)}{y_i^2}.
\]  \hspace{1cm} (4.113)

Note that \( \alpha \) and \( \beta \) have been used as a short-hand notation for a complete set of quantum numbers \( \{\ell, \ell', L, M\}. \) Note furthermore that \( x_j \) and \( y_j \) are functions
4.3 Configuration-space Faddeev–Yakubovsky equations

of \( x_i, y_i, \) and \( \nu_i \), and finally that Eq. (4.111) is an infinite set of coupled two-dimensional integro-differential equations, due to the fact that the two lengths \( x_i \) and \( y_i \) do not fully describe the three-body triangle.

By comparing the two representations introduced in this subsection, we find the following correspondence:

\[
\psi^{L_{\ell \ell'}}_{i_{\mathcal{M} \mathcal{L}}}(x_i, y_i, \vartheta_i) = \sum_{\ell, \ell'} \psi^{L_{\ell \ell'}}_{i_{\mathcal{M} \mathcal{L}}}(x_i, y_i) \frac{1}{x_i y_i} [Y_{\ell_0}(0, 0) \otimes \tilde{Y}_{\ell'_0}(\vartheta_i, 0)]_{\mathcal{L} \mathcal{M}}. \tag{4.114}
\]

This identity can be obtained by recognizing that a bipolar harmonic is an eigenstate of the total-angular-momentum operator, and can therefore be rotated using a Wigner \( \mathcal{D} \) rotation matrix.

4.3.2 Boundary conditions

We can now attempt to derive the boundary conditions for the configuration-space Faddeev equations in a manner similar to the one used in the two-body problem. The three-body problem is, however, much more complicated, and hence also the corresponding analysis of the boundary conditions. Noyes [1969] was the first to analyze this problem in detail. I will start with this analysis in a somewhat altered form, and then continue with two methods which have been used to approximate the exact boundary conditions.

Regions in configuration space

As was mentioned before, the Faddeev equations decouple outside the range of the potential \( (R) \). Outside this range the Faddeev amplitudes satisfy the free Schrödinger equation. To be able to calculate the Faddeev amplitudes inside the range of the potential, the permuted Faddeev amplitudes are needed in the region \( [x_i, y_i] \in [0, R] \times [0, \infty) \). This region corresponds to a region in the permuted coordinate system bounded by

\[
y_j \leq |\csc \mu^i_{k} x_j| + \cos \mu^i_{k} y_j .
\]

For the following, it is useful to distinguish between three areas in configuration space. I will distinguish between the interaction region and the free region, \( \nu \), the region described by \( x_i < R \) and the region described by \( x_i \geq R \). It is useful to split the interaction region into an interior region and a final-state region, \( \nu \), a region where all three interactions act and a region where only one of the interactions is nonzero. The interior, final-state, and free regions will be denoted by \( \Omega^i \), \( \Omega^i \nu \), and \( \Omega^i \nu \), respectively:

\[
\Omega^i = \{(x_i, y_i) \mid x_i \leq R \wedge y_i \leq |\csc \mu^i_{k} x_i| + \cos \mu^i_{k} y_i \}, \tag{4.116}
\]

\[
\Omega^i \nu = \{(x_i, y_i) \mid x_i \leq R \wedge y_i > |\csc \mu^i_{k} x_i| + \cos \mu^i_{k} y_i \}, \tag{4.117}
\]

\[
\Omega^i \nu = \{(x_i, y_i) \mid x_i > R \}. \tag{4.118}
\]
These regions are depicted in Fig. 4.2. Note that $\Omega'_I$ contains the region where all three interactions are active:

$$
\Omega'_I \supset \{ (x_i, y_i) \mid x_i \leq R \land x_j(x_i, y_i) \leq R \},
$$

and that $\Omega'_I \cup \Omega''_I$ contains the set $\{ (x_i, y_i) \mid x_j(x_i, y_i) \leq R \}$ (i.e., $\Omega'_I \cup \Omega''_I$).

These results imply that the Faddeev equations, using the notation $\psi_i^\alpha = \psi_i^{\alpha I} + \psi_i^{\alpha II} + \psi_i^{\alpha III}$ where the roman numerals denote the support ($\Omega'_I$, $\Omega''_I$, or $\Omega''_I$) of the different parts of the Faddeev components, can be written as

$$
\psi_i^\alpha(x_i, y_i) = \phi_i^\alpha(x_i, y_i) + \int_0^R \int_0^\infty dx'_i dy'_i G_i(E; x_i, y_i, x'_i, y'_i) V_i(x'_i)

\times \sum_{j \neq i} \sum_{\beta} \int_0^1 d\nu_i K_{ij}^{\alpha \beta}(x'_i, y'_i, \nu_i)(\psi_j^{\beta I} + \psi_j^{\beta III}),
$$

