Size effects in cellular solids
Tekoğlu, Cihan

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Document Version
Publisher's PDF, also known as Version of record

Publication date:
2007

Link to publication in University of Groningen/UMCG research database

Citation for published version (APA):

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3 Micropolar Modelling of Size Effects

In this chapter, we fit the elastic micropolar constants of the microstructures that we modelled discretely in chapter 2, by comparing the discrete and micropolar solutions for the simple shear problem in terms of the best agreement in the macroscopic shear stiffness. We develop a strain mapping procedure to be able to estimate the displacement and strain fields for the discrete calculations, and to compare the discrete and the analytical solutions for the local response under simple shear. Finally, we give the analytical solution for the pure bending problem of a plane-strain micropolar beam and critically address the limitations of the micropolar continuum theory.
3.1 Introduction

In the previous chapter, we have shown the size effects in the mechanical behaviour of two-dimensional cellular solids (i.e. honeycombs with square, hexagonal and Voronoi microstructures), for which we took the discreteness of the cellular morphology into account by modelling individual cell walls as beam elements. Although the mechanical properties calculated this way are in good agreement with experiments, this method is computationally expensive for more complex (especially three-dimensional) microstructures. Another approach to account for these size effects is to use a generalised continuum theory. Here, we use the micropolar (Cosserat) theory for this purpose. For an overview of generalized continuum theories, including the micropolar theory, the reader is referred to chapter 1.

The elasticity matrix of a micropolar solid includes new constants relating microrotations to antisymmetric stresses, and couple stresses to curvatures (gradient of microrotations). In the literature, several homogenization/averaging techniques are proposed to obtain these constitutive coefficients for the micropolar theory. In most of these studies, cellular solids are represented by lattice frameworks (e.g. Aşkar and Çakmak [1968], Banks and Sokolowski [1968], Bazant and Christensen [1972], Ostoja-Starzewski et al. [1996], Adachi et al. [1998], Chen et al. [1998]). Micropolar homogenization approaches for irregular microstructures suffer from a dependence of the resulting effective constants on the specific choice of boundary conditions and RVE (representative volume element) size (e.g. Dendievel et al. [1998], Onck [2002]). Therefore, we do not use a homogenization technique, but instead, fit the elastic constants of the microstructures modelled discretely in chapter 2 by comparing the discrete solutions for the simple shear problem with the analytical micropolar solution, in terms of the best agreement in the macroscopic shear stiffness.

In section 3.2, we give the two-dimensional constitutive relations for micropolar solids. In section 3.3, we solve the simple shear problem for a micropolar continuum. After fitting the elastic constants of each microstructure in section 3.4.1, we develop a strain mapping procedure in section 3.4.2. In section 3.4.3, by means of this strain mapping procedure, we analyse the microrotation and strain fields for the discrete structures and compare them to the corresponding continuum fields of the micropolar theory. In section 3.5, we solve the pure bending problem of a plane-strain micropolar beam, and section 3.6 concludes this chapter by discussing the limitations of Cosserat-type continuum theories.
3.2 Constitutive equations

The three-dimensional elastic constitutive relations of a centro-symmetric micropolar (Cosserat) solid read

\[
\sigma_{ij} = C_{ijkl} \gamma_{kl}, \quad m_{ij} = D_{ijkl} k_{kl}. \tag{3.1}
\]

For the most general case of a two-dimensional anisotropic micropolar solid, the in-plane elastic constitutive relations can be written in matrix-notation as

\[
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{12} \\
m_{13} \\
m_{23}
\end{bmatrix} =
\begin{bmatrix}
C_{1111} & C_{1122} & C_{1112} & C_{1121} & 0 & 0 \\
C_{1212} & C_{2222} & C_{2212} & C_{2221} & 0 & 0 \\
C_{1112} & C_{2212} & C_{1212} & C_{2221} & 0 & 0 \\
0 & 0 & 0 & 0 & D_{1313} & D_{1323} \\
0 & 0 & 0 & 0 & D_{1323} & D_{2323}
\end{bmatrix}
\begin{bmatrix}
\gamma_{11} \\
\gamma_{22} \\
\gamma_{12} \\
\gamma_{21} \\
k_{13} \\
k_{23}
\end{bmatrix}, \tag{3.2}
\]

and in terms of the symmetric and the antisymmetric stresses and strains as

\[
\begin{bmatrix}
s_{11} \\
s_{22} \\
s_{12} \\
m_{13} \\
m_{23}
\end{bmatrix} =
\begin{bmatrix}
A_{1111}^{(1)} & A_{1122}^{(1)} & A_{1112}^{(1)} & A_{1121}^{(2)} & 0 & 0 \\
A_{1222}^{(1)} & A_{1212}^{(1)} & A_{1221}^{(2)} & A_{1212}^{(2)} & 0 & 0 \\
\frac{1}{2} A_{2212}^{(1)} & \frac{1}{2} A_{2222}^{(1)} & A_{2212}^{(1)} & A_{2221}^{(2)} & 0 & 0 \\
\frac{1}{2} A_{1112}^{(1)} & \frac{1}{2} A_{2212}^{(2)} & A_{1212}^{(2)} & A_{1221}^{(2)} & 0 & 0 \\
0 & 0 & 0 & 0 & D_{1313} & D_{1323} \\
0 & 0 & 0 & 0 & D_{1323} & D_{2323}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{12} \\
\beta_{13} \\
\beta_{23}
\end{bmatrix}, \tag{3.3}
\]

where

\[
s_{ij} = \frac{\sigma_{ij} + \sigma_{ji}}{2}, \quad \tau_{ij} = \frac{\sigma_{ij} - \sigma_{ji}}{2}, \tag{3.4}
\]

\[
\varepsilon_{ij} = \frac{\gamma_{ij} + \gamma_{ji}}{2}, \quad \beta_{ij} = \frac{\gamma_{ij} - \gamma_{ji}}{2} = \varepsilon_{ijk} (\omega_k - \phi_k),
\]

with \(\omega_k = (\varepsilon_{ijk} u_{ji})/2\) the classical macrorotation vector. By substituting \(\sigma_{ij} = s_{ij} + \tau_{ij}\) and \(\gamma_{ij} = \varepsilon_{ij} + \beta_{ij}\) into Equations (3.2) and solving for the symmetric \(s_{ij}\) and antisymmetric \(\tau_{ij}\) stresses, the coefficients of Equations (3.3) are found in terms of those of Equations (3.2) as
According to the constitutive Equations (3.3), a two-dimensional anisotropic micropolar solid has 13 independent constants. The discrete structures analyzed in chapter 2 fall in two classes: in-plane orthotropic (the square structure) and in-plane isotropic (hexagons, perturbed hexagons and Voronoi structures). By incorporating the specific symmetries of these structures, it turns out that they can both be written as (Nowacki [1986], Dendievel et al. [1998], Wang and Stronge [1999], Warren and Byskov [2002])

\[
A_{111}^{(1)} = C_{1111}, \quad A_{1122}^{(1)} = C_{1122}, \quad A_{1112}^{(1)} = C_{1112} + C_{1121},
A_{1112}^{(2)} = C_{1112} - C_{1121}, \quad A_{2222}^{(2)} = C_{2222}, \quad A_{2212}^{(2)} = C_{2212} + C_{2221},
A_{2212}^{(3)} = C_{2212} - C_{2221}, \quad A_{1212}^{(3)} = \frac{C_{1212} + 2C_{1221} + C_{2121}}{2},
A_{1212}^{(2)} = \frac{C_{1212} - C_{2121}}{2}, \quad A_{1212}^{(3)} = \frac{C_{1212} - 2C_{1221} + C_{2121}}{2}.
\]

