Chapter 1

Skeletal Curves and Scope of Thesis

This chapter briefly introduces (with references to historic origins) skeletal curves, which are either medial axes or linear skeletons. (Detailed definitions follow in the next chapter.) The chapter informs then about motivation, intentions, and structure of this thesis.

1.1 Skeletal Curves

The linear skeleton has been introduced into topology by (Listing 1861), defined by continuous contractions of a set into a one-dimensional (1D) subset, but not by intentions of having “centered lines”. On the opposite, the medial axis is defined by centers of maximal disks contained in a set; it was informally specified by (Blum 1962).

1. Definition. A skeletal curve in the continuous space is either a (not uniquely defined) linear skeleton or a (uniquely defined) medial axis, aiming at topologic or geometric studies.

Curves in general have been defined by P. Urysohn and K. Menger in the 1920s and 1930s [see (Urysohn 1923) and (Menger 1932)], also introducing branch or end points of curves. A curve finally defines, at a more abstract level, an undirected graph, whose nodes are identified with branch or end points, and edges symbolize existing arcs. A curve is a 1D (or linear) set, and a skeletal curve is an abstract disambiguation of a (biomedical) skeleton (see Figure 1.1).

Both concepts of skeletal curves have been “digitized”. Digital medial axes go back on (Rosenfeld and Pfaltz 1966), who defined them based on distance transforms. For a given connected set, they are not necessarily connected. Digital linear skeletons are defined by topology-preserving thinning [see (Rosenfeld 1970)], which is a repeated removal of “deletable elements”; today those are called simple pixels for two-dimensional (2D) pictures, or simple voxels for three-dimensional (3D) pictures.

The calculation of skeletal curves proved to be a very useful preprocessing step in image analysis, and it is still an active field of research – now already for nearly
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A skeleton in its biomedical meaning is not 1D or linear; it is composed of 3D volume elements (copy of a painting by Kurt Grosjean from www.deviantart.com).

40 years! A large number of papers has been published on digital medial axes (distance transforms) or digital linear skeletons (thinning, or simple pixels or voxels), see (Klette and Rosenfeld 2004).

Algorithms for calculating skeletal curves are typically also based on heuristic arguments. For example, when calculating a linear skeleton, a “modified thinning” should stop such that a resulting linear “stick figure represents all body parts” (see Figure 1.2), and will not continue with removing “arms or legs”. Thinning is the operation which implements topologic contraction.

Figure 1.1: A skeleton in its biomedical meaning is not 1D or linear; it is composed of 3D volume elements (copy of a painting by Kurt Grosjean from www.deviantart.com).

Figure 1.2: The upper row is contracted into the middle row, and then further into the bottom row. All three “stick figures” on the left will finally each contract into a single pixel, defining a linear skeleton. Both stick figures on the right contract into more complex linear skeletons.
1.2 Motivation and Scope of Thesis

Modern imaging techniques produce digital images of high resolution, and support for a (partially) automated exploitation of those huge amounts of data requires ongoing research for improvements of applied algorithms. Shape analysis is one of the important issues. In 2D image analysis, the extraction of features from skeletons is used in the context of pattern recognition, for example for character recognition, finger print recognition, prenatal diagnosis, or biological cell studies. Skeletonization methods in two dimensions and their applications have been extensively studied, and ongoing work still appears frequently in image processing literature. Approaches in three or higher dimensions are an order of magnitude more difficult to formalize or to describe, and implementations are becoming very complex. Skeletonization results can be curves (i.e., linear skeletons), but also (say, as an intermediate result, or for a “better match” with given shapes) “thin” or “elongated” volume parts.

Figure 1.3: Left: two digital squares representing maximal disks in the Minkowski metric $L_1$, define a disconnected medial axis (both black nodes). Right: all black nodes are centers of maximal disks (two disks are shown), defined by the same metric.

Applied heuristics for the calculation of medial axes aim on preserving connectedness (on the expense of allowing redundancy), and on ensuring results which are 1D curves (i.e., which do not contain parallel digital line segments; see Figure 1.3).

Algorithms for calculating skeletal curves can be characterized as being ill-posed: Minor differences between algorithms or input data can deliver totally different results, also with respect to run-time complexity, or properties derived from skeletal curves.