(4.120)

where the bipolar expansion Eq. (4.110) has been used. (The analysis for the total-angular-momentum representation can be performed along the same line.) Note that $\nu_i = \dot{x}_i \cdot \dot{y}_i$, that the integration kernel $K_{ij}^{\alpha \beta}$ is defined by Eq. (4.112), and that $x_j$ and $y_j$ are functions of $x_i$, $y_i$, and $\nu_i$. In contrast to the two-body problem, where the finite range of the interaction guarantees that the integration is limited to a finite interval, allowing the scattering problem to be solved by matching at a finite range, the potential does not cut off the integration in the $y_i$ direction. The convergence of the integrals must therefore be questioned. The kernel $K_{ij}^{\alpha \beta}$ is $O(y_i^{-1})$ when $y_i \to \infty$ and $x_i$ is fixed, so that the convergence must come from the asymptotic behavior of the Faddeev amplitude. Actually, it is quite simple to show that on $\Omega''_I$ the Faddeev amplitude has the following asymptotic behavior

$$
\psi_i^\alpha(\rho, \theta_i) \sim \frac{A_i^\alpha(\theta_i)}{\rho^\frac{1}{2}} e^{\frac{iK\rho}{\rho^\frac{1}{2}}},
$$

(4.121)
where $K = \sqrt{E}$, where polar coordinates,

\begin{align}
  x_i &= \rho \cos \theta_i, \\
  y_i &= \rho \sin \theta_i,
\end{align}

(4.122a, 4.122b)

were introduced, and where $A_i^\alpha$ is some smooth bounded function. When $y_i \to \infty$ and $x_i$ is fixed, the permuted coordinates $x_j$ and $y_j$ both go to infinity, so that the asymptotic behavior (4.121) does indeed hold. Consequently, the source term is $O(y_i^{-3/2})$ for $y_i \to \infty$ and the integral is convergent. Note that it was necessary to use the asymptotic behavior of the permuted Faddeev amplitudes to prove the convergence. This is in accordance with the observation that the Faddeev kernel itself is not compact, but that its once iterated form is. Noyes [1969, 1970] explains this phenomenon in terms of something he refers to as the *eternal triangle effect*: scattering in channel $j$ ($j = 1, 2, 3$) leads to outgoing waves which interfere with the interaction in the $i$ ($i \neq j$) channel.

Equation (4.120) shows that the Faddeev components in the region $\Omega_i^{II}$ are not coupled to the Faddeev components on the rest of the domain, except of course, for matching conditions. Something similar can be said for the behavior on $\Omega_i^{III}$: it is a solution of the free Hamiltonian which must satisfy matching conditions at $x_i = R$. Summarizing; on $\Omega_i^{I}$ (i.e., the region where the three interactions overlap) the Faddeev amplitude is governed by three-body dynamics, on $\Omega_i^{II}$ (the region where only one of the three interactions is present) it is primarily governed by two-body dynamics, and on $\Omega_i^{III}$ the equation is essentially free. This implies that the three-body problem need only be solved with its full complexity on $\Omega_i^{I}$, provided we are able to formulate the matching conditions on the borders between the different regions.

**The exterior wave function**

The exterior Faddeev amplitude (i.e., $\psi_i^{II} + \psi_i^{III}$) can be determined once the interior Faddeev amplitude $\psi_i^j$ is known, by solving a one-dimensional integral equation. This equation can be derived from Eq. (4.120), by using one of following two forms for the Green’s function:

\begin{align}
  G_i(E; x, y, x', y') &= \frac{2}{\pi} \int_0^\infty dq \, \hat{J}^e(qy) \left( -\frac{1}{p_q} \psi_{\epsilon, p_q}(x_<) \hat{F}_{\epsilon, p_q}(x_) \right) \hat{J}^e(qy'), \\
  &= \sum_n \psi_{\epsilon, n}^*(x) \left( -\frac{1}{q_n} \hat{J}^e(q_n y_<) \hat{h}_{\epsilon, n}^{(+)}(q_n y_+) \right) \psi_{\epsilon, n}(x') \right)
  + \frac{2}{\pi} \int_0^\infty dp \, \psi_{\epsilon, p}^*(x) \left( -\frac{1}{q_p} \hat{J}^e(q_p y_<) \hat{h}_{\epsilon, p}^{(+)}(q_p y_+) \right) \psi_{\epsilon, p}(x'),
\end{align}

(4.123a, 4.123b)
where \( \psi_{\ell,n} \) is the \( n \)th \((n = 1, 2, \ldots)\) normalized bound state of the two-body subsystem, and \( \psi_{\ell,p} \) is the normalized regular solution:

\[
\psi_{\ell,p}(x) = \frac{\phi_{\ell,p}(x)}{f_\ell(p)}. \tag{4.124}
\]

(Note that this function is identical to the one defined in Sec. 3.1.3.) The momenta \( p_q, q_n, \) and \( q_p \) are defined by

\[
p^2_q + q^2 = q^2_n + E_n = q^2_p + p^2 = E, \tag{4.125}
\]

where \(-E_n\) is the binding energy of \( \psi_{\ell,n}. \) (All momenta lie on the positive real or imaginary axis.) The definitions of the regular solution \( \psi_{\ell,k} \), the Jost function \( f_\ell \), and the Jost solution \( \mathcal{T}_\ell^{+} \) can be found in Sec. 3.1.4.