For general in-plane orthotropic structures there are three independent classical constants, \(C_{1111}, C_{1122}\) and \(A_{1112}^{(1)}\), while for in-plane isotropic structures only two of these are independent. In addition, there are two micropolar constants that need to be determined, \(A_{1212}^{(3)}\) and \(D_{1313}\). These constants must be larger than zero to satisfy positive definiteness of the strain energy density (Eringen [1999]).

In the following we will obtain closed-form expressions for the classical constants in terms of the microstructural parameters \(t\) (cell wall thickness), \(d\) (cell size), \(E_s\) (Young’s modulus of the cell wall material) and \(\nu_s\) (Poisson’s ratio of the cell wall material). To do so, we start with the general expression of Hooke’s law for an orthotropic linear elastic classical material:
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\[
\begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{33} \\
\varepsilon_{12} \\
\varepsilon_{13} \\
\varepsilon_{23}
\end{bmatrix} = \begin{bmatrix}
\frac{1}{E_1} & -\frac{v_{21}}{E_2} & -\frac{v_{31}}{E_3} & 0 & 0 & 0 \\
-\frac{v_{12}}{E_2} & \frac{1}{E_2} & -\frac{v_{32}}{E_3} & 0 & 0 & 0 \\
-\frac{v_{13}}{E_3} & -\frac{v_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2G_{12}} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2G_{13}} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2G_{23}}
\end{bmatrix} \begin{bmatrix}
s_{11} \\
s_{22} \\
s_{33} \\
s_{12} \\
s_{13} \\
s_{23}
\end{bmatrix}, \quad (3.7)
\]

with

\[
\frac{v_{21}}{E_2} = \frac{v_{12}}{E_1}, \quad \frac{v_{31}}{E_3} = \frac{v_{13}}{E_1}, \quad \frac{v_{32}}{E_3} = \frac{v_{23}}{E_2}. \quad (3.8)
\]

Throughout this thesis, the in-plane properties are those in the \(x_1-x_2\) plane (as shown in Fig. 2.1), and the properties in the direction parallel to the axis of the prismatic cells, that is in the \(x_3\) direction, are referred to as the out-of-plane properties. The Poisson’s ratio \(\nu_{ij}\) is defined as the negative of the normal strain in the \(j\) direction divided by the normal strain in the \(i\) direction, for normal loading in the \(i\) direction \((\nu_{ij} = -\varepsilon_{jj}/\varepsilon_{ii}, \text{no summation over the indices } i \text{ and } j)\).

For squares it can be deduced that

\[
E_1 = E_2 = E_s \frac{t}{d}, \quad E_3 = 2E_s \frac{t}{d},
\]

\[
G_{12} = \frac{1}{2} E_s \left( \frac{t}{d} \right)^3 \frac{1}{1 + (2.4 + 1.5\nu_s)(t/d)^2},
\]

\[
v_{12} = v_{21} = 0, \quad v_{13} = v_{23} = 0.5 \nu_s, \quad v_{31} = v_{32} = \nu_s,
\]

with \(d\) the length of an edge of a square and with shear deformations taken into account. The elastic constants of Equations (3.6) can be obtained in terms of the microstructural parameters \((t, d, E_s, \nu_s)\) by inverting Equations (3.7) after substituting Equations (3.9). Assuming plane stress conditions \((s_{33} = s_{13} = s_{23} = 0)\), the elastic constants read

\[
C_{1111} = E_s \frac{t}{d}, \quad C_{1122} = 0, \quad A_{1212}^{(1)} = 2G_{12} = E_s \left( \frac{t}{d} \right)^3 \frac{1}{1 + (2.4 + 1.5\nu_s)(t/d)^2}. \quad (3.10)
\]
Similarly, for plane strain ($\varepsilon_{33} = \varepsilon_{13} = \varepsilon_{23} = 0$):

\[
C_{1111} = E_s \frac{t}{d} \frac{1 - 0.5\nu_s^2}{1 - \nu_s^2}, \quad C_{1122} = E_s \frac{t}{d} \frac{0.5\nu_s^2}{1 - \nu_s^2},
\]

\[
A_{1212}^{(l)} = 2G_{12} = E_s \left( \frac{t}{d} \right)^3 \frac{1}{1 + (2.4 + 1.5\nu_s)(t/d)^2}.
\]  

(3.11)

The hexagonal microstructures (both regular and perturbed) and Voronoi tessellations are transverse isotropic, i.e., there is a plane of isotropy (the $x_1-x_2$ plane), but the material properties are not symmetric with respect to the out-of-plane, or transverse ($x_3$), direction. For transverse isotropy we can therefore write:

\[
E_1 = E_2 = E_p \quad \text{(in-plane Young's modulus)}
\]

\[
E_3 = E_t \quad \text{(out-of-plane/transverse Young's modulus)}
\]

\[
\nu_{12} = \nu_{21} = \nu_p \quad \text{(in-plane Poisson's ratio)}
\]

\[
\nu_{13} = \nu_{23} = \nu_{pt} \quad \text{(Poisson's ratio associated with the effect of an in-plane loading on the out-of-plane direction)}
\]

\[
\nu_{31} = \nu_{32} = \nu_{tp} \quad \text{(Poisson's ratio associated with the effect of an out-of-plane loading on the in-plane direction)}.
\]  

(3.12)

Because of symmetry, we have in addition:

\[
\frac{\nu_{tp}}{E_t} = \frac{\nu_{pt}}{E_p}.
\]  

(3.13)

For the strain energy density to be positive definite, the elastic constants must satisfy (Hibbitt et al. [2001], Padovani [2002]):

\[
E_p, E_t > 0, \quad \left| \nu_{pt} \right| < \left( \frac{E_p}{E_t} \right)^{1/2}, \quad \left| \nu_{p} \right| < 1.
\]  

(3.14)

Note that the in-plane Poisson’s ratio $\nu_p$ can be larger than 0.5, which is not possible for isotropic materials. For the transverse isotropic microstructures that we analyzed, the main deformation mechanism is bending of cell walls in case of in-plane loading. When they are loaded in the out-of-plane direction, on the other hand, the cell walls deform by stretching, and consequently the out-of-plane Young’s modulus ($E_t$) is much larger than the in-plane Young’s modulus ($E_p$) (see also Table 2.1). As a consequence, the Poisson’s ratio associated with the effects of the in-plane loads in
the out-of-plane direction is very low, as shown by Gibson and Ashby [1997], and we take it to be zero:

\[ v_{pt} = \frac{E_p}{E_t} v_s \approx 0. \]  

