Chapter 3

Polarization theory

In this chapter the theory for some of the processes introduced in chapter 2 is presented in detail. The following aspects are covered: First, the scattering asymmetry coefficients for Compton, Möller and Bhabha scattering and annihilation in flight. Second, the polarization transfer in Compton scattering as well as in bremsstrahlung and pair production. Third, the depolarization of a photon, electron or positron due to Compton, Möller and Bhabha scattering as well as bremsstrahlung.

By using Stokes parameters the theory is put into a form which makes it possible to follow the polarization development of an electromagnetic shower. The parameters are introduced using classical electromagnetism as an example. The results of this chapter are implemented in the GEANT code in chapter 4.

3.1 The Stokes parameters

The most general monochromatic homogeneous plane-wave for the electric field vector \( \vec{E} \) satisfying Maxwell’s equations in a source free, uniform and infinite medium is given by [80]

\[
\vec{E}(\vec{x}, t) = (a_1 \vec{E}_1 + a_2 \vec{E}_2)e^{i(\vec{k} \cdot \vec{x} - \omega t)},
\]

(3.1)

where \( \vec{k} \) is the wave vector pointing in the direction of propagation. Its magnitude \( k \) is related to the speed \( v \) of the wave in the medium and to the frequency \( \omega \) of the oscillation by \( k = \omega / v \).
The real or complex unit vectors $\vec{E}_1$ and $\vec{E}_2$ are perpendicular to $\vec{k}$ (the wave is transverse as required by Maxwell’s equations) and form the 'polarization basis'. For real vectors the basis is formed by two linear or plane polarization directions which are chosen to be orthogonal to each other for convenience (linear basis). For complex vectors the basis is elliptical. Usually the left and right circular polarization vectors are chosen in this case.

The polarization of the wave depends on the magnitude and relative phase of the complex coefficients $a_1$ and $a_2$ and the choice of basis. If, for example, $a_1$ and $a_2$ are in phase and the basis is linear then the polarization is linear. For a phase difference of 90°, the polarization is circular, and for all other cases the polarization is elliptical.

The four parameters introduced by Stokes in 1852 are quadratic in $a_1$ and $a_2$ and can be used to determine the polarization state of an electromagnetic wave by means of intensity measurements only [14, 80]. With the help of a linear polarizer and a quarter-wave plate (or an equivalent of these), the transverse and circular components of the electric field can be disentangled. By introducing an orthogonal right handed coordinate system in which the wave is travelling in the positive z-direction, the Stokes parameters can be defined as

$I_0$, the intensity of the wave.

$P_1$, the degree of linear polarization with respect to the x- and y-axis. If the intensities are $I_x$ and $I_y$, $P_1 = (I_x - I_y)/I_0$.

$P_2$, the degree of linear polarization with respect to the axes oriented at 45° to the right of the previous x- and y-axis. The intensities are $I_{45}$ and $I_{135}$, $P_2 = (I_{45} - I_{135})/I_0$.

$P_3$, the degree of circular polarization. If the intensities for left and right circular polarization are $I_l$ and $I_r$, respectively, $P_3 = (I_l - I_r)/I_0$.

Expressed in terms of $a_1$ and $a_2$ and taking the linear basis \{\vec{E}_1, \vec{E}_2\} the Stokes parameters become

\[
\begin{align*}
I_0 &= a_1^* a_1 + a_2^* a_2, \\
P_1 &= a_1^* a_1 - a_2^* a_2 / I_0, \\
P_2 &= a_1^* a_2 + a_2^* a_1 / I_0, \\
P_3 &= -i(a_1^* a_2 - a_2^* a_1) / I_0.
\end{align*}
\] (3.2)
In a compact way this is written as the Stokes vector \((I, \vec{P})\). The Stokes parameters are real and satisfy the relation: \(P_1^2 + P_2^2 + P_3^2 = 1\). It should be kept in mind that their actual value depends on the chosen polarization basis. Originally, they were introduced in a classical background. The next step will be the introduction of the Stokes operators for the electromagnetic field.

### 3.2 Polarization of photons

The Maxwell equations can be written using a four vector \(A^\mu = (\phi, \vec{A})\), where \(\phi\) is a scalar potential and \(\vec{A}\) a vector potential defined up to a gauge transformation [81, 82]. Since the electromagnetic field has only two independent components, two of the four components of \(A^\mu\) are superfluous in describing the field. The two redundant components can be eliminated by making use of the gauge freedom. In the so called radiation or Coulomb gauge this is done by choosing \(\phi = 0\) and \(\text{div} \vec{A} = 0\).