Using the representation (4.123a) for \( G_i \), we find the following integral equations for the Faddeev amplitudes:

\[
\psi^{\alpha\beta\gamma}_{i}(x_i, y_i) = \phi^\alpha_i(x_i, y_i) - \frac{2}{\pi} \int_0^\infty dq \frac{1}{p_q} \mathcal{T}_{\ell, p_q}^{+}(x_i) j_{i\ell}(q y_i) \times \int_0^\infty dx'_i \int_0^\infty dy'_i j_{i\ell}(q y'_i) \psi_{\ell, p_q}(x'_i) V_i(x'_i) S^\alpha_i(x'_i, y'_i). \tag{4.126}
\]

Here \( S^\alpha_i \) is written to denote the permuted Faddeev components:

\[
S^\alpha_i(x_i, y_i) = \sum_{j \neq i} \sum_{\beta} \int_0^1 d\nu_i K_{i,j}^{\alpha\beta}(x_i, y_i, \nu_i) \psi^\beta_j(x_j, y_j). \tag{4.127}
\]

Since \( \psi^{\alpha\beta\gamma}_{i} \) is governed by the free Hamiltonian, it can equivalently be written in the following form:

\[
\psi^{\alpha\beta\gamma}_{i}(x_i, y_i) = \phi^\alpha_i(x_i, y_i) - \frac{2}{\pi} \int_0^\infty dq \mathcal{T}_{\ell}^{\alpha}(q) \hat{h}_{\ell}^{(+)}(p_q x_i) j_{i\ell}(q y_i). \tag{4.128}
\]

Note that the \( y_i \) dependence may only contain a regular solution at positive energies in order to satisfy the boundary condition at \( y_i = 0 \) and \( y_i = \infty \), and that the \( x_i \) dependence must be a purely outgoing wave, due to the fact that the Green's operator is evaluated at energy \( E + i0 \) (which in turn is just a consequence of the physical requirement that the scattered part contains outgoing waves only).

Comparing Eqs. (4.126) and (4.128) leads to the following form for \( \mathcal{T}_{\ell}^{\alpha} \):

\[
\mathcal{T}_{\ell}^{\alpha}(q) = \frac{1}{p_q} \int_0^\infty dx_i \int_0^\infty dy_i j_{i\ell}(q y_i) \psi_{\ell, p_q}(x_i) V_i(x_i) S^\alpha_i(x_i, y_i), \tag{4.129}
\]

which can be simplified by using the following identity:

\[
\psi_{\ell,p}(x)V(x) = -\frac{2p}{\pi} \int_0^\infty dk k \hat{j}_{\ell}(kx) \hat{t}_{i\ell}^{d}(p^2; k, p), \tag{4.130}
\]
where \( t_i^\ell(p^2; k, p) \) is the half off-shell two-body \( T \) matrix. The result is:

\[
\tilde{T}_i^\alpha(q) = -\int_0^\infty dk k t_i^\ell(p_2^2; k, p_q) \sum_{j \neq i} i \langle k q \alpha | \psi_j \rangle.
\] (4.131)

I will now continue to exploit this relationship between the external wave function and the Faddeev amplitudes to define the boundary conditions for the internal wave function.

**The exact boundary conditions**

An integral equation can be derived for \( \tilde{T}_i^\alpha \) by decomposing the Faddeev component \( \psi_j \) into \( \psi_j^1 + \psi_j^{H} \) (cf. Eq. (4.120)) and using Eq. (4.128):

\[
\tilde{T}_i^\alpha(q) = I_i^\alpha(q) + \sum_{j \neq i} \sum_\beta \int_0^\infty dq' Q_{ij}^{\alpha \beta}(q, q') \tilde{T}_j^\beta(q'),
\] (4.132)

where the kernel \( Q_{ij}^{\alpha \beta} \) is defined by:

\[
Q_{ij}^{\alpha \beta}(q, q') = \int_{-1}^1 d\nu \, k t_i^\ell(p_2^2, k, p_q) f_\ell(p', p_q) K_{ij}^{\alpha \beta}(k, q, \nu),
\] (4.133)

where \( k \) and \( p' \) are functions of \( q, q', \) and \( \nu, \) and where \( f_\ell \) is defined to be the momentum-space representation of \( \mathcal{T}_{\ell, p'} \) on the interval \([R, \infty)\):

\[
f_\ell(p, p') = \frac{2}{\pi} \int_R^\infty dx \, j_\ell(px) \mathcal{T}_{\ell, p'}(x).
\] (4.134)

Note that this function has a singularity at \( p = p' \). Integrals involving \( f_\ell \) must be considered as principal-value integrals. The driving term \( I_i^\alpha \) is defined by

\[
I_i^\alpha(q) = -\int_0^\infty dk k t_i^\ell(p_2^2; k, p_q) \sum_{j \neq i} i \langle k q \alpha | (\psi_j - \phi_j)^1 \rangle
\]

\[-\int_0^\infty dk k t_i^\ell(p_2^2; k, p_q) \sum_{j \neq i} i \langle k q \alpha | \phi_j \rangle,
\] (4.135)

where the superscript \( I \) is used to denote that the specific wave function is zero outside \( \Omega_j^i \).