(3.15)

Now, if we insert Equations (3.12) and (3.15) into Equations (3.7), assume plane strain and invert, the in-plane equations follow to be

\[
\begin{bmatrix}
  s_{11} \\
  s_{22} \\
  s_{12}
\end{bmatrix} = \frac{E_p}{1 - \nu_p^2} \begin{bmatrix}
  1 & \nu_p & 0 \\
  \nu_p & 1 & 0 \\
  0 & 0 & (1 - \nu_p)
\end{bmatrix} \begin{bmatrix}
  \varepsilon_{11} \\
  \varepsilon_{22} \\
  \varepsilon_{12}
\end{bmatrix},
\]

(3.16)

which would be exactly the same if plane stress was assumed. Thus, for the case of \( v_{pt} = 0 \), the plane-stress and plane-strain in-plane constitutive relations for a transverse isotropic solid are the same and given by (3.16). Comparison of (3.16) with (3.6) yields

\[
C_{1111} = C_{2222} = \frac{E_p}{1 - \nu_p^2}, \quad C_{1122} = \frac{\nu_p E_p}{1 - \nu_p^2}, \quad A_{4212}^{(i)} = \frac{E_p}{1 + \nu_p}. \]

(3.17)

For the regular hexagonal structure, \( E_p \) and \( \nu_p \) (taking shear deformations into account) can be directly written in terms of the microstructural parameters as (Gibson and Ashby [1997])

\[
E_p = \frac{4}{\sqrt{3}} E_s (t/l)^3 \frac{1}{1 + (5.4 + 1.5\nu_s)(t/l)^2}, \quad \nu_p = \frac{1 + (1.4 + 1.5\nu_s)(t/l)^2}{1 + (5.4 + 1.5\nu_s)(t/l)^2}. \]

(3.18)

For the irregular structures (perturbed hexagons and Voronoi structures) we will perform finite element calculations on large periodic unit-cells to obtain the elastic constants.

The in-plane constitutive equations for an isotropic and a transverse isotropic (with \( v_{pt} = 0 \)) micropolar solid can both be written in the form of Equation (3.6). For an isotropic material (with \( E \) and \( \nu \)) in plane-strain, the coefficients in the first three equations of (3.6) read

\[
C_{1111} = C_{2222} = \frac{E (1 - \nu)}{(1 + \nu)(1 - 2\nu)}, \quad C_{1122} = \frac{E\nu}{(1 + \nu)(1 - 2\nu)}, \quad A_{4212}^{(i)} = \frac{E}{1 + \nu}. \]

(3.19)
They can also be written in terms of the classical Lamé constants $\lambda$ and $\mu (=G)$ as

$$C_{1111} = C_{2222} = \lambda + 2\mu, \quad C_{1122} = \lambda, \quad A_{1212}^{(1)} = 2\mu,$$  \hspace{1cm} (3.20)

For isotropic materials the Lamé constants are related to Young’s Modulus and Poisson’s ratio as $\mu = E/2(1+\nu)$ and $\lambda = 2E\nu/(2(1+\nu)(1-2\nu))$. For transverse isotropic materials with $\nu_{pt} = 0$ (see Equations (3.16)) the moduli can also be written in terms of ‘in-plane’ Lamé constants $\mu_p$ and $\lambda_p$, similar to Equations (3.20), with $\mu_p$ and $\lambda_p$ defined as $\mu_p = E_p/(1+\nu_p)$ and $\lambda_p = 2E_p\nu_p/(2(1+\nu_p)(1-\nu_p))$. Therefore, if we know the solution of a plane-strain boundary value problem for an isotropic material expressed in terms of $\lambda$ and $\mu$, we can obtain the solution for a transverse isotropic material (with $\nu_{pt}=0$) simply by substituting $\mu_p$ for $\mu$ and $\lambda_p$ for $\lambda$. We will make use of this conversion at several places in the thesis.

### 3.3 Analytical solution of the simple shear problem

In this section, we analyze the simple shear of a sandwich panel having a micropolar material as a core (see also Diebels and Steeb [2002], Tekoğlu and Onck [2003a, b] and [2005]). The boundary conditions are similar as for the discrete structures analyzed in chapter 2, i.e. $u_1 = -\gamma H$, $u_2 = \phi_3 = 0$ at the top ($x_2 = H$) and $u_1 = u_2 = \phi_3 = 0$ at the bottom ($x_2 = 0$). The panel is assumed to be infinitely wide in the horizontal ($x_1$) and out-of-plane ($x_3$) directions and is under a state of plane strain. For this situation the kinematic equations reduce to:

$$\varepsilon_{11} = 0, \quad \varepsilon_{22} = u_{2,2},$$
$$\varepsilon_{12} = \frac{1}{2} u_{1,2}, \quad \beta_{2} = -\frac{1}{2} \left( u_{1,2} + 2\phi_3 \right),$$
$$k_{13} = 0, \quad k_{23} = \phi_{3,2},$$

while the equilibrium equations read

$$s_{12,2} - \tau_{12,2} = 0, \quad s_{22,2} = 0, \quad m_{23,2} + 2\tau_{12} = 0.$$  \hspace{1cm} (3.22)

Inserting (3.21) into (3.22) via the constitutive equations (3.6), yields

$$\left( m + 1 \right) u_{1,22} + 2m\phi_{3,2} = 0,$$  \hspace{1cm} (3.23)
$$u_{2,22} = 0.$$  \hspace{1cm} (3.24)
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\[ l_c^2 \phi_{3,2} - 2m \phi_3 - mu_{1,2} = 0, \quad (3.25) \]

with

\[ l_c^2 = \frac{D_{1313}}{A_{1212}^{(i)}}, \quad m = \frac{A_{1212}^{(3)}}{A_{1212}^{(i)}}. \quad (3.26) \]

The parameter \( l_c \) is a material length, and it is often referred to as the characteristic length. The constant \( m \), on the other hand, is a dimensionless term that can be seen as a coupling factor between the microrotation and macrorotation fields. For the limit \( m \to \infty \), the micro- and macrorotation fields are fully coupled, and the micropolar theory reduces to the couple stress theory. We will explore the effect of the two higher order constants, \( m \) and \( l_c \), in the following. The solution of the differential equations reads

\[ u_2 = A_1x_2 + A_2, \quad (3.27) \]
\[ \phi_3 = C_1 + C_2 e^{\omega x_1} + C_3 e^{-\omega x_1}, \quad (3.28) \]
\[ u_1 = C_4 - 2C_1x_2 - PC_2 e^{\omega x_1} + PC_3 e^{-\omega x_1}, \quad (3.29) \]

with eigenvalue \( \omega \) and coefficient \( P \) defined as

\[ \omega = \frac{1}{l_c} \sqrt{\frac{2m}{m+1}}, \quad P = l_c \sqrt{\frac{2m}{m+1}}. \quad (3.30) \]