The quantization of the field in this gauge is straightforward [82] and leads to a vector field in terms of creation, \(a^{(\lambda)\dagger}(k)\), and annihilation, \(a^{(\lambda)}(k)\), operators

\[
\vec{A}(x) = \int \frac{d^3 k}{(2\pi)^3 2k_0} \sum_{\lambda=1}^{2} \vec{A}^{(\lambda)}(k) [a^{(\lambda)}(k)e^{-ikx} + a^{(\lambda)\dagger}(k)e^{ikx}],
\]

where \(k\) is the four momentum satisfying \(k^2 = 0\) or \(k_0 = |\vec{k}|\). The gauge condition \(\text{div} \vec{A} = 0\) leads to \(\vec{k} \cdot \vec{A}^{(\lambda)}(k) = 0\). The two polarization vectors \(\vec{A}^{(\lambda)}(k)\) are perpendicular to \(\vec{k}\) and will be chosen to be orthogonal to each other for convenience.

A precise understanding of equation 3.3 is not necessary in the present context. However, from the equation it can be seen that for a certain momentum \(k\) two independent orthogonal photon states can be created from the vacuum \(|0\rangle\). An arbitrary photon state with momentum \(k\) is a linear combination of these two independent states. The polarized states \(\phi_1 = a^{(1)\dagger}(k)|0\rangle\) and \(\phi_2 = a^{(2)\dagger}(k)|0\rangle\) form a so called complete set in terms of which a general state can always be written as

\[
\phi = a_1 \phi_1 + a_2 \phi_2,
\]

where \(|a_1|^2\) and \(|a_2|^2\) are the relative probabilities of finding the photon in either one of the basis states (the states \(\phi_1\) and \(\phi_2\) are added coherently). A
single photon is always in a pure polarization state. Its Stokes parameters are given by \(3.2\).

The vector \(\vec{P}\) can be considered as the polarization vector for the state. By choosing a coordinate system in the laboratory, the \(x, y\) and \(z\) components of \(\vec{P}\) do not have a physical interpretation yet. Because the photon is massless its helicity is always \(\pm 1\) \[83\]. As a consequence the vector field \(\vec{A}\) must always be perpendicular to the direction of motion of the photon. The physical interpretation of \(\vec{P}\) follows in section \(3.4\).

### 3.3 Polarization of electrons

The quantum mechanical wave function for an electron can be found by solving the Dirac equation \[5, 81\]. The solutions are four-component wave functions; two independent solutions for an electron and two for a positron. If the solutions for the electron are \(\phi_1\) and \(\phi_2\) the general wave function \(\phi\) for an electron can again be written as \(3.4\). In the rest frame of the electron the four-component solutions of the Dirac equation reduce to the two-component spinors. In this frame the polarization of an electron is defined as the expectation value of the Pauli spin operators

\[
\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

Thus the polarization vector \(\vec{S}\) in the rest frame of the electron is given by

\[
S_x = \langle \phi | \sigma_x | \phi \rangle = (a_1^* a_2^*) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = a_1^* a_2 + a_2^* a_1,
\]

\[
S_y = \langle \phi | \sigma_y | \phi \rangle = (a_1^* a_2^*) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = -i(a_1^* a_2 - a_2^* a_1),
\]

\[
S_z = \langle \phi | \sigma_z | \phi \rangle = (a_1^* a_2^*) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = a_1^* a_1 - a_2^* a_2.
\]

Comparing this with the definition of the Stokes parameters it follows that \(P_1 = S_z\), \(P_2 = S_x\) and \(P_3 = S_y\). If \(I\) is taken as

\[
I = \langle \phi | I | \phi \rangle = (a_1^* a_2^*) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = a_1^* a_1 + a_2^* a_2.
\]

(3.7)
the Stokes parameters can be used to describe the polarization of an electron just as they are used to describe the polarization of a photon. However, for the electron the meaning of the \( x, y \) and \( z \) components of \( \vec{P} \) is fixed by the choice of a basis in the laboratory frame because the polarization direction of an electron does not necessarily have a connection with the direction of motion of the electron.

### 3.4 Mixed states

Until now only pure states were considered, i.e. states for which there exists a definite direction in which the polarization is unity. To describe a partially polarized beam of electrons or photons the density matrix formalism is used (see for example [84]). In this formalism a beam of electrons or photons is represented by a \( 2 \times 2 \) Hermitian matrix \( \rho \) with positive or zero eigenvalue and trace 1 (the state is normalized). For the pure state 3.4 the matrix is

\[
\rho = \begin{pmatrix} a_1^* a_1 & a_2^* a_1 \\ a_1^* a_2 & a_2^* a_2 \end{pmatrix},
\]

which can be brought into the form

\[
\rho = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},
\]

by a unitary transformation. A mixed state is obtained by an incoherent superposition of pure states. In this case \( \rho \) can be diagonalized but in general both diagonal matrix elements are nonzero. The resulting density matrix is an incoherent superposition of an unpolarized and a polarized state

\[
\rho = \begin{pmatrix} \rho_a & 0 \\ 0 & \rho_b \end{pmatrix} = (1 - p) \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} + p \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},
\]

where \( p = \rho_a - \rho_b \) is the degree of polarization. This is to be compared with a pure state which is a coherent superposition of \( \phi_1 \) and \( \phi_2 \).