Equation (4.132) is a one-dimensional integral equation, which has a unique solution for a given driving term (for a proof, see Noyes [1969]). By assuming a finite-dimensional basis set on \( \Omega_j^i \), we find a finite set of solutions of the integral equation, which can be used to calculate the boundary conditions. In this manner, the three-body problem consists of solving the configuration-space Faddeev
equations on a finite domain with boundary conditions which are obtained from solving a one-dimensional integral equation. Note that this integral equation has a complexity which is similar to the Faddeev integral equations for a separable interaction (cf. Chapter 5).

Note that although in principle $\tilde{T}_i^\alpha(q) \equiv \tilde{T}_i^\alpha$ is defined for all $q$, in practice only an interval somewhat larger than $[0, K]$, where $K = \sqrt{\tilde{E}}$, is needed, since the components with $q$ larger than $K$ are suppressed by the exponential decay of $\hat{h}_\ell^{(+)}(p_q)$. Cutting off the integration interval at some finite $q_{\text{max}}$ will therefore lead to an error which can be made arbitrarily small by increasing $q_{\text{max}}$ or by increasing $R$.

The above method leads to an exact implementation of the boundary conditions, but it does have practical difficulties: (i) the integral equation must in principle be solved for all basis states on the region $\Omega_i$, (ii) it may not be easy to implement the resulting boundary conditions in the differential equations in an efficient manner, and (iii) the singularities in the kernel $Q_{\alpha\beta}^{ij}$ and the inhomogeneous term $I_i^\alpha$ require some care.

Asymptotic boundary conditions

An alternative approach to solving the Faddeev equations is matching the Faddeev amplitudes to their asymptotic form at some large but finite hyperradius. This leads to an approximate solution, which will tend to the exact solution when the matching radius goes to infinity. The asymptotic form of the Faddeev amplitudes for $s$ waves only was derived by Glöckle [1974]. I will summarize and generalize his results here.

First, let us consider the simplest case: $x_i \to \infty$. Remember that at first sight the leading behavior of the Faddeev amplitude is given by Eq. (4.121). This result can be improved by giving an accurate error estimate and finding an expression for the unknown function $A_i^\alpha$. Starting from Eq. (4.126), using $\tilde{T}_i^\alpha$ as defined in Eq. (4.129), ignoring the exponentially decreasing parts in the Faddeev component, substituting $K \sin \phi_i$ for $q$, and exploiting the fact that the integrand is an even function of $q$, we obtain

$$\psi_i^\text{III}(x_i, y_i) = \frac{iK}{\pi} \int_{-\pi}^{\pi} d\phi_i F_i^\alpha(\rho, \theta_i, \phi_i) e^{iK \rho \cos(\phi_i - \theta_i)} + O(x_i^{-2}), \quad (4.136)$$

for $x_i \to \infty$, where

$$F_i^\alpha(\rho, \theta_i, \phi_i) = \frac{\hat{e}_\ell^{(+)}(x_i)}{e^{ip_q x_i}} \frac{\hat{h}_\ell^{(+)}(q y_i)}{e^{iq y_i}} \cos \phi_i \tilde{T}_i^\alpha(K \sin \phi_i). \quad (4.137)$$

Note that the $O(x_i^{-2})$ error is entirely due to the neglect of the contribution of the interval $[K, \infty)$ to the integral with respect to $q$. The remaining integral can
be evaluated using the method of steepest descent (see e.g., Erdélyi [1956] or de Bruijn [1958]), using the stationary point $\phi_i = \theta_i$:

\[ \psi^{\alpha \text{III}}_i(x_i, y_i) = \sqrt{\frac{2i}{\pi K}} e^{iK\rho} \left\{ F(\rho, \theta_i, \phi_i) - \frac{i}{2} \frac{\partial^2}{\partial \phi^2} F(\rho, \theta_i, \phi_i) + \cdots \right\} \phi_i = \theta_i + O(x_i^{-2}) \]

\[ = \sqrt{\frac{2i}{\pi K}} e^{iK\rho - i\frac{x}{2}(\ell + c')} \cos \theta_i \frac{\tilde{T}^\alpha}{(K\rho)^{\frac{3}{2}}} (K \sin \theta_i) + O(\rho^{-\frac{3}{2}}). \] (4.138)

The final step involved using the asymptotic forms of the Jost solution and the Riccati–Hankel function. Note that this expression is not valid for $y_i \to \infty$ and $x_i$ fixed, i.e., $\theta_i = 0$.