The remaining constants are determined by inserting the boundary conditions into the solutions (3.27), (3.28) and (3.29) and are given by

\[ A_1 = A_2 = 0, \quad (3.31) \]
\[ C_1 = \frac{\gamma H(e^{\omega H} - e^{-\omega H})}{2P(2 - e^{\omega H} - e^{-\omega H}) + 2H(e^{\omega H} - e^{-\omega H})}, \quad (3.32) \]
\[ C_2 = -\frac{\gamma H(1 - e^{-\omega H})}{2P(2 - e^{\omega H} - e^{-\omega H}) + 2H(e^{\omega H} - e^{-\omega H})}. \quad (3.33) \]
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\[ C_3 = \frac{\gamma H (1 - e^{\alpha H})}{2P(2 - e^{\alpha H} - e^{-\alpha H}) + 2H(e^{\alpha H} - e^{-\alpha H})}, \]  
\[ C_4 = -\frac{\gamma HP(2 - e^{\alpha H} - e^{-\alpha H})}{2P(2 - e^{\alpha H} - e^{-\alpha H}) + 2H(e^{\alpha H} - e^{-\alpha H})}. \]  

(3.34)

(3.35)

The external work done by the surface tractions and couples can be calculated as shown on the left side of the virtual work expression (see Equation (1.3)). The microrotations and vertical displacements are zero on the surface, so that the surface couples \( Q_3 \) and vertical tractions \( t_2 \) do not contribute to the external work per unit-out-of-plane thickness, \( W^{ex} \), which reduces to

\[ W^{ex} = \int_S t_1 u_1 \ dS = (r_{12} - s_{12}) L \gamma H = F \gamma H. \]  

(3.36)

Here, \( F \) is the shear force per unit out-of-plane thickness and \( L \) the width of the sample (introduced to connect the continuum results of this chapter to the discrete results of the previous chapter). By substituting the constitutive equations (3.6), one can write

\[ F = -LG \left(1 + m\right) u_{1,2} = -LG \left( l_c^2 \phi_{3,22} + u_{1,2} \right), \]  

(3.37)

where \( A_{212}^{(1)} = 2G \) has been substituted. In the second expression the boundary condition \( \phi_3 = 0 \) has been used, while for the third expression Equation (3.25) has been incorporated. These equations clearly emphasize the dependence on the two additional micropolar constants present in the theory. For classical elasticity we have \( l_c = m = 0 \) and \( u_{1,2} = -\gamma \), so that \( F = LG\gamma \). This force will be used to normalize the results for the micropolar solid.

Figure 3.1a shows the normalized shear force (or normalized shear stiffness), \( \frac{F}{(LG\gamma)} \), plotted against the relative specimen size, \( H/d \), for different values of \( l_c/d \) and a fixed coupling factor \((m = 1)\). For convenience we normalize \( H \) and \( l_c \) with the cell size \( d \) to facilitate comparison with the discrete results. For \( l_c/d \rightarrow 0 \), the solution for the micropolar continuum theory reduces to the solution of the same problem in the classical continuum theory, irrespective of the coupling factor \( m \). We see that with increasing \( l_c/d \), the stiffening behaviour in the small \( H/d \) regime becomes more pronounced, and the convergence rate of the macroscopic shear stiffness, \( F/(L\gamma) \), to the shear modulus, \( G \), decreases. In the limiting case of \( l_c/d \rightarrow \infty \), it can be easily deduced from Equations (3.23), (3.24) and (3.25) that \( u_2 = \phi_3 = 0 \) and \( u_1 = -\gamma x_2 \) is the solution of the problem and that the normalized shear modulus does not depend on \( H \).
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(a) $\frac{F}{L G \gamma}$ vs $H/d$

$b)$ $\frac{F}{L G \gamma}$ vs $H/d$

(c) $\frac{X_2}{H}$ vs $\phi_3/\phi_3'$

(d) $\frac{X_2}{H}$ vs $\phi_3/\phi_3'$

(e) $\frac{X_2}{H}$ vs $\varepsilon_{12}/\varepsilon_{12}'$

(f) $\frac{X_2}{H}$ vs $\varepsilon_{12}/\varepsilon_{12}'$

Increasing $l_c/d$

Increasing $m$
Figure 3.1: (a) The normalized shear force (or normalized shear stiffness), $F/(L\gamma)$, plotted against the relative specimen size, $H/d$, for different values of $l_c/d$ and a fixed coupling factor ($m=1$). (b) The effect of $m$ on $F/(L\gamma)$ plotted against $H/d$ for $l_c/d=1$. The normalized microrotation $\phi_3/\phi_3'$ ($\phi_3'=\gamma/2$) through the thickness of a specimen with $H/d=10$, for: (c) a fixed $m$ and increasing $l_c/d$. (d) a fixed $l_c/d$ and increasing $m$. The normalized symmetric shear strain $\varepsilon_{12}/\varepsilon_{12}'$ ($\varepsilon_{12}'=-\gamma/2$), through the thickness of a specimen with $H/d=10$, for: (e) a fixed $m$ and increasing $l_c/d$. (f) a fixed $l_c/d$ and increasing $m$. The normalized antisymmetric strain $\beta_{12}/\varepsilon_{12}'$ ($\varepsilon_{12}'=-\gamma/2$), through the thickness of a specimen with $H/d=10$, for: (g) a fixed $m$ and increasing $l_c/d$. (h) a fixed $l_c/d$ and increasing $m$.

and is given by $F/(L\gamma)=1+m$ (see the second expression in (3.37)).

Figure 3.1b shows the effect of $m$ on $F/(L\gamma)$ plotted against $H/d$ for $l_c/d=1$. In this case, the stiffening behaviour in the small $H/d$ regime becomes more and more pronounced as $m$ approaches to the couple stress limit at $m\to\infty$. However, in contrast to Fig. 3.1a, the convergence rate of the shear stiffness to the classical shear modulus, $G$, increases with increasing $m$. For $m\to0$, the macroscopic shear stiffness does not depend on $H/d$ and is equal to the classical value $F/(L\gamma)=1$.

Next, we will explore the local fields through the thickness of the specimen, responsible for the overall behaviour shown in Figs. 3.1a and b. Figure 3.1c shows the normalized microrotation $\phi_3/\phi_3'$ ($\phi_3'=\gamma/2$) through the thickness of a specimen with $H/d=10$ and $m=1$. A boundary layer is formed at the top and bottom, characterized by the constrained microrotations. The thickness of the boundary layer increases with increasing $l_c/d$. Figure 3.1d, on the other hand, shows that for a fixed $l_c/d$, the thickness of the boundary layer decreases with increasing $m$ and converges to a thickness of approximately $l_c$ for $m\to\infty$. Figures 3.1e and f show the normalized symmetric shear strain $\varepsilon_{12}/\varepsilon_{12}'$, through the thickness of the specimen, for a fixed $m$ and increasing $l_c/d$, and for a fixed $l_c/d$ and increasing $m$, respectively. Here $\varepsilon_{12}'$ is the classical shear strain, given as $\varepsilon_{12}'=-\gamma/2$. Similar to the case of the normalized
rotations, we see that for a fixed $m (l_c/d)$ the thickness of the boundary layer increases (decreases) with increasing $l_c/d (m)$.