Another way of writing 3.8 [84] shows the connection with the polarization vector \( \vec{S} \)

\[
\rho = \frac{1}{2} (I + \vec{S} \cdot \vec{\sigma}),
\]

where \( \vec{\sigma} \) is the vector of polarization operators.
where \( I \) is given by 3.7, \( \vec{S} \) by 3.6 and \( \vec{\sigma} \) by 3.5. For a pure state \( |\vec{S}| = 1 \), while for a mixed state \( |\vec{S}| = p < 1 \), where \( p \) is the degree of polarization used in 3.10.

Likewise a beam can be described by a density matrix \( \rho \), a polarization sensitive detector can be characterized by a density matrix \( \rho_{\text{det}} \). The probability for the detector responding when placed in a beam is given by

\[
W = \text{Tr}(\rho_{\text{det}}),
\]

or, using the Stokes vectors

\[
W = \frac{1}{2}(1, \vec{S}_{\text{det}}) \left( \begin{array}{c} I \\ \vec{S} \end{array} \right) = \frac{1}{2}(1, \vec{P}_{\text{det}}) \left( \begin{array}{c} I \\ \vec{P} \end{array} \right). \tag{3.13}
\]

As mentioned, the values of the Stokes parameters depend on the polarization basis one has chosen. A change of basis is accompanied by a transformation of the parameters. If both the initial and final basis are orthogonal the transformation of the Stokes parameters is described by a rotation matrix \( M \). For example, a rotation over an angle \( \theta \) about the \( P_3 \)-axis is given by (see the appendix of [15])

\[
M = \left( \begin{array}{cccc} 1 & 0 & 0 & 0 \\ \cos \omega & \sin \omega & 0 & 0 \\ -\sin \omega & \cos \omega & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right), \tag{3.14}
\]

where \( \omega = \theta \) for electrons and \( \omega = 2\theta \) for photons. For electrons the \( z \)-axis will be chosen as the direction of propagation. A rotation \( \theta \) around this \( P_1 \) axis is given by

\[
M = \left( \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos \theta & \sin \theta \\ 0 & 0 & -\sin \theta & \cos \theta \end{array} \right). \tag{3.15}
\]

Using the Stokes parameters, the effect of an electromagnetic interaction can be written as a \( 4 \times 4 \) matrix, \( T \), working on the Stokes vector 3.2. The resulting 4-vector gives the Stokes parameters of the final state:

\[
\left( \begin{array}{c} I' \\ \vec{P}' \end{array} \right) = T \left( \begin{array}{c} I \\ \vec{P} \end{array} \right). \tag{3.16}
\]
Table 3.1: Interpretation of components of the Stokes vector if the polarization basis for the photon is chosen as explained in the text.

<table>
<thead>
<tr>
<th>Stokes parameter</th>
<th>Photon</th>
<th>Electron</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I$</td>
<td>Intensity</td>
<td>Intensity</td>
</tr>
<tr>
<td>$P_1$</td>
<td>Plane polarization in the direction of $\vec{A}_1$ (+1) and $\vec{A}_2$ (-1)</td>
<td>Spin in the $z$-direction ($S_3$)</td>
</tr>
<tr>
<td>$P_2$</td>
<td>Plane polarization in the basis 45° to the right of $\vec{A}_1$</td>
<td>Spin in the $x$-direction ($S_1$)</td>
</tr>
<tr>
<td>$P_3$</td>
<td>Left (+1) and right (-1) circular polarization</td>
<td>Spin in the $y$-direction ($S_2$)</td>
</tr>
</tbody>
</table>

The probability of finding the beam in a state $(1, \vec{D})$ after the interaction is given by 3.13

$$ W = \frac{1}{2} (1 \vec{D}) T \begin{pmatrix} I \\ \vec{p} \end{pmatrix}. $$

(3.17)

In the following sections reduced matrices will be presented. This means that the cross section for an unpolarized beam detected by a detector that is not sensitive to polarization is normalized to unity i.e., the upper left matrix element of $T$ is equal to one.

An interaction makes it possible to define the physical meaning of the Stokes parameters for the photon. The components $\vec{A}_1$ and $\vec{A}_2$ of the vector field $\vec{A}$ are chosen perpendicular to and in the reaction plane, respectively. They form, together with the unit vector $\vec{n}$ in the direction of motion of the photon, a right handed coordinate system $(\vec{A}_1/|\vec{A}_1|, \vec{A}_2/|\vec{A}_2|, \vec{n})$. The directions of $\vec{A}_1$ and $\vec{A}_2$ are as required by the Maxwell equations. Relative to this basis the Stokes parameters for the photon can be interpreted as given in table 3.1. In the laboratory frame this basis will have a different orientation for every interaction. If the photon direction of motion is changed due to the interaction the basis for the outgoing and incoming photon will also have a different orientation as seen from the laboratory frame.