The analysis for $y_i \to \infty$ is a little bit more involved. Applying the form (4.123b) to the Faddeev differential equations gives

\[ \psi_i(x_i, y_i) - \phi_i^\alpha(x_i, y_i) = -\sum_n \psi_{\ell,n}(x_i) \tilde{h}_{\ell}^{(+)}(q_n y_i) T_{i,n}^\alpha - \frac{2}{\pi} \int_0^K dp \psi_{\ell,p}^*(x_i) \tilde{h}_{\ell}^{(+)}(q_p y_i) T_i^\alpha(p) \]

\[ - A_i^\alpha \frac{e^{iK y_i}}{y_i^{3/2}} \left( \sum_n C_{\ell,n} \psi_{\ell,n}(x_i) + \frac{2}{\pi} \int_0^\infty dp C_{\ell}(p) \psi_{\ell,p}^*(x_i) \right) + O(y_i^{-2}), \] (4.139)

with

\[ T_{i,n}^\alpha = \frac{1}{q_n} \int_0^{\infty} dx_i \int_0^{y_i} dy_i j_\ell(q_n y_i) \psi_{\ell,n}(x_i) V_i(x_i) S_i^\alpha(x_i, y_i), \] (4.140)

\[ T_i^\alpha(p) = \frac{1}{q_p} \int_0^{\infty} dx_i \int_0^{y_i} dy_i j_\ell(q_p y_i) \psi_{\ell,p}(x_i) V_i(x_i) S_i^\alpha(x_i, y_i), \] (4.141)

\[ C_{\ell,n} = \frac{1}{E_n} \int_0^{\infty} dx_i x_i V_i(x_i) \psi_{\ell,n}(x_i), \] (4.142)

\[ C_{\ell}(p) = \frac{1}{p^2} \int_0^{\infty} dx_i x_i V_i(x_i) \psi_{\ell,p}(x_i). \] (4.143)

The values for $C_{\ell,n}$ and $C_{\ell}(p)$ were determined using the asymptotic form

\[ S_i^\alpha(x_i, y_i) \sim A_i^\alpha x_i \frac{e^{iK y_i}}{y_i^{3/2}} (1 + O(y_i^{-1})), \] (4.144)

where the factor $A_i^\alpha$ is a linear combination of the factors $A_j^\beta$ evaluated at $\theta_j = |\frac{x}{2} - |\mu_j^i||$, and hence a well-defined bounded number.

The leading error term in Eq. (4.139) is again due to the neglect of the contribution of the interval $[K, \infty)$ to the integral with respect to $k$. The remaining
integral can again be evaluated using the method of steepest descent. The final result is

\[
\psi_i(x_i, y_i) = \phi_i^\alpha(x_i, y_i) - \sum_n \psi_{\ell,n}(x_i) \hat{h}^{(+)}(q_n y_i) T_{i,n}^\alpha
\]

\[
- \sqrt{\frac{2i}{\pi}} K e^{iK\rho - \frac{i\pi}{2}(\ell + \ell')} \sin\theta_i T_i^\alpha(K \cos\theta_i) + O(\rho^{-\frac{3}{2}}). \quad (4.145)
\]

Comparing Eqs. (4.138) and (4.145), remembering that due to their definitions, \( \tilde{T}_i^\alpha \) and \( T_i^\alpha \) are related by \( q \tilde{T}_i^\alpha(p) = p T_i^\alpha(q) \), provided \( p^2 + q^2 = K^2 \), and noting that the bound-state wave functions vanish exponentially when \( x_i \to \infty \), we see that the two expressions are identical if both \( x_i \) and \( y_i \) go to infinity, and hence Eq. (4.145) is valid provided \( 0 < \theta_i < \frac{\pi}{2} \). Careful analysis of the method of steepest descent for the cases \( \theta_i = 0 \) and \( \theta_i = \frac{\pi}{2} \) gives a different leading behavior than given here. However, Eq. (4.145) is valid for these directions as well. The reason is simply that the leading term is of order \( \rho^{-3/2} \) for both cases, and is therefore not in conflict with Eq. (4.145). (Note that \( \sin\theta T(K \cos\theta) \) behaves as \( \sin 2\theta \) near \( \theta = 0 \) and \( \theta = \frac{\pi}{2} \), so that its leading contribution for either \( x_i \to \infty \) or \( y_i \to \infty \) is actually \( O(\rho^{-3/2}) \).)

A few final remarks are in place here. The method of steepest descent yields a power series in \( \rho^{-1} \) of which I have given only the leading order. Higher orders can be calculated, but involve derivatives of \( T(q) \). For \( \theta = 0 \) and \( \theta = \frac{\pi}{2} \) the leading order cannot be given exactly without calculating derivatives of \( T(q) \). Matching the Faddeev amplitudes to the asymptotic form given by Eq. (4.145) is sufficient for practical calculations, since it leads to an error in the Faddeev amplitude which is \( O(\rho^{-3/2}) \) for all angles. The breakup amplitude which can be extracted from this Faddeev amplitude will have an error of \( O(\rho^{-1}) \) for all angles. Note that including the second-order term of the power series generated by the method of steepest descent is possible in principle, improving the accuracy of the Faddeev amplitude to \( O(\rho^{-2}) \). Going beyond this involves taking into account the contribution of the interval \( [K, \infty) \) in the momentum integrations. In the strip where the bound-state wave function contributes significantly, the breakup component can in principle be ignored since it is of the order \( \rho^{-3/2} \), whereas the elastic contributions are \( O(1) \).