Note that when $l_c$ goes to zero (for fixed $m$) or when $m$ goes to zero (for fixed $l_c$), the overall response is size-independent and the shear stiffness is equal to its classical counterpart (see Figs. 3.1a and b), while the symmetric shear strains are uniform over the thickness (see Figs. 3.1e and f) for both cases. However, these two cases are not equivalent as can be nicely exemplified by analyzing the antisymmetric strains $\beta_{12}$ (see Figs. 3.1g and h). For the case of $m = 0$, the microrotation field is not coupled with the displacement field (see Equations (3.23) and (3.25)). Incorporating the boundary conditions yields $\phi_3 = 0$ (see Fig. 3.1d) and $u_1 = -\gamma x_2$ (or $\varepsilon_{12} = -\gamma/2$, see Fig. 3.1f). As a result, the antisymmetric shear strains are not zero as for the $l_c/d = 0$ case (see Fig. 3.1g), but become equal to $\beta_{12} = \gamma/2$ (see Fig. 3.1h). For different boundary conditions for $\phi_3$, however, the microrotation field throughout the specimen would be different, whereas the displacement field and the macroscopic shear stiffness would still coincide with their classical counterparts. The case of zero coupling, $m = 0$, is analyzed by Lakes [1985] as well, for a quasistatic torsion problem of a circular cylindrical rod of an isotropic micropolar solid. He showed that for $m = 0$, a given value of applied torque can be realized in more than one way by using different boundary conditions. This does not correspond to a failure of uniqueness in the usual sense, since different local boundary tractions are associated with different displacement and microrotation fields, even though the total microscopic load is the same. However, $m = 0$ is a pathological case in the sense that the macroscopic end load does not uniquely determine the displacement and microrotation fields. Therefore, the classical elasticity is only recovered as a special case of the micropolar theory when both $m$ and $l_c$ vanish.

For the case of $m \to \infty$ (couple stress solution), the microrotations become equal to the macrorotations, $\phi_3 = (u_{2,1} - u_{1,2})/2$, and the antisymmetric part of the shear strain vanishes (see e.g. Fig. 3.1h), so that the strain tensor becomes symmetric and equal to that in the classical continuum theory. The governing differential equations for the shear problem follow from combining Equations (3.23) and (3.25) and substituting $\phi_3 = -u_{1,2}/2$, yielding

$$\frac{l_c^2}{2} u_{1,222} - u_{1,22} = 0,$$

supplemented by Equations (3.24), which stays the same. The external work done on the material per unit out-of-plane thickness reduces to

$$W^{ex} = \int_{S} t_1 u_1 \ dS = \gamma HGL \frac{l_c^2}{2} u_{1,222}. \tag{3.39}$$
Note that in the limit that $l_c$ goes to zero, $W^x$ converges to $HGL\gamma^2$, the solution for the classical problem.

### 3.4 Comparison with the discrete results

In this section, we will compare the discrete results for the simple shear problem (see chapter 2) with the analytical solution for the micropolar continuum theory, and fit the micropolar constants, $m$ and $l_c$, to give the best overall agreement. We will, in addition, develop a strain mapping procedure that enables a comparison between the discrete and continuum deformation fields.

#### 3.4.1 Macroscopic response

Figures 3.2a-f show the best fit of the micropolar solution to the average value of the discrete results for the macroscopic shear stiffness, $F/(L\gamma)$, for the microstructures that we analyzed. To reflect the scatter, we plot the upper and lower bounds as well. For all cases, the couple stress solution ($m \to \infty$) gives the best agreement. We see that the characteristic length $l_c$ depends on the cell orientation in the case of the perfect hexagonal microstructure ($l_c=0.15d$ for the default and $l_c=0.28d$ for the rotated orientations), whereas the difference is very small for the perturbed case ($l_c=0.55d$ versus $l_c=0.47d$). The stiffening behaviour is the largest for the Voronoi microstructure, followed by the square, perturbed hexagonal and perfect hexagonal microstructure, respectively, which is reflected in a larger value for the characteristic length $l_c$. Clearly, the characteristic length not only scales with the cell size, it also depends on the cellular morphology.
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Figure 3.2: The best fit of the micropolar solution to the average value of the discrete results for the macroscopic shear stiffness, $F/(L\gamma)$, for the: (a) Square microstructure. (b) Perfect hexagonal
In this section, we will develop a strain mapping procedure that will be used throughout the thesis for different boundary value problems. The finite element calculations performed on the discrete models (see chapter 2), provide the displacements and rotations of cell vertices, i.e. the locations at which the cell-walls meet. Figure 3.3, for instance, shows a Voronoi sample, indicated with red lines, having cell vertices that are indicated by black dots. We construct a square background grid (black dashed-lines), having grid nodes that are indicated with open squares. We take the average of the displacements and rotations at the cell vertices that are located inside a square, and assign these average values to each node of the square. Here, we only show a single sample for clarity, but in case there are multiple samples, the averaging is performed over the vertices of all the samples that are located in that square. If a node is shared by \( n \) squares, the values of the displacements and the rotations at that node are given by the arithmetic average of the values coming from each of these \( n \) squares. After finding the displacements and the rotations for each grid node, we divide each square in two triangles and calculate the strains for each triangle using standard finite element techniques. The strains within a triangle

![Figure 3.3: A square mesh for the strain mapping procedure plotted on top of a Voronoi sample.](image-url)
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with nodes $p$, $r$, and $q$, respectively (numbered counterclockwise), are given as

\[
\begin{align*}
\{\varepsilon_{11}\} &= \begin{bmatrix} N^p,1 & 0 & N^r,1 & 0 & N^q,1 & 0 \\
0 & N^p,2 & 0 & N^r,2 & 0 & N^q,2 \\
N^p,2 & N^p,1 & N^r,2 & N^r,1 & N^q,2 & N^q,1 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} u_1^p \\
u_2^p \\
u_1^r \\
u_2^r \\
u_1^q \\
u_2^q \end{bmatrix}, \\
\{\varepsilon_{22}\} &= \begin{bmatrix} 0 & N^p,1 & 0 & N^r,1 & 0 & N^q,1 \\
N^p,2 & 0 & N^r,2 & 0 & N^q,2 \\
N^p,2 & N^p,1 & N^r,2 & N^r,1 & N^q,2 & N^q,1 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} u_1^p \\
u_2^p \\
u_1^r \\
u_2^r \\
u_1^q \\
u_2^q \end{bmatrix},
\end{align*}
\]

where $N^i$ ($i = p, r, q$) are the shape functions for a constant strain triangle (see Zienkiewicz and Taylor [2000]). Finally, the strains within a square are obtained by taking the average of the two triangles.