Skeletal curves are very useful tools for understanding the shape or structure of 2D or 3D objects. The thesis discusses briefly the generation of such skeletal curves, and focuses then on their analysis by “digitizing” concepts of the theory of curves in Euclidean spaces (as established by Menger and Urysohn).
Figure 1.4: A volume scan composed of 42 slices of 256 × 256 density images. This 3D view was produced using commercial software.

This thesis was initiated in 2002 by one particular research project at the Medical School of The University of Auckland (but developed then on its own). Figure 1.4

Figure 1.5: Isosurface generated by a marching cubes algorithm.
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Figure 1.6: Left: Normal control human hippocampus CA1 area, right: MTLE-damaged human hippocampus CA1 area

shows a 3D (volume) sample of human brain tissue. The volume is composed of slices, being confocal microscope images, and it shows astrocytes (star-shaped cells of the nervous system which provide nutrients, support and insulation for neurons). Medical experts developed the hypothesis that features (such as the number, “distribution” or “structural complexity”) of astrocytes in such a volume scan are sufficient (or, at least essential) for defining states between normal or abnormal tissue.

Figure 1.5 shows the isosurface of segmented astrocytes as calculated (by one co-supervised student) at an early stage of this project. Figures 1.4 and 1.5 provide a good insight into the topologic and geometric complexity of these volume scans. Astrocytes enclose blood vessels; Figure 1.4 contains a Y-shaped blood vessel (from lower left corner to upper right), and Figure 1.5 shows a diagonally intersecting, slightly curved blood vessel.

Medical experts are able to evaluate the state of neurological diseases of patients just by looking at such images. However, these evaluations are based on experience, and they are influenced by subjective judgments. It proved to be difficult to describe explicitly those observed shapes or structures that differentiate normal tissue from tissue of people with neurological diseases. Figure 1.6 shows normal control tissue on the left and (mesial temporal lobe epilepsy) MTLE-damaged tissue on the right. The structure in the picture on the right can be described as “more dense” or “more complex” compared to the picture on the left. We need to define those descriptions.

Image analysis appears as a suitable way to deliver objective features, which might be useful for those evaluations.

This PhD project was initiated by the understanding that topologic, geometric,
or graph-theoretic properties for such volume scans are needed for further progress in those studies. The medical project itself still requires further preparation (e.g., each volume scan as illustrated by Figures 1.4 and 1.5 requires weeks of preparation and expensive scanning, and obtained volume scans are often still unsatisfactory for subsequent automated analysis).

This thesis is about the mathematics and algorithmic challenges related to the calculation of properties of skeletal curves. It discusses problems related to skeletal curves on a much more general level than just in the context of one particular application.

Curve-like structures appear frequently in 3D biomedical image analysis (e.g., analysis of blood vessels, of neurons, or in ultrasound medical imaging), material sciences (e.g., analysis of porous media where cavities define curve-like structures to be studied), or different disciplines in physics (e.g., bubble chamber data). Studies of moving 3D objects (e.g., of beating hearts) also lead to models defined in four-dimensional (4D) spaces. Besides our main interest in 2D or 3D applications, the thesis often formulates in a general $n$-dimensional ($n$D) way, with $n \geq 2$. At least, this unifies 2D and 3D.

The thesis is restricted to studies of linear skeletons, and we do not consider skeletons whose components are also surfaces or volumes.

Furthermore, the thesis prefers the term picture rather than image, because considered data in multidimensional grids may have resulted from other processes than just imaging (e.g., measurements or calculations). The common representation of considered pictures $P$ are $n$D arrays, whose elements $p, q, r, \ldots$ are called pixels for $n = 2$, voxels for $n = 3$, and (in general) picture elements for $n \geq 2$.

We only consider pictures defined on a regular orthogonal grid (and not, for example, a triangular or hexagonal grid for $n = 2$). Picture elements are labeled, and labels are denoted by $P(p), P(q), P(r), \ldots$. We assume only scalar discrete labels in the sense of measured density. In particular, we use (after picture segmentation) binary pictures with $P(p) \in \{0, 1\}$. $P(p) = 1$ indicates an object element $p$, and $P(p) = 0$ a non-object element $p$. (More basic definitions follow in Chapter 2.)