The physical observables in three-body scattering are the numbers \( |T_{i,n}^\alpha|^2 \), which describe the elastic process and the transition to other bound states (excitation, rearrangement), and a function \( T(\alpha, \beta) \) where \( \alpha \) and \( \beta \) are two angles describing the geometry of the breakup process. This breakup amplitude is a combination of the functions \( T_i^\alpha \).

By matching the Faddeev amplitudes to the asymptotic form Eq. (4.145), an approximation of the exact solution is obtained. The boundary conditions can be implemented in a rather straightforward manner in a numerical method. However, one clearly needs to go to a rather large cutoff radius to obtain a reasonably
4.3 Configuration-space Faddeev–Yakubovsky equations

accurate result,\(^3\) which is a serious disadvantage of this method.

I will conclude this section by describing a method which allows the large domain needed to accurately solve the configuration-space Faddeev equations to be reduced to a long but narrow strip.

Exact boundary conditions outside interaction region

In this subsection I describe a different approach to the implementation of the boundary conditions in a practical scheme, based on an idea by Carbonell and Gignoux [1994], who express the wave function outside the range of the interaction using the integral expression

\[
\psi^{\text{III}}_i(x, y) = \int_0^\infty dy' K(R; y, y') \psi_i(R, y'),
\]

(4.146)

where \(K\) is a known analytical function. This function can be defined as follows:

\[
\psi^{\text{III}}_i(x, y) = \int_0^\infty dy' \int_0^\infty dq \hat{j}_c(qy') \frac{\hat{h}^+(p_q y')}{\hat{h}^+(p_q R)} \int_0^\infty dy'' \hat{j}_c(qy'') \psi_i(R, y'').
\]

(4.147)

The boundary condition for \(y_i \to \infty\) is not fixed by this approach. In practice this is resolved by matching the Faddeev amplitudes to the asymptotic form Eq. (4.145). This method has the advantage of being relatively easy to implement in a numerical solution scheme, and requiring a solution on a narrow strip only. Unfortunately, the strip must be rather long, in order to ensure an accurate solution.

The interior wave function

When solving the differential equation on \(\Omega_1^i\), the permuted Faddeev amplitudes are needed (due to the right-hand side of the Faddeev equations) on a region bounded as follows:

\[
\begin{align*}
0 & \leq x_j \leq (1 + 2|\cos \mu_k^{ij}|)R, \\
0 & \leq y_j \leq (|\csc \mu_k^{ij}| + |\cot \mu_k^{ij}|)R.
\end{align*}
\]

(4.148)

(4.149)

Therefore, if the interior region is extended to contain at least this region, it is unnecessary to evaluate the exterior wave function accurately to determine the interior wave function. (Of course, the exterior solution still enters implicitly through the boundary conditions, as described before.)

\(^3\) It has even been argued, based on some trial forms for \(T(q)\), that for physically interesting potentials an impractically large cutoff radius must be used. (Kuruoglu and Levin [1987] found significant deviations at roughly \(10^4\) times the range of the interaction for a scattering at an energy corresponding to a wavelength approximately equal to the range of the potential). However, the trial breakup amplitudes used in [Kuruoglu and Levin, 1987] turned out to contain unphysical singularities, and the claim was refuted [Göckle and Payne, 1992].
Conclusion

The three-body scattering problem in configuration space is well defined if the Faddeev equations are used. The boundary conditions can be obtained asymptotically, but convergence is in general rather slow. Alternatively, the boundary conditions can be calculated exactly in either the $x$ variable only or in both variables, leading to a set of coupled two-dimensional partial-differential equations defined on a long but narrow strip with simple boundary conditions, or a set of two-dimensional partial-differential equations on a small rectangular region with rather involved boundary conditions, respectively. The first two methods have been used in the past, the third has not, as it was deemed to be prohibitively difficult (i.e., numerically expensive) to calculate the boundary conditions.

4.3.3 Long-ranged potentials

All is well for short-ranged potentials, but one of the reasons for using configuration space is the hope to be able to handle long-ranged potentials. The descriptions given in this section until now do not apply to long-ranged potentials. In fact, the Faddeev equations are mathematically not well founded for long-ranged potentials, and need modification to ensure compactness. This is caused by an interplay of the singularities of the Coulomb potential and those of the resolvent $G_i$: below the breakup threshold $G_i$ has a single pole associated with the subsystem bound state, above breakup there is a cut. It is possible to deal with the pole and the cut for short-ranged potentials as shown earlier, but in combination with the Coulomb potentials these singularities are modified and cannot be treated as easily as for the short-ranged problem.

In configuration space, this can be seen as follows. Looking at the (unmodified) Faddeev equations, Eq. (4.92), in which $V_i$ now includes a Coulomb potential, we observe that (i) although in absence of the right-hand side and below the breakup threshold, the Faddeev amplitudes decrease exponentially when $x_i \to \infty$ they do not decrease when $y_i \to \infty$, and as a consequence, that (ii) in a strip centered around $\theta_i = \pi - |\mu^j |$ the term $V_i \psi_j$ does not decrease exponentially when $x_i \to \infty$, so that (iii) the right-hand side may not be ignored at any distance. It is this combination of nonvanishing of the Faddeev amplitudes in all directions and the $\rho^{-1}$ behavior of the Coulomb potentials which causes the problems.