The accuracy and the scatter in the estimated displacement and strain fields depend, obviously, on the number of samples (NS) included in the strain mapping procedure. The samples that we analyze are cut from a large block of material, and each one of them corresponds to a different boundary configuration and cell distribution. The difference between two samples depends on the cutting step size (CSS, see section 2.3.1) and we will see that this affects the number of samples required for a converged displacement or strain field. The grid size (GS), i.e. the length of the edge of a square of the background grid (see Fig. 3.3), plays an important role as well. In the following, we explore the effect of these parameters on the strain maps of a uniaxial strain field.

Figure 3.4a shows a Voronoi sample with a length of $40d$ and a height of $24d$. The sample is compressed in the $x_1$-direction by prescribing symmetry conditions at the left and a uniform horizontal displacement field at the right. Figures 3.4b, c and d show the strain field $\varepsilon_{11}$ normalized with the applied strain $\varepsilon_{\text{appl}}$ for the sample of Fig. 3.4a, for 25 samples, and for 200 samples, respectively. The grid size is taken to be equal to the average cell size and the cutting step size to half a cell size (GS = $d$ and CSS = $d/2$). We see that in the case of a single sample, there are large fluctuations in the value of $\varepsilon_{11}$ (Fig. 3.4b), whereas with increasing number of samples the fluctuations decrease (Fig. 3.4c), resulting in a smooth field for 200 samples (Fig. 3.4d). It can be observed that the strain is underestimated in an edge layer parallel to the left and right boundaries. To investigate whether these layers are a consequence of the strain mapping procedure or related to the discrete behaviour of the cellular material, we applied the strain mapping procedure to a linearly increasing displacement-field. The same edge layers were observed as for the discrete structure (Fig. 3.4d). The underlying reason of this behaviour is that the squares located at the left (right) boundary share only their right-side (left-side) nodes with other squares. As a result, the displacements are almost the same for all nodes and consequently, the
Figure 3.4: (a) A Voronoi sample with a length of $40d$ and a height of $24d$. The strain field $\varepsilon_{11}$ normalized with the applied strain $\varepsilon_{\text{appl}}$ for: (b) the sample shown in (a), (c) 25 samples, and (d) 200 samples. Strains are much lower for these elements. This is a bias of the strain mapping algorithm and results in edge layers with a thickness on the order of the grid size. Note that this artefact does not develop at the top and bottom, where the strain $\varepsilon_{11}$ is calculated from the displacement gradients in a direction parallel instead of perpendicular to the edges.

Figure 3.5: The effect of the number of samples $\text{NS}$ on the normalized strain $\varepsilon_{11}/\varepsilon_{\text{appl}}$, plotted through the thickness at the line $x_1 = L/2$, for a cutting step size (CSS) of (a) $d/10$, (b) $d/2$. 
Figures 3.5a and b show the normalized strain $\varepsilon_{11}/\varepsilon_{appl}$ through the thickness at the line $x_1 = L/2$, for a cutting step size (CSS) of $d/10$ and $d/2$, respectively. In both cases, the fluctuations in the $\varepsilon_{11}/\varepsilon_{appl}$ value decreases with increasing number of samples NS, as it would be expected. The convergence rate, however, is much larger for the case of CSS = $d/2$: The field averaged over 50 samples (NS = 50) for CSS = $d/2$ (Fig. 3.5b) is almost the same as the field averaged over 200 samples (NS = 200) for CSS = $d/10$ (Fig. 3.5a).

\[ \frac{\varepsilon_{11}}{\varepsilon_{appl}}\]

\[ x_2/d \]

**Figure 3.6:** (a) The effect of the grid size GS on the normalized strain $\varepsilon_{11}/\varepsilon_{appl}$, plotted through the thickness at the line $x_1 = L/2$ for CSS = $d/2$ and NS = 200. The normalized rotation $\phi_3/\phi_3'$ ($\phi_3' = \gamma/2$) (b) and the normalized shear strain $\varepsilon_{12}/\varepsilon_{12}'$ ($\varepsilon_{12}' = -\gamma/2$) (c) through the thickness, for samples with $H = 10d$ and $L = 100d$.

Figure 3.6a shows the effect of the grid size GS on the normalized strain $\varepsilon_{11}/\varepsilon_{appl}$, plotted through the thickness at the line $x_1 = L/2$ for CSS = $d/2$ and NS = 200.
The results are as expected: The larger the grid size, the smoother the strain field. However, to be able to capture the gradients in the field variables, the grid size should be small enough. To investigate that, we plot the rotations and strains through the thickness for the Voronoi samples tested in shear in chapter 2 (section 2.3.1, Fig. 2.11f). Figures 3.6b and c show the normalized rotation $\phi_3 / \phi_3'$ ($\phi_3' = \gamma / 2$) and the normalized shear strain $\varepsilon_{12} / \varepsilon_{12}'$ ($\varepsilon_{12}' = -\gamma / 2$), respectively, through the thickness, for samples with $H = 10d$ and $L = 100d$. The rotation $\phi_3$ and the shear strain $\varepsilon_{12}$ corresponding to an $x_2$ coordinate are the average values over all the nodes with the same $x_2$ coordinate along the length $L$ of the samples. We see that the gradients near the top and bottom boundaries can be better picked up with small grid sizes, at the expense of an increased scatter in the central region.

We can summarize the effects of the different strain mapping parameters as follows.

- The number of samples (NS) should be large enough to smooth out the fluctuations in the field variables. An increase in the cutting step size (CSS) can significantly reduce the number of samples NS required for a converged solution.
- The optimal grid size (GS) is the one that is small enough to capture the gradients in non-uniform regions and large enough to provide relatively smooth fields in uniform regions.

### 3.4.3 Local response

In section 3.4.1 we found that the best fit to the global results (i.e. the macroscopic stiffness) was made by the couple stress theory ($m \to \infty$). Each specific cellular microstructure resulted in a specific value for the characteristic length, which turned out to be on the order of the cell size (ranging from $l_c = 0.15d$ to $0.9d$). In this section, our aim is to see how accurate the corresponding local continuum fields correspond to the discrete results. To do so, we analyze the normalized rotations $\phi_3 / \phi_3'$ ($\phi_3' = \gamma / 2$) and the normalized shear strains $\varepsilon_{12} / \varepsilon_{12}'$ ($\varepsilon_{12}' = -\gamma / 2$) through the thickness of samples with height $H = 5d$ and $10d$.