In this thesis we assume that binary pictures define the input. We do not discuss picture segmentation algorithms for the creation of binary pictures, and refer to related text books such as (Haralick and Shapiro 1992, Sonka et al. 1999).

### 1.3 Outline of this Thesis

Chapter 2 reviews basic concepts and definitions which are then used throughout the thesis. Chapter 3 reports about mathematical models for linear skeletons which
have been proposed for continuous spaces. This concludes Part 1 of the thesis, which is on fundamentals.

Two classes of algorithms are considered for the generation of curve-like structures. Distance transforms are fundamental tools for generating distance maps. Those allow to calculate digital medial axes or distance skeletons, and are suitable for elongated objects in input pictures $P$. The choice of the metric influences the result and the computational complexity of the algorithms. The review of existing algorithms for distance transforms in Chapter 4 creates a grouping. Each group of algorithms represents a different strategy for the computation of distance maps. Algorithms of the same group differ by applied metrics or by applied picture models\(^1\). The Chapter provides a short but concise description of a linear-time Euclidean distance transform which calculates exact Euclidean distance maps. A new proposition provides a justification for the selection of elements of the medial axis.

The second class of algorithms for the generation of skeletal curves are iterative thinning processes, which are characterized by removals of simple elements. The removal of a simple element does not change the topology of a given picture. Iterative removals continue until the result is 1D (i.e., a set of digital arcs or curves). The application of different forms of characterizing simple elements can change the time complexity of a thinning algorithm. We prove several new theorems which emphasize that a number of existing characterizations of simple elements are actually equivalent. We show that the new characterization of non-simple elements is of benefit. The literature offers a large number of thinning algorithms. Chapter 5 explains theoretical fundamentals of such approaches. It reviews four algorithms which have been originally developed for 2D pictures and it describes the resulting curves. Each of those algorithms represents a prototype which has been further developed for 2D pictures and for higher dimensional pictures in many publications. We combine extended versions of two prototypes with the new characterization of non-simple elements. This concludes Part 2 of the thesis, which is on both basic strategies (i.e., either geometric distance transforms or topologic thinning).

Chapter 6 classifies all elements (i.e., pixels or voxels in 2D or 3D) of skeletal curves based on branching indices. It reviews the definitions of a branching index and a branching point for the continuous space, and it introduces analogous concepts in digital space. One option is based on the use of the adjacency set (or of the smallest non-trivial neighborhood) of a curve element. A second option uses topologic properties of components which are generated in a larger (global) neighborhood of a curve element. This chapter provides a classification of elements in skeletal curves which is used for a new method to generate abstract curve graphs for subsequent

\(^1\)Some authors use non-square or non-cubic grid models such as for example hexagonal grids or elongated voxel grids (Sintorn and Borgefors 2004).
property calculation.

The analysis of digital curves or arcs is a subject of research since (Freeman 1961). Chapter 7 proposes and discusses a diversity of properties for describing skeletal curves. It also contains examples illustrating the use of the proposed properties for volume scans of astrocytes. It proposes features of connected components by computing properties of the approximated skeleton adapted from graph theory. This concludes Part 3 of the thesis, which is on curve segmentation at branch elements and subsequent property measurements.

Chapter 8 provides our conclusions and offers ideas for further developments.

1.4 Summary

The chapter gave a brief introduction into three alternative points of view (i.e., topologic, geometric, or graph-theoretic) when defining or studying digital approximations of skeletal curves. Major statements are as follows: the thesis

- is about methodological aspects rather than on an application;
- it recalls and groups existing algorithms for the approximative and exact computation of the Euclidean distance transform;
- it provides a new algorithm for thinning which is based on characterizing non-simple elements;
- the objects of interest are digital approximations of skeletal curves in (digital) \( n \)-D pictures \( n \geq 2 \), with a focus on \( n = 2 \) and \( n = 3 \);
- a digital concept of branch nodes is introduced for resulting digital curves;
- the generation of abstract curve graphs is explained to adapt properties known from graph theory;
- the thesis discusses concepts to analyze digital approximations of skeletal curves with the special aim to support various ways of quantitative evaluations;
- it informs about conclusions derived from theoretical studies and experiments.