The development of Faddeev equations for long-ranged potentials started with the work of Dollard [1964]. In this paper time-dependent $N$-body scattering with Coulomb interactions was formulated by defining new Möller wave operators, and their existence was proved. A first attempt at incorporating the Coulomb potential in a time-independent framework was by Noble [1967], who suggests the following modified form of the Faddeev equations:

\[
\left( E - \nabla_{x_i}^2 - \nabla_{y_i}^2 + \sum_l \frac{2s_l}{x_i} - V_i \right) \psi_i = V_i (\psi_j + \psi_k),
\]
where $V_i$ are the short-ranged parts of the interactions. In this formulation the free Hamiltonian has been replaced by the free Hamiltonian plus the sum of all Coulomb interactions. However, the the form of the associated Green’s function is not known (Noble needed quite a lot of approximations to be able to use the Faddeev–Noble equations as a practical tool), and as such these equations are not suitable to define the formal scattering theory of three charged particles.

The first correct and practical formulation of the three-body Coulomb problem below the breakup threshold was given by Veselova [1970], who succeeded in a reformulation of the Faddeev equations based on replacing the Coulomb potentials by screened (Yukawa) potentials and letting the screening radius go to infinity. The divergent part of the wave function is separated out and treated on the two-body level. However, this method cannot be applied above the breakup threshold, since there the wave function does not decrease exponentially in any direction, so that it is no longer possible to separate off a single term which will lead to an exponentially decreasing remainder. (Below the breakup threshold $G_i$ has a single pole associated with the subsystem bound state, but above breakup there is a cut. A pole can easily be accounted for, a cut is much more difficult to treat.)

A general solution of the three-body Coulomb problem was developed by Merkuriev [1980]. I will briefly describe his solution here. As a first step, the potentials are written as a sum of a short-ranged and a long-ranged part, as follows:

$$
V_i(x_i, y_i) = V_i^S(x_i, y_i) + V_i^L(x_i, y_i) = V_i(x_i)\chi_i(x_i, y_i) + V_i(x_i)[1 - \chi_i(x_i, y_i)],
$$

(4.151)

where $\chi_i$ is a smooth bounded function satisfying

$$
\chi_i(x_i, y_i) = \begin{cases} 
1, & \text{if } x_i < a_0(1 + y_i)\nu', \\
0, & \text{if } x_i > a'_0(1 + y_i)\nu',
\end{cases}
$$

(4.152)

for given constants $a_0$, $a'_0$, $\nu$, and $\nu'$ satisfying $a_0 < a'_0$ and $\nu < \nu' < \frac{1}{2}$. The modified Faddeev equations in configuration space have the following form:

$$
(E - \nabla^2_{r_i} - \nabla^2_{r_j} - \sum_i V_i^L - V_i^S)\psi_i = V_i^S(\psi_j + \psi_k).
$$

(4.153)

Note the similarity with the Faddeev–Noble equations (4.150). As in the case of the Faddeev–Noble equations, it is impossible to construct the resolvent for the left-hand side analytically. To resolve this problem, Merkuriev expressed these resolvents using a Lippmann–Schwinger type equation. After showing the existence and uniqueness of these resolvents, he was able to formulate the asymptotic behavior of the wave functions and to show that they uniquely specify solutions of the modified Faddeev equations.
The asymptotic form can be given as follows:

\[ \psi_i(x, y_i) = a_0 \psi(x_i) e^{i q_i y_i} - i \gamma_i \log 2 q_i y_i + A_0 \frac{e^{i q_i y_i} - i \gamma_i(\theta_i) \log 2 q_i y_i}{\rho^{1/2}}, \]  

(4.154)

where \( \gamma_i \) and \( \gamma(\theta_i) \) are the generalizations of Sommerfeld's parameter \( \gamma = -s/k \), where now the strength \( s \) depends on the spatial configuration, and hence the angle. More details can be found in [Merkuriev, 1981; Kostrykin et al., 1989]. This is just what one would expect from a quick glance at the left-hand side of the modified equations (4.153): the Coulomb potential can be written as

\[ V_i(x_i, y_i) = -\frac{2s_i(\theta_i)}{\rho}, \]  

(4.155)

where \( s_i \) is a function depending on the three strengths of the Coulomb potentials, as well as the spatial configuration, and hence \( \theta_i \). If \( s_i \) were a constant, then the asymptotic form (4.154) plays a role equivalent to the asymptotic form (4.145) for the short-ranged case. The fact that \( s_i \) varies with \( \theta_i \) complicates matters, but it can be shown that this function does indeed constitute the leading term.

Practical calculations for the three-body Coulomb scattering problem are very scarce. Alt et al. [Alt et al., 1978; Alt and Rauh, 1994] use the screening method above breakup. Merkuriev et al. [Merkuriev, 1981; Kuperin et al., 1983] use the configuration-space formulation with matching to asymptotic boundary conditions. Since in principle the screening method is only valid below breakup, and no independent calculations in configuration space exist, there is strong demand for new accurate, independent, Coulomb scattering calculations.