Figures 3.7a and b show the normalized microrotation $\phi_3 / \phi_3'$ for the square microstructure (corresponding to Fig. 3.2a). To accurately account for the fields we repeated the calculations from chapter 2 using a smaller step size, CSS = $d/10$ instead of $d/5$. The discrete data correspond to the rotations of cell vertices (i.e. the locations where four cell walls meet). The fit is very good for both sample thicknesses. These figures clearly show that the thickness of the strong boundary layers located at the top and bottom of the samples is the same for both sample heights, approximately one cell size. As a result, the ratio of the boundary layer thickness to the total specimen thickness is smaller for the $H = 10d$ case (Fig. 3.7b), explaining why the stiffening for the $H = 10d$ case is lower as well (cf. Fig. 3.2a).
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Figs. 3.8a and b), whereas for the shear strain (Figs. 3.8c and d), all values at the ordinate are shown in addition to the averages. We see that

Figure 3.7: The normalized rotation \( \phi_3/\phi'_3 \) (\( \phi'_3 = \gamma/2 \)) for the square microstructure with: (a) \( H = 5d \). (b) \( H = 10d \).

Figures 3.8a (c) and b (d) show the normalized microrotation \( \phi_3/\phi'_3 \) (the normalized shear strain \( \varepsilon_{12}/\varepsilon'_{12} \)) through the thickness, for the Voronoi microstructures with \( H = 5d \) and \( H = 10d \), respectively. The discrete data is obtained by applying the strain mapping procedure as explained in section 3.4.2 with \( NS = 100 \) and \( GS = d/2 \) (note that the 100 samples were cut with \( CSS = d/10 \)). For the microrotation, only the average value at each \( x_2 \)-coordinate is shown (dashed lines in Figs. 3.8a and b), whereas for the shear strain (Figs. 3.8c and d), all values at the nodes with the same \( x_2 \)-coordinate are shown in addition to the averages. We see that the couple stress theory is able to nicely pick up the local fields and the boundary layer thickness for each case. For both the rotations and strains for the \( H/d = 5 \) case the average discrete fields are slightly overestimated, although the continuum solution is
Figure 3.8: The normalized rotation $\phi_3 / \phi_3'$ ($\phi_3' = \gamma / 2$) through the thickness, for the Voronoi microstructures with: (a) $H = 5d$. (c) $H = 10d$. The normalized shear strain $\varepsilon_{12} / \varepsilon_{12}'$ ($\varepsilon_{12}' = -\gamma / 2$) through the thickness, for the Voronoi microstructures with: (b) $H = 5d$. (d) $H = 10d$. (e) Comparison of the strain maps for the couple stress solution ($l_c / d = 0.9$) and discrete analyses with the best fit for the couple stress theory for Voronoi tessellations. All the strain maps are for a grid size $GS = d/2$.

well located within the scatter band of the discrete results. For the $H/d = 10$ case, the agreement is very good. To investigate the role of the strain mapping procedure in the discrepancy between the discrete and continuum solution for the $H/d = 5$ case, strain maps are constructed of the analytical (exact) displacement fields. As input for the strain map procedure we take the displacement values given by the analytical solution at the locations of the grid points for a grid size $GS = d/2$. Figure 3.8e shows that the strain map for the analytical solution overestimates the field variables in the boundary
layers as well. In addition, the analytical strain map slightly underestimates the analytical shear strains in the central (core) region. Thus, the difference between the discrete and couple stress results is partly due to the inaccuracy in the strain mapping procedure, and partly due to the inaccuracy in the couple stress prediction.

### 3.5 Analytical solution of the pure bending problem

In this section, we will solve the pure bending problem of a straight plane-strain beam (i.e. a plate that is infinitely wide in the out-of-plane direction) for an isotropic, centro-symmetric micropolar material. The midplane of the plate is set as the $x_1$-$x_3$ plane, the $x_3$ direction coinciding with the out-of-plane direction. The pure bending assumption states that “transverse plane sections (that are parallel to the $x_2$-$x_3$ plane, in our case) remain plane and normal to the longitudinal fibres”, see Fig. 3.9.

![Figure 3.9: Notation and geometry of the plane-strain pure bending problem.](image)

The most general displacement field that satisfies this can be written as

$$u_1 = \frac{1}{R} x_1 x_2, \quad u_2 = -\frac{1}{2R} x_1^2 + f(x_2) + P, \quad u_3 = 0,$$  \hspace{1cm} (3.41)

where $R$ is the radius of curvature, $f$ a function of $x_2$ alone and $P$ is an integration constant. The kinematic equations reduce to:

$$\varepsilon_{11} = \frac{1}{R} x_2, \quad \varepsilon_{22} = f_{,2}, \quad \varepsilon_{12} = 0,$$

$$\beta_{12} = -\left(\frac{1}{R} x_1 + \phi_3\right), \quad k_{13} = \phi_{3,1}, \quad k_{23} = \phi_{3,2},$$  \hspace{1cm} (3.42)

while the equilibrium equations are similar to the shear problem and given in Equations (3.22). Since the shear strains $\varepsilon_{12}$ are zero, the first equilibrium equation (Equation 3.22a) combined with the constitutive behaviour (see Equations (3.6)) and
the kinematics states that $k_{23}$ must vanish. As a result, the third equilibrium equation (Equation 3.22c) corresponds to $\tau_{12}=\beta_{12}=0$, so that the only expression for the microrotations that satisfies equilibrium is given by

$$\phi_3 = -\frac{1}{R} x_1. \quad (3.43)$$

Finally, from the second equilibrium equation (Equation 3.22b) and the requirement that the surface tractions $t_i$ and surface couples $Q_i$ must vanish at the top and bottom, the remaining unknowns in Equations (3.41) can be determined, yielding

$$u_1 = \frac{1}{R} x_1 x_2, \quad u_2 = -\frac{1}{2R} \left(x_1^2 + \frac{C_{1122}}{C_{1111}} x_2^2\right), \quad u_3 = 0, \quad (3.44)$$

which is equal to the solution of the bending problem in classical elasticity. Using the virtual work expression (1.2), we can find the bending moment per unit out-of-plane thickness acting on the beam as

$$M = \int_{-H/2}^{H/2} s_{11} x_2 \, dx_2 - \int_{-H/2}^{H/2} m_{13} \, dx_2 = \left(C_{1111} - \frac{C_{1122}}{C_{1111}}\right) \frac{H^3}{12R} + D_{1313} \frac{H}{R} \quad (3.45)$$

Multiplying the bending moment $M$ with the radius of curvature $R$ yields the bending stiffness $B$. By substituting the moduli for an isotropic material (Equations (3.19)) and using the definition of the characteristic length (Equation (3.26a)), the normalized bending stiffness $B/B_{\text{class}}$ can be written as

$$\frac{B}{B_{\text{class}}} = 1 + 12(1-\nu) \left(\frac{l_c}{H}\right)^2, \quad (3.46)$$

with the classical bending stiffness $B_{\text{class}}$ defined as

$$B_{\text{class}} = \frac{EH^3}{12(1-\nu^2)}. \quad (3.47)$$

The above results agree with the solutions given by Koiter [1964] and Gauthier and Jahsman [1975]. Similarly, by substituting the moduli for the transverse isotropic
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material (Equations (3.17)), we find

\[
\frac{B}{B_{\text{class}}} = 1 + \frac{12}{1 + \nu_p} \left( \frac{l_c}{H} \right)^2,
\]

(3.48)

with

\[
B_{\text{class}} = \frac{E_p H^3}{12}.
\]

(3.49)

Since we have \( \nu_p = 0 \) for the transverse isotropic materials, there is no coupling with the out-of-plane dimension and the classical bending stiffness (Equation (3.49)) is lower than the classical bending stiffness for isotropic materials (Equation (3.47)).