Summarizing: the parameters first introduced by Stokes describe the polarization of a beam of electrons and photons in a unified way and make it possible to incorporate interactions by means of matrices.
3.5 Compton scattering

3.5.1 Scattering asymmetry

The cross section for Compton scattering of a polarized photon (Stokes vector $\vec{P}$) by a polarized electron (polarization vector $\vec{S}$) is proportional to \[ 1, 45, 46 \]

\[
\frac{d\sigma}{d\Omega} \propto (1 + \cos^2 \theta) + (k_0 - k)(1 - \cos \theta) + P_1 \sin^2 \theta - P_2(1 - \cos \theta) \vec{S} \cdot (\vec{k}_0 \cos \theta + \vec{k}),
\]

where $\vec{k}_0 = k_0 \vec{n}_0$ and $\vec{k} = k \vec{n}$ are the momentum vectors of the incoming and outgoing photon of momentum $k_0$ and $k$ (in units of $mc$) in the directions given by the unit vectors $\vec{n}_0$ and $\vec{n}$, respectively, and where $\theta$ is the scattering angle between the incoming and outgoing photon in the laboratory system. The relation between $k_0$, $k$ and $\theta$ is $k = k_0/(1 + k_0(1 - \cos \theta))$. The polarization insensitive part of 3.18 is the Klein-Nishina cross section.

The Compton scattering asymmetry ratio between photons that are linearly polarized in the direction $\vec{A}_1 (P_1 = 1)$ to those polarized in the direction $\vec{A}_2 (P_1 = -1)$ is

\[
R_{\text{lin}} = \frac{k_0^2 + k^2}{k_0^2 + k^2 - 2k_0k \sin^2 \theta}.
\]  

Figure 3.1a shows $R_{\text{lin}}$ as a function of the scattering angle of the photon for various energies. The degree of linear polarization of the photons is measured by changing the position of the photon detector in such a way that the reaction plane turns over an angle of $90^\circ$.

The asymmetry ratio for left-circular ($P_3 = 1$) to right-circular ($P_3 = -1$) polarized photons scattered from electrons polarized in the direction of the momentum of the incoming photons ($S_3 = 1$), is given by

\[
R_{\text{lr}} = \frac{(1 + \cos^2 \theta) + (1 - \cos \theta)[(k_0 - k)(1 - (k_0 + k) \cos \theta)]}{(1 + \cos^2 \theta) + (1 - \cos \theta)[(k_0 - k)(1 + (k_0 + k) \cos \theta)].}
\]  

Figure 3.1b shows this ratio as a function of the scattering angle of the photon. The ratio of the cross section for left circular ($P_3 = 1$) polarized photons to the cross section for unpolarized ($P_3 = 0$) photons scattered off electrons polarized in the direction of the momentum ($S_3 = 1$) of the incoming photons is shown in 3.1c. In a polarimeter the circular polarization of the photons is measured by reversing the magnetization of the (iron) target ($\vec{S}$ in 3.18; see chapter 2).
Figure 3.1: a) Ratio of the intensities of Compton-scattered photons polarized perpendicular to the scattering plane and those polarized in the scattering plane. b) Ratio of left and right circular polarized photons Compton scattered from electrons polarized in the direction of the momentum of the incoming photons. c) Ratio of left polarized and unpolarized photons Compton scattered from electrons polarized in the direction of the momentum of the incoming photons. The curves are labeled by the photon energy in MeV.

3.5.2 Depolarization

The reduced interaction matrix describing the polarization state of the photons after Compton scattering with unpolarized electrons is given by [15, 46, 85]

\[ T_{CS\gamma} = \begin{pmatrix} 1 & A & 0 & 0 \\ A & B & 0 & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & D \end{pmatrix}, \]  

(3.21)

where

\[ A = \sin^2 \theta / I, \]
\[ B = (1 + \cos^2 \theta) / I, \]
\[ C = 2 \cos \theta / I, \]
\[ D = (2 \cos \theta + (k_0 - k)(1 - \cos \theta) \cos \theta) / I, \]
\[ I = (1 + \cos^2 \theta) + (k_0 - k)(1 - \cos \theta). \]

This matrix describes the depolarization of photons due to Compton scattering with unpolarized electrons. The polarization of the photon before and
after the scattering is given relative to reference frames as defined at the end of section 3.4. The matrix for Compton scattering with polarized electrons can be found in the literature [15, 46, 85]. It will not be used in this thesis.

3.5.3 Polarization transfer

The matrix describing the polarization transfer to the scattered electron can be deduced from [46] as

$$ T_{Cse} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & E \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & F \end{pmatrix}, $$

where

$$ E = ((\cos \theta - 1)((k_0 + k) \cos \theta + G(k \cos \theta - k_0))) / I, $$
$$ F = ((\cos \theta - 1)(1 + G)k \sin \theta) / I, $$
$$ G = (1 + \cos \theta)(k_0 + k) / (k_0 - k + 2). $$

The matrix gives the polarization of the electron in the reference frame of the outgoing photon. The electron polarization is rotated from this reference frame to its own reference frame with the matrices 3.14 and 3.15.

A discussion on the use of Compton scattering to create beams of polarized electrons is presented by Tolhoek and Lipps [1, 45, 46].

3.6 Møller scattering

3.6.1 Scattering asymmetry

The cross section for polarized electrons scattered by polarized (target) electrons is

$$ \frac{d\sigma}{d\Omega} = \frac{d\sigma_0}{d\Omega}(1 + \sum_{i,j=x,y,z} m_{ij}S_i^j S_j^i), $$

---

1. This result is in agreement with the formulas found in the extensions made to the EGS4-code by Flöttmann [16].
where $S'_i$ ($S'_j$) are the components of the beam (target) polarization vector in the rest frame of the electrons. $d\sigma_o/d\Omega$ is the cross section for unpolarized electron electron scattering and $m_{ij}$ are the asymmetry coefficients. The following relations hold for an electron moving along the z-axis and the xz-plane being the scattering plane [48, 49, 86]

\[
\frac{d\sigma_0}{d\Omega} = \frac{r_0^2}{4\gamma^2} \frac{m_0}{(\gamma^2 - 1)^2 \sin^4 \theta},
\]

\[
m_0 = \sin^2 \theta (\gamma^2 - 1)^2 (4 + \sin^2 \theta) + (2\gamma^2 - 1)^2 (4 - 3 \sin^2 \theta),
\]

\[
m_{xx} = -\sin^2 \theta \{ \sin^2 \theta (\gamma^4 - 1) + (2\gamma^2 - 1) \}/m_0,
\]

\[
m_{yy} = \sin^2 \theta \{ \sin^2 \theta (\gamma^2 - 1)^2 - (4\gamma^2 - 3) \}/m_0,
\]

\[
m_{xz} = m_{zx} = \sin^2 \theta (\gamma^2 - 1) \sin 2\theta / m_0,
\]

\[
m_{xy} = m_{yx} = m_{yz} = m_{zy} = 0.
\]

In these formulas $\theta$ and $\gamma$ are the scattering angle and the total energy in units of $mc^2$ of the electron, respectively, both in the center-of-mass frame. In a fixed laboratory frame the scattering intensities are dependent on the polar angle $\phi$ around the beam axis. This angle does not show up in the formulas because the coefficients are defined relative to the scattering plane.

Figure 3.2 shows the asymmetry coefficients as a function of the laboratory energy and scattering angle. Numerical values for the asymmetry coefficients $m_{ij}$ were given by Holzwarth [86].

### 3.6.2 Depolarization

To calculate the depolarization of an electron beam due to Möller scattering the Stokes interaction matrix will be used again. The incoming electron is moving along the z-axis, the xz-plane is the plane of scattering and the target
electron is unpolarized. The reduced matrix is given by\(^2\)

\[
T_M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & C & D & 0 \\ 0 & D & E & 0 \\ 0 & 0 & 0 & F \end{pmatrix},
\]

\(^2\)McMaster [15] has given this matrix but with some errors. According to his matrix it is possible to obtain polarized electrons after Møller scattering from unpolarized electrons which is not possible. The matrix given here is calculated from matrix elements presented by Stehle [87]. Stehle deduced matrix elements for all combinations of possible incoming and outgoing polarization states.
where
\[
C = 2 \cos \theta (2 \gamma^2 - 1)(2 \gamma^2 - 1 - \gamma^2 \sin^2 \theta) / I,
\]
\[
D = 2 \gamma (2 \gamma^2 - 1) \sin \theta \cos \theta / I,
\]
\[
E = 2 \cos \theta (2 \gamma^2 - 1)(2 \gamma^2 - 1 - \sin^2 \theta) / I,
\]
\[
F = 2[(2 \gamma^2 - 1)^2 - (2 \gamma^4 - 1) \sin^2 \theta] / I,
\]
\[
I = \frac{1}{2}[(2 \gamma^2 - 1)^2(4 - 3 \sin^2 \theta) + (\gamma^2 - 1)^2(\sin^4 \theta + 4 \sin^2 \theta)],
\]
and where \( \gamma \) and \( \theta \) are the electron energy in units of \( mc^2 \) and the scattering angle of the electron, respectively, both in the center-of-mass frame. The corresponding laboratory values \( \gamma' \) and \( \theta' \) are:
\[
\gamma' = 2 \gamma^2 - 1,
\]
\[
\cos \theta = \frac{2 - (\gamma' + 3) \sin^2 \theta'}{2 + (\gamma' - 1) \sin^2 \theta'}.
\] (3.26)

The relation between the solid angle \( d\Omega \) and its laboratory equivalent \( d\Omega' \) is
\[
d\Omega = \frac{8(\gamma' + 1) \cos \theta'}{[2 + (\gamma' - 1) \sin^2 \theta']^2} d\Omega'.
\] (3.27)

The factor \( \frac{1}{2} \) in the expression for \( I \) is due to the fact that the scattering and scattered electron are indistinguishable in Møller scattering [88]. In this way the highest-energy member of the scattered particles is associated with the original incoming electron. The polarization of the lowest-energy electron, interpreted as the original target electron, is not calculated.

The resulting polarization is given in the rest frame of the electron with has its z-axis parallel to the direction of motion of the electron in the center-of-mass frame. Rotation matrices can be applied to obtain the polarization in the rest frame of the electron with has its z-axis parallel to the direction of motion in the laboratory frame.

### 3.7 Bhabha scattering

#### 3.7.1 Scattering asymmetry

The cross section for electron-positron scattering can also be written in the form 3.23. Replacing \( m_0 \) and \( m_{ij} \) by \( b_0 \) and \( b_{ij} \) to avoid confusion the asym-
metry coefficients \( b_{ij} \) become [86]

\[
\frac{d\sigma_0}{d\Omega} = \frac{r_0^2}{16\gamma^4 (\gamma^2 - 1)^2 \sin^4(\theta/2)} b_0
\]

\[
b_0 = 2\gamma^2 + 7\gamma^4 - 14\gamma^6 + 9\gamma^8 + 2\gamma^2(-4 + \gamma^2 + 3\gamma^4) \cos \theta + \\
(1 + 4\gamma^2 - 5\gamma^4 - 6\gamma^6 + 6\gamma^8) \cos^2 \theta + 2(\gamma^2 - 1)^3 \cos^3 \theta + \\
(\gamma^2 - 1)^4 \cos^4 \theta,
\]

\[
b_{xx} = (\gamma^2 - 1)(\cos \theta - 1)\left[3\gamma^4 + \gamma^6 + (-4\gamma^2 + \gamma^4 + \gamma^6) \cos \theta + \\
(1 + \gamma^2 - \gamma^4 - \gamma^6) \cos^2 \theta + (-1 + \gamma^2 + \gamma^4 - \gamma^6) \cos^3 \theta \right] / b_0,
\]

\[
b_{yy} = (\gamma^2 - 1)(\cos \theta - 1)\left[2\gamma^2 + 5\gamma^4 - \gamma^6 + (-6\gamma^2 + 7\gamma^4 - \gamma^6) \cos \theta + \\
(1 - \gamma^2 + \gamma^4 - \gamma^6) \cos^2 \theta + (-1 + 3\gamma^2 - 3\gamma^4 + \gamma^6) \cos^3 \theta \right] / b_0,
\]

\[
b_{zz} = (\gamma^2 - 1)\left[3\gamma^4 - 7\gamma^6 + (-4\gamma^2 + 6\gamma^4) \cos \theta + \\
(1 + 5\gamma^2 - 10\gamma^4 + 6\gamma^6) \cos^2 \theta + \\
(-2 + 2\gamma^4) \cos^3 \theta + (1 - \gamma^2 - \gamma^4 + \gamma^6) \cos^4 \theta \right] / b_0,
\]

\[
b_{xz} = b_{zx} = \frac{2\gamma(\gamma^2 - 1)\sin \theta(\cos \theta - 1)\left[\gamma^2 + (-1 + \gamma^4) \cos \theta + \\
+ (1 - 2\gamma^2 + \gamma^4) \cos^2 \theta \right] / b_0},
\]

\[
b_{xy} = b_{yx} = b_{yz} = b_{zy} = 0.
\]

where \( \gamma \) and \( \theta \) are again center of mass variables. Numerical values for the asymmetry coefficients for laboratory energies and scattering angles are shown in figure 3.3.

### 3.7.2 Depolarization

The matrices for the depolarization due to Bhabha scattering are calculated using the matrix elements given by Stehle [87]. For an unpolarized target electron the reduced matrix for the outgoing positron is

\[
T_B = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & D & E & 0 \\
0 & E & F & 0 \\
0 & 0 & 0 & C
\end{pmatrix},
\]

\[I = B2^2 + (B1^2 + B4^2 + B6^2 + B7^2) / 4,
\]
Figure 3.3: Asymmetry coefficients for Bhabha scattering as a function of the laboratory kinetic energy and scattering angle of the positron.

\[
C = [(B1B7 + B6B4)/2 + B2^2]/I,
D = (B1^2 - B4^2 - B6^2 + B7^2)/4I,
E = B2(B1 + B4 + B6 + B7)/2I,
F = (B6B4 - B1B7)/2I,
\]

with [87]:

\[
B1 = -\cot^2(\theta/2)[1 + 2\beta^2\gamma^2 \cos^2(\theta/2)],
B2 = \gamma \cot(\theta/2)[1 - 2\beta^2 \sin(\theta/2)],
\]
\[ B4 = [-1 + 2\beta^2 \gamma^2 \sin^2(\theta/2)], \]
\[ B6 = -(1 + 2\beta^2 \gamma^2 \cos^2(\theta/2))/\gamma^2, \]
\[ B7 = -\beta^2(2\gamma^2 + 1) - \cot^2(\theta/2)[(2\gamma^2 - 1) + 2\beta^2 \sin^2(\theta/2)], \]

and \( \beta = (\gamma^2 - 1)/\gamma^2 \). The variables \( \gamma \) and \( \theta \) are given in the center-of-mass frame. A series of rotations has to be applied to obtain the polarization of the positron in the rest frame with the \( z \)-axis parallel to the direction of motion in the laboratory frame.

### 3.8 Annihilation in flight

#### 3.8.1 Scattering asymmetry

The cross section for annihilation in flight can be written in the form 3.23 similar as has been done for Bhabha and Møller scattering. The asymmetry coefficients will be denoted by \( a_{ij} \). Using the results of Page [54] leads to the following formulas

\[
\frac{d\sigma_o}{d\Omega} = \frac{\gamma_o^2}{4\beta \gamma^2 (1 - \beta^2 \cos^2 \theta)^2} \frac{a_0}{a_0}
\]

\( a_0 = 1 - \beta^4 + 2\beta^2 \sin^2 \theta - \beta^4 \sin^4 \theta, \)

\[ a_{xx} = \frac{-(1 + \beta^2 + 2\beta^2 \sin^2 \theta - \beta^2 \sin^2 \theta)}{a_0}, \]

\[ a_{yy} = \frac{-(1 - \beta^2)^2 - \beta^4 \sin^4 \theta}{a_0}, \]

\[ a_{zx} = a_{xz} = 0, \]

\[ a_{xy} = a_{yx} = a_{yy} = 0. \] (3.29)

In these equations \( \beta^2 = (\gamma^2 - 1)/\gamma^2 \). Both \( \gamma \) and the scattering angle \( \theta \) of one of the photons are in the center-of-mass system. Numerical values are shown in figure 3.4 for kinetic energies and scattering angles in the laboratory. The fact that \( a_{xz} \) and \( a_{zx} \) are zero is due to a cancelation of terms depending on \( P_x P_z^\sigma \) and \( P_z P_x^\sigma \) when the polarization states of the outgoing particles are summed over. The asymmetry coefficients are the same as calculated by Corriveau [37].

The polarization transfer to the annihilation photons is left outside the scope of this thesis.
Figure 3.4: Asymmetry coefficients for annihilation in flight as a function of the laboratory kinetic energy and scattering angle of the photon.

3.9 Bremsstrahlung

The matrices 3.30, 3.31 and 3.33 presented in this section and the next one on pair production were previously calculated by Flöttmann [16]. The results given below are in agreement with his calculations.
3.9.1 Polarization transfer

The polarization transfer of an electron or positron to a photon due to bremsstrahlung was first calculated by Olsen and Maximon [89] taking into account Coulomb and screening effects. From their work the following transfer matrix can be deduced for the polarization vector of the photon

\[
T_{\text{brem}, \gamma} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
D & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & L & T & 0
\end{pmatrix},
\]

where

\[
I = (\epsilon_1^2 + \epsilon_2^2)(3 + 2, ) - 2\epsilon_1\epsilon_2(1 + 4u^2\xi^2, ), \\
D = 8\epsilon_1\epsilon_2u^2\xi^2, /I, \\
T = -4k\epsilon_2\xi(1 - 2\xi), u/I, \\
L = k[(\epsilon_1 + \epsilon_2)(3 + 2, ) - 2\epsilon_2(1 + 4u^2\xi^2, )]/I, \\
\epsilon_1 = \text{total energy of the incoming electron/positron in units } mc^2, \\
\epsilon_2 = \text{total energy of the outgoing electron/positron in units } mc^2, \\
\vec{p} = \text{electron/positron initial momentum in units } mc, \\
\vec{k} = \text{photon momentum in units } mc, \\
\vec{u} = \text{component of } \vec{p} \text{ perpendicular to } \vec{k}, \\
u = |\vec{u}|, \\
k = \epsilon_1 - \epsilon_2 \text{ energy of the photon in units } mc^2, \\
\xi = 1/(1 + u^2).
\]

The polarization vector of the incoming electron/positron must be rotated into the frame defined by the scattering plane (xz-plane) and the direction of the outgoing photon (z-axis) with the rotation matrices 3.14 and 3.15 prior to applying 3.30. The resulting polarization vector of the bremsstrahlung photon is also given in this frame, contains the Coulomb and screening effects

\[
, = \ln(1/\delta) - 2 - f(Z) + \mathcal{F}(\delta/\xi), \\
\delta = k/(2\epsilon_1\epsilon_2),
\]
Table 3.2: $F$ for intermediate values of the screening factor $\Delta$ [92].

<table>
<thead>
<tr>
<th>$\Delta$</th>
<th>$-F(\delta/\xi)$</th>
<th>$\Delta$</th>
<th>$-F(\delta/\xi)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.0145</td>
<td>40.0</td>
<td>2.001</td>
</tr>
<tr>
<td>1.0</td>
<td>0.0490</td>
<td>45.0</td>
<td>2.114</td>
</tr>
<tr>
<td>2.0</td>
<td>0.1400</td>
<td>50.0</td>
<td>2.216</td>
</tr>
<tr>
<td>4.0</td>
<td>0.3312</td>
<td>60.0</td>
<td>2.393</td>
</tr>
<tr>
<td>8.0</td>
<td>0.6758</td>
<td>70.0</td>
<td>2.545</td>
</tr>
<tr>
<td>15.0</td>
<td>1.126</td>
<td>80.0</td>
<td>2.676</td>
</tr>
<tr>
<td>20.0</td>
<td>1.367</td>
<td>90.0</td>
<td>2.793</td>
</tr>
<tr>
<td>25.0</td>
<td>1.564</td>
<td>100.0</td>
<td>2.897</td>
</tr>
<tr>
<td>30.0</td>
<td>1.731</td>
<td>120.0</td>
<td>3.078</td>
</tr>
<tr>
<td>35.0</td>
<td>1.875</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

where $f(Z)$ is the Coulomb correction term calculated by Davies, Bethe and Maximon [90]. An approximated version [91] which is accurate up to 4 digits for $a = aZ$ up to $2/3$ (Uranium) is

$$f(Z) = a^2 [(1 + a^2)^{-1} + 0.20206 - 0.0369a^2 + 0.0083a^4 - 0.002a^6].$$

The screening is included in $F$. Whether or not screening is important depends on the value of $\Delta = 6Z^2\xi/(121\delta)$

$$F(\delta/\xi) = \ln(111\delta/\xi Z^\delta)$$

for $\Delta \geq 120$; complete screening,

$$F(\delta/\xi) = 0,$$

for $\Delta \leq 0.5$; no screening,

$$F(\delta/\xi) = \text{linear interpolations between values of table 3.2}$$

for $0.5 < \Delta < 120$; intermediate screening.

The use of matrix 3.30 is only valid under certain conditions. Olsen and Maximon state that the results are restricted to high-energy bremsstrahlung i.e., $\epsilon_1, \epsilon_2, k \gg 1$ and the angle $(\theta)$ under which the radiation is emitted needs to be small: $u = p \sin \theta \approx p\theta \sim 1$ (in units of $mc$). The first constraint means that the results fail near the high frequency limit of the bremsstrahlung spectrum where $\epsilon_2 \sim 1$. The error in calculating the total cross section obtained with these approximations is [92, 93] $(Z/137)^2 (\ln \epsilon_1)/\epsilon_2$ which is around 1% at 10 MeV energy and $Z=26$. The region of applicability is discussed further in the next chapter.
3.9.2 Depolarization

The polarization vector for the outgoing electron/positron is not given by Olsen and Maximon. However, their results can be used to calculate the following transfer matrix

$$T_{\text{brem},e} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & G + H & F & 0 \\ 0 & E & G & 0 \\ 0 & 0 & 0 & G \end{pmatrix},$$

(3.31)

where

$$E = 4k\xi, \quad F = 4k\xi, \quad G = 4\epsilon_1\epsilon_2[(1 + \gamma) - 2u^2\xi^2], \quad H = k^2[1 + 8, (\xi - 0.5)^2]/I.$$  

(3.32)

Both the polarization vector of the incoming and outgoing electron/positron are given relative to the scattering plane (xz-plane) and the direction of the outgoing photon (z-axis). The rotation matrices 3.14 and 3.15 are used to transform the polarization vector from or to the electron or positron system.

3.10 Pair production

3.10.1 Polarization transfer

It follows from the work by Olsen and Maximon [89] that the polarization vector for an electron or positron after pair production, making approximations similar to the ones in the previous section, is

$$T_{\text{pair},e} = \begin{pmatrix} 1 & D & 0 & 0 \\ 0 & 0 & 0 & L \\ 0 & 0 & 0 & T \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

(3.33)

where

$$D = 8\epsilon(\epsilon - k)u^2\xi^2, /I,$$
Here, $\epsilon$ is the energy of the observed electron or positron. The matrix 3.33 for pair production is the transpose of matrix 3.30 with $\epsilon_2$ replaced by $-\epsilon_2$ ($k = \epsilon_1 + \epsilon_2$). This reflects the inverse nature of bremsstrahlung and pair production. The polarization vector of the electron or positron is given relative to the scattering plane (xz-plane) and the direction of the photon (z-axis). The rotation matrices 3.14 and 3.15 are again used to transform the polarization vector to the electron or positron system.