### 4.3.4 Scattering below breakup

Below the three-body breakup threshold, three-body scattering is considerably simpler than above the threshold. This can be seen by looking at the asymptotic form of the Faddeev amplitudes. When \( E < 0 \), the breakup part of the Faddeev amplitude decreases exponentially. This implies that below the breakup threshold, a solution to the scattering problem can be found in an efficient manner, simply by matching the Faddeev amplitude to its asymptotic form at some, reasonably small, hyperradius. Note that as the energy increases, and the threshold is approached, the exponential decay becomes slower, and a larger matching radius will be needed.

The bound-state problem is even simpler, because for \( E < E_b \), where \( E_b \) is the lowest lying two-body bound state, the elastic and rearrangement terms decrease exponentially as well. Consequently, setting the Faddeev amplitude equal to zero outside some hyperradius will lead to an approximation of the actual bound state. The error decreases exponentially with the cutoff radius and can therefore be made arbitrarily small. If the three-body bound-state energy lies close to the two-body bound-state energy, the exponential decay in the interaction region will be slow.
This is the case for very lightly bound systems, such as Efimov states. This will be discussed in Chapter 5.

The situation is similar for long-ranged potentials: (i) above breakup the problem is extremely difficult, (ii) below breakup the problems can be resolved by treating the two-body bound state pole separately, and (iii) for the three-body bound states, no real problems exist, and the unmodified Faddeev equations can be used (due to the exponential decrease of the Faddeev amplitudes in all directions).

### 4.3.5 $N$-body equations

The configuration-space Yakubovsky equations can be derived from Eq. (4.88), by repeated application of the Yakubovsky equations for the $T$ matrices:

$$
\psi_{a_i}^{k} = G_0 T_{a_i(1)}^{k} X_{a_i}^{k} \psi_{a_i}^{k},
$$

$$
= G_0 T_{a_i(2)}^{k}(1 + X_{a_i(1)}^{k}) T_{a_i(1)}^{k} X_{a_i}^{k} \psi_{a_i}^{k},
$$

$$
= G_0 T_{a_i(2)}^{k}(X_{a_i}^{k} + X_{a_i(1)}^{k}) \psi_{a_i}^{k},
$$

(4.156)

which leads to

$$
\psi_{a_i}^{k} = G_0 T_{a_i(N-k)}^{k} \sum_{j=0}^{N-k} X_{a_i(j)}^{k} \psi_{a_i}^{k},
$$

(4.157)

Writing out the components of the vector $\psi_{a_i}^{k}$ leads to

$$
\psi_{a_i}^{a} = G_{a N-1} V_{a N-1} \sum_{j=0}^{N-k} \sum_{k=0}^{N-k} \psi_{a_{k+1} \cdots a_{k-1} a_{k+j+1}}^{a_{k+1} \cdots a_{k+j+1}} ,
$$

(4.158)

or:

$$
[E - H_{a N-1}] \psi_{a_i}^{a} - V_{a N-1} \sum_{b_{k+1} \cdots a_{k+1}} [1 - \delta(a_{N-1}, b_{N-1})] \psi_{a_i}^{a_{b_{k+1}}},
$$

$$
= V_{a N-1} \sum_{b_{k+1} \cdots a_{k+1}} \psi_{b_{k+1}}^{b_{k+1}} .
$$

(4.159)

Note that one can immediately check the sum rule Eq. (4.89) explicitly.

Equation (4.159) describes a set of coupled partial-differential equations, which must be supplemented with boundary conditions in order to specify a unique solution. In principle, it is possible to derive the boundary conditions using methods similar to the three-body problem. However, the resolvent of the left-hand side is much more complicated in this case (it involves the solutions of the $(N-1)$-body problem), and I will not attempt such an analysis. For the bound-state case the Yakubovsky amplitudes must clearly decrease exponentially in all directions, so that it suffices to require that they are zero at infinity. (For a more general discussion on this subject, see Merkuriev and Yakovlev [1983].)
4.4 Conclusion

In this chapter I have studied the \( N \)-body scattering problem from a theoretical perspective. The aim was to understand the problems with and the consequences for the configuration-space \( N \)-body scattering formalism, and to describe the mathematically correct framework of Faddeev and Yakubovsky.

Using the Faddeev formalism, I have described the boundary conditions for the three-body scattering problem. The boundary conditions can be formulated in at least three different, but equivalent, ways. The first is the one used almost without exception until now: that of fitting to some asymptotic behavior. A second method was introduced quite recently, and treats the boundary condition for large values of \( x \) exactly. Finally, I propose a third method, which was in fact suggested by Noyes, and which deals with the boundary conditions exactly, using a one-dimensional integral equation.

As a spinoff from the investigation of the boundary conditions, two different formulations of the Faddeev equations in configuration space were described, based on the bipolar-harmonic expansion of the angular variable (which is the traditional method), and on a treatment of the angular variable as a third continuous variable. Both representations were used for practical calculations, and I will give some results in the next chapter.