Figure 3.10a shows the effect of the characteristic length \( l_c \) on the normalized bending stiffness for a transverse isotropic material with \( \nu_p = 0.9 \), which is a representative value for the cellular microstructures that we analyzed. We see that, as in the case of simple shear (see Fig. 3.1a), for \( l_c/d \to 0 \), the solution for the micropolar continuum theory reduces to the solution of the same problem in the classical continuum theory. There is a stiffening in the small \( H/d \) regime \((B/B_{\text{class}}>1)\), which becomes more pronounced with increasing \( l_c/d \). Note that our results for the discrete models (see chapter 2), on the contrary, show weakening, i.e. a lower bending rigidity for small heights. For larger heights, the bending stiffness converges to its classical counterpart. Figure 3.10b shows the effect of the Poisson’s ratio on the stiffening behaviour; the stiffening is larger for a lower Poisson’s ratio.

**Figure 3.10:** (a) The effect of the characteristic length \( l_c \) on the normalized bending stiffness for a transverse isotropic material with \( \nu_p = 0.9 \). (b) The effect of the Poisson’s ratio \( \nu_p \) on the normalized bending stiffness for \( l_c/d = 0.9 \).
3.6 Summary and discussion

In this chapter, we have solved the simple shear and the pure bending problems analytically for the micropolar continuum theory. We investigated the effect of the two additional micropolar constants: the coupling factor $m$ and the characteristic length $l_c$. By comparing the analytical and the discrete solutions for the simple shear problem in terms of the macroscopic shear stiffness, we found that it was possible to fit all discrete results only if the coupling factor $m$ was taken to be very large ($m \to \infty$). In this limit, the micropolar theory reduces to the couple stress theory in which microrotations are no longer independent degrees of freedom but are constrained to be equal to the macroscopic rotations, $(u_{2,1}-u_{1,2})/2$. From the comparison it was found that $l_c$ scales with the cell size $d$ and depends on the cellular morphology, with $l_c/d$ in the range $0.15 - 0.9$.

We developed a strain mapping procedure to be able to obtain the strain fields based on the displacement data given by the discrete calculations. For the square and hexagonal microstructures, we used this strain mapping procedure to compare the discrete microrotation and strain fields for the simple shear problem with their continuum counterparts given by the analytical couple stress solution. It was shown that the characteristic length $l_c$ obtained by fitting the macroscopic shear stiffness resulted in an excellent agreement between the discrete and continuum microrotation and strain fields.

Hexagonal materials have six-fold symmetry in the $x_1$-$x_2$ plane, which makes them transversely (in-plane) isotropic both in classical and micropolar continuum theories (see also section 2.2.1). This means there is only one $l_c$ value for the regular hexagonal structure, irrespective of its orientation. This is clearly in contradiction with Fig. 3.2b and c, yielding two different values for $l_c$ for the default ($l_c=0.15d$) and the rotated ($l_c=0.28d$) orientation. However, the large scatter in the discrete results and the relatively low stiffening makes it hard to find a unique fit; i.e. taking $l_c=0.28d$ would also make a reasonable fit for the default hexagons that falls well within the scatter band (see Figs. 2.11b and 3.2b). Random imperfections in the hexagonal structure not only reduce the scatter, it also decreases the difference between the two orientations (Figs. 2.11d and e and Figs. 3.2d and e). As a result, the fitting procedure is more accurate, leading to values of $l_c$ that are close together. The results for the fully random structures are orientation-independent with a limited amount of scatter (Figs. 2.11f and 3.2f). By taking into account the isotropy of the structures, good fits are obtained for $l_c/d=0.28$ for the regular hexagons, $l_c/d=0.47$ for the perturbed hexagons and $l_c/d=0.9$ for the Voronoi structures. These characteristic lengths also reflect the increasing stiffening with increasing randomness.
The current study enables a comparison with existing homogenization studies on regular lattices. For square structures, Banks and Sokolowski [1968] and Adachi et al. [1998] found \( l_c/d = 0.289 \) for couple stress theory \((m \to \infty)\). Warren and Byskov [1997] and Chen et al. [1998] found \( m=1, \ l_c/d = 0.289 \) and \( m=1, \ l_c/d = 0.577 \), respectively. For regular hexagonal structures, Warren and Byskov [2002], Pradel and Sab [1998] and Wang and Stronge [1999] found \( m=0.5 \) and \( l_c/d = 0.22 \). By comparing the continuum solutions based on these parameters with the discrete results and the best couple stress fits (Figs. 3.2a-c), it follows that the ‘homogenization parameters’ have a strong tendency to underestimate the discrete stiffening effect. It should be noted, however, that the scatter in the small \( H/d \) regime is too large to make any conclusive statements.

The system parameters that determine the overall elastic response to a shear deformation in couple stress theory are \( H, \ E_p, \ \nu_p \) and \( D_{1313} \). The overall stiffness \( S (=F/(Ly)) \) can be written in terms of the system parameters as

\[
S \left( H, E_p, \nu_p, D_{1313} \right) = G g \left( \frac{H}{l_c} \right), \tag{3.50}
\]

with \( 2G = E_p/(1+\nu_p) \), \( 2l_c^2 = D_{1313}/G \), and the function \( g \) being dependent on \( H/l_c \) only. As discussed in section 2.3.1 a similar expression for the average discrete results was obtained in terms of the microstructural parameters:

\[
S \left( t, d, H, E_s \right) = G f \left( \frac{H}{d} \right), \tag{3.51}
\]

with

\[
G = cE_s \left( \frac{t}{d} \right)^3, \tag{3.52}
\]

and \( c \) a dimensionless constant. By taking \( G \) to be the classical shear stiffness of the specific cellular microstructure under consideration in both Equation (3.50) and (3.51), the function \( g \) can be fitted to the discrete function \( f \), which was found to yield \( l_c = \alpha d \), with \( \alpha \) a dimensionless constant between 0.28 and 0.9, depending on cellular morphology. By using the definition of \( l_c \), the new micropolar constant \( D_{1313} \) can be written in terms of the microstructural parameters as

\[
D_{1313} = 2cE_s \left( \frac{t}{d} \right)^3 (\alpha d)^2, \tag{3.53}
\]
with $\alpha$ and $c$ dimensionless constants specific for the cellular microstructure under consideration.

For pure bending the discrete results show softening while micropolar theory predicts stiffening for non-zero values of the characteristic length (independent from the value of the coupling factor $m$). Clearly, free edge effects leading to softening cannot be captured by the micropolar theory, giving the opposite effect